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ON ESSENTIAL ABSOLUTE CONTINUITY

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Throughout this paper D will denote a bounded domain in Euclidean n-space R^n, and T will be a bounded, continuous, single-valued transformation from D into R^n. For such transformations, concepts of essential bounded variation and essential absolute continuity have been defined and studied by Rado and Reichelderfer ([3], IV. 4). In this paper a characterization of essential absolute continuity will be given. The characterization suggests a definition of uniform essential absolute continuity and some of the consequences of this definition will be investigated.

1. For every point x in R^n a multiplicity function $K(x, T, D)$ is defined ([3], II. 3.2). T is said to be essentially of bounded variation (briefly eBV) in D provided $K(x, T, D)$ is Lebesgue summable in R^n ([3], IV. 4.1, Definition 1). Let $X_\infty = X_\infty (T, D)$ denote the set of points x in R^n for which $K(x, T, D)$ is infinite. Thus if T is eBV in D, then $\mathscr{L}X_\infty = 0$ (if A is a subset of R^n, then $\mathscr{L}A$ denotes its exterior Lebesgue measure). Since $K(x, T, D)$ is a lower semicontinuous function of x ([3], II. 3.2, Remark 10), X_∞ is a Borel set and, by Theorem 1 of [3], IV. 1.1, the set $T^{-1} X_\infty$ is also a Borel set.

2. If x is a point in R^n and C is a component of $T^{-1}x$ which is closed relative to R^n, then C is termed a maximal model continuum (x, T, D) ([3], II. 3.1, Definition 1). Denote by $\mathfrak{C} = \mathfrak{C}(T, D)$ the class composed of all sets C for which TC is a point in R^n and C is a maximal model continuum for (TC, T, D). Let $\mathfrak{E} = \mathfrak{E}(T, D)$ be the subset of $\mathfrak{C}$ consisting of those elements C each of which is an essential maximal model continuum (briefly e.m.m.c.) for (TC, T, D) ([3], II. 3.3, Definition 1); the set $E = E(T, D) = \cup C, C \in \mathfrak{E}$ ([3], II. 3.6). Let $\mathfrak{E}_i = \mathfrak{E}_i(T, D)$ be the subset of $\mathfrak{E}$ consisting of those elements C each of which is an essentially isolated e.m.m.c. (briefly e.i. e.m.m.c.) for (TC, T, D) ([3], II. 3.3, Definition 2); the set $E_i = E_i(T, D) = \cup C, C \in \mathfrak{E}_i$ ([3], II. 3.6.). Finally, let $\mathfrak{E}_i^p = \mathfrak{E}_i^p(T, D)$ be the subset of $\mathfrak{E}_i$ consisting of those elements of $\mathfrak{E}_i$ which consist of single points; the set $E_i^p = E_i^p(T, D) = \cup C, C \in \mathfrak{E}_i^p$ ([3], II. 3.6). The sets E, E_i and E_i^p are Borel sets ([3], II. 3.6, Theorem 1).

If T is eBV in D, then a necessary and sufficient condition that T be essentially absolutely continuous (briefly eAC) in D ([3], IV. 4.2) is

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that T satisfies the condition (N) on the set $E(T, \boldsymbol{D})$ ([3], IV. 4.2, Theorem 3) i.e., if $S \subseteq E$ and $\mathscr{L}S = 0$, then $\mathscr{L}TS = 0$.

DEFINITION 1. T will be said to satisfy the (ε, δ) condition on a subset A of $\boldsymbol{D}$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $S \subseteq A$ and $\mathscr{L}S < \delta$, then $\mathscr{L}TS < \varepsilon$. Clearly if T satisfies the (ε, δ) condition on each of a finite number of subsets of $\boldsymbol{D}$, then T satisfies the (ε, δ) condition on any subset of their union. Also, if A is a Borel subset of $\boldsymbol{D}$, then T satisfies the (ε, δ) condition on A if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that if S is a Borel subset of A and $\mathscr{L}S < \delta$, then $\mathscr{L}TS < \varepsilon$.

THEOREM 1. *Suppose T is eBV in $\boldsymbol{D}$. Then a necessary and sufficient condition that T be eAC in $\boldsymbol{D}$ is that T satisfies* the (ε, δ) *condition on the set $E(T, \boldsymbol{D})$.*

Proof. Since T is assumed to be *eBV* in $\boldsymbol{D}$ it suffices to prove that a necessary and sufficient condition that T satisfies the condition (N) on the set E is that T satisfies the (ε, δ) condition on E. Since the proof of the sufficiency is immediate, we proceed to a proof of the necessity. If T satisfies the condition (N) on E, then, by Lemma 4 of [3], IV. 4.2, $\mathscr{L}T(E - E_i^p) = 0$ and so T clearly satisfies the (ε, δ) condition on $E - E_i^p$. Since T is *eBV* in $\boldsymbol{D}$, $\mathscr{L}X_\infty = 0$ and so T satisfies the (ε, δ) condition on $T^{-1}X_\infty$. Since E is a subset of the union of the sets $E - E_i^p$, $T^{-1}X_\infty$ and $E_i^p - T^{-1}X_\infty$, in view of the remarks following Definition 1 it remains only to be shown that T satisfies the (ε, δ) condition on $E_i^p - T^{-1}X_\infty$ whenever T satisfies the condition (N) on E. Assume then that T does not satisfy the (ε, δ) condition on $E_i^p - T^{-1}X_\infty$. The proof will be completed by showing that T does not satisfy the condition (N) on E. Since E_i^p and $T^{-1}X_\infty$ are Borel sets, their difference is a Borel set. Thus the assumption that T fails to satisfy the (ε, δ) condition on $E_i^p - T^{-1}X_\infty$ implies, in view of the remarks following Definition 1, that there is an $\varepsilon_0 > 0$ such that for every positive integer k there is a Borel set $S_k \subseteq E_i^p - T^{-1}X_\infty$ such that $\mathscr{L}S_k < 1/2^k$ and $\mathscr{L}TS_k \geqq \varepsilon_0$. Let $S^* = \limsup S_k$ $(= \bigcap_{n=1}^\infty \bigcup_{k \geqq n} S_k)$. S^* is a subset of $E_i^p - T^{-1}X_\infty$ and so

$$S^* \subseteq E. \tag{1}$$

For every positive integer n, $S^* \subseteq \bigcup_{k \geqq n} S_k$ and so $\mathscr{L}S^* \leqq 1/2^{n-1}$. Hence

$$\mathscr{L}S^* = 0. \tag{2}$$

Let k be a positive integer and suppose $x \in TS_k$. Since $S_k \subseteq E_i^p - T^{-1}X_\infty$, $K(x, T, \boldsymbol{D}) < \infty$ and there is a point u in E_i^p such that $Tu = x$, Since $K(x, T, \boldsymbol{D}) < \infty$ there are at most a finite number of e.m.m.c.s. for $(x, T, \boldsymbol{D})$ ([3], II. 3.3, Definition 1 and II. 3.4, Theorem 3). But for every point u in E_i^p such that $Tu = x$ the set consisting of the point u is an e.m.m.c. for $(x, T, \boldsymbol{D})$. Thus there are at most a finite number of points u in $E_i^p - T^{-1}X_\infty$ for which $Tu = x$. Thus it has been shown that

(3) For every integer k, if x is in TS_k then $(E_i^p - T^{-1}X_\infty) \bigcap T^{-1}x$ is a finite set.

Since $\bigcup S_k \subseteq E_i^p - T^{-1}X_\infty$ it is easy to show that (3) implies that lim sup $TS_k = T(\limsup S_k)$ and so

$$\mathscr{L}(\limsup TS_k) = \mathscr{L}TS^*. \tag{4}$$

By Theorem 4 of [3], IV. 1. 1, the sets TS_k are measurable. Since T is a bounded transformation, $\mathscr{L}(\cup TS_k)$ is finite. Thus ([5], p. 17)

$$\mathscr{L}(\limsup TS_k) \geqq \limsup \mathscr{L}TS_k. \tag{5}$$

But $\mathscr{L}TS_k \geqq \varepsilon_0 > 0$ for all k and so

$$\limsup \mathscr{L}TS_k > 0 \tag{6}$$

By (4), (5) and (6),

$$\mathscr{L}TS^* > 0 \tag{7}$$

Now (1), (2) and (7) imply that T does not satisfy condition (N) on E.

3. Definition 2. For every positive integer j let $\boldsymbol{D}_j$ be a bounded domain in R^n and let T_j be a bounded, continuous, single-valued transformation from $\boldsymbol{D}_j$ into R^n. The transformations T_j will be termed uniformly essentially absolutely continuous (briefly UEAC) provided:

(i) For each j, $T_j \, eBV$ in $\boldsymbol{D}_j$ and

(ii) Given any $\varepsilon > 0$, there is a $\delta > 0$, depending only on ε, such that for all j the following is true: if S is a subset of $E(T_j, \boldsymbol{D}_j)$ and $\mathscr{L}S < \delta$, then $\mathscr{L}T_jS < \varepsilon$.

Note that if the transformations T_j are UEAC, then, by Theorem 1, for each j, T_j is eAC in $\boldsymbol{D}_j$.

Each point u in $\boldsymbol{D}$ is contained in a unique component of $T^{-1}Tu$ denoted by C_u. A subset U of $\boldsymbol{D}$ is termed a T set if $u \in U$ implies $C_u \subseteq U$ ([4], 1).

Theorem 2. *Let $\boldsymbol{D}$ be a bounded domain in Euclidean n-space R^n and let T be a bounded, continuous, single-valued transformation from $\boldsymbol{D}$ into R^n. For every positive integer j let $\boldsymbol{D}_j$ be a bounded domain in R^n and let T_i be a bounded, continuous, single-valued transformation from $\boldsymbol{D}_j$ into R^n,* *If*

(i) *The mappings* T_j *are* UEAC

(ii) *The mappings* T_j *converge to* T *uniformly on compact subsets of* $\boldsymbol{D}$ ([3], II. 3. 2, Remark 9) *and*

(iii) A *is a* T *set contained in* $E(T, \boldsymbol{D})$ *and* $\mathscr{L}A = 0$,

then $\mathscr{L}TA = 0$.

Proof Let $\varepsilon > 0$ be given and let δ be the corresponding positive number in (ii) of Definition 2. Since A is a subset of the open set $\boldsymbol{D}$ and $\mathscr{L}A = 0$, there is an open set O, containing A and contained in $\boldsymbol{D}$, such that $\mathscr{L}O < \delta$. Let $x \in TA$. Since $A \subseteqq E(T, \boldsymbol{D})$, there is a set C, e.m.m.c. for $(x, T, \boldsymbol{D})$, such that C meets A. $C \subseteqq A$ since A is a T set and so $C \subseteqq O$. By Definition 1 in [3], II. 3.3 there is a set D, which contains C and whose closure $\mathscr{K}D$ is contained in O, such that D is an indicator domain for $(x, T, \boldsymbol{D})$ ([3], II. 3.2). By definition $\mathscr{K}D \subseteqq \boldsymbol{D}$, x is not in $T\mathscr{B}D$ (where $\mathscr{B}D$ denotes the boundary of $\boldsymbol{D}$) and the topological index $\mu(x, T, D)$ ([3], II. 2) is not zero. Since $T\mathscr{B}D$ is compact, the ecart of x from $T\mathscr{B}D$, $e(x, T\mathscr{B}D)$, is positive ([3], I.1.4, Exercise 3). Since $\mathscr{K}D \subseteqq \boldsymbol{D}$, by (ii) there is a positive integer j_x such that, for $j > j_x$, $\mathscr{K}D \subseteqq \boldsymbol{D}_j$ and $\rho(T, T_j, \mathscr{K}D)$ the deviation of T_j from T on $\mathscr{K}D$ ([3], I. 1.5, Definition 5) is less than $e(x, T\mathscr{B}D)$. Clearly $\rho(T, T_j, \mathscr{B}D) \leqq \rho(T, T_j, \mathscr{K}D)$. Thus, for $j > j_x$, $\mathscr{K}D \subseteqq \boldsymbol{D} \cap \boldsymbol{D}_j$ and $\rho(T, T_j, \mathscr{B}D) < e(x, T\mathscr{B}D)$. By Theorem 6 of [3], II. 2.3, $\mu(x, T_j, D)$ is defined and equals $\mu(x, T, D)$. Thus D is an indicator domain for $(x, T_j, \boldsymbol{D}_j)$ and, by Lemma 4 of [3], II. 3.3, there is a set C_j, e.m.m.c. for $(x, T_j, \boldsymbol{D}_j)$, such that $C_j \subseteqq D$. Now $C_j \subseteqq O \cap E(T_j, \boldsymbol{D}_j)$ and $T_jC_j = x$. Thus $x \in T_j[O \cap E(T_j, \boldsymbol{D}_j)]$ for all $j > j_x$ and hence $x \in \liminf T_j[O \cap E(T_j, \boldsymbol{D}_j)]$. Since x was any point in TA, it has been shown that $TA \subseteqq \liminf T_j[O \cap E(T_j, \boldsymbol{D}_j)]$ and so

$$\mathscr{L}TA \leqq \mathscr{L}\liminf T_j[O \cap E(T_j, \boldsymbol{D}_j)]. \tag{1}$$

Since $E(T_j, \boldsymbol{D}_j)$ is a Borel set, $O \cap E(T_j, \boldsymbol{D}_j)$ is also a Borel set and so $T_j[O \cap E(T_j, \boldsymbol{D}_j)]$ is Lebesgue measurable. Thus ([5], p. 17)

$$\mathscr{L}\liminf T_j[O \cap E(T_j, \boldsymbol{D}_j)] \leqq \liminf \mathscr{L}T_j[O \cap E(T_j, \boldsymbol{D}_j)]. \tag{2}$$

Now

$$\mathscr{L}[O \cap E(T_j, \boldsymbol{D}_j)] \leqq \mathscr{L}O < \delta. \tag{3}$$

By the choice of δ, (3) implies that $\mathscr{L}T_j[O \cap E(T_j, \boldsymbol{D}_j)] < \varepsilon$ and hence

$$\liminf \mathscr{L}T_j[O \cap E(T_j, \boldsymbol{D}_j)] \leqq \varepsilon. \tag{4}$$

By (1), (2) and (4)

$$(5) \qquad \mathscr{L}TA \leqq \varepsilon.$$

Since (5) has been proved for an arbitrary $\varepsilon > 0$, it follows that $\mathscr{L}TA = 0$.

4. Theorem 2 suggests the question: under the hypotheses of Theorem 2 does T satisfy the condition (N) on $E(T, \boldsymbol{D})$? Note that T does satisfy the condition (N) on $E_i^p(T, \boldsymbol{D})$. In the remainder of the paper some results pertinent to this question will be presented.

Reichelderfer introduced the concept of the T magnification ([4], 6). It will be useful to have the definition repeated here.

Let $\mathfrak{D}^* = \mathfrak{D}^*(T, \boldsymbol{D})$ be the class composed of all domains D for each of which $\mathscr{K}D$ is contained in $\boldsymbol{D}$ and there exists an open oriented n-cube Q in R^n such that D is a component of $T^{-1}Q$. If C is a maximal model continuum for $(x, T, \boldsymbol{D})$ for some point x in R^n, for every positive number ε define

$$\bar{d}(C, \mathscr{L}T, \varepsilon) = \text{l.u.b.}\ \mathscr{L}TD/\mathscr{L}D,\ C \sqsubseteq D \in \mathfrak{D}^*,\ \delta TD \leqq \varepsilon$$

and

$$\underline{d}(C, \mathscr{L}T, \varepsilon) = \text{g.l.b.}\ \mathscr{L}TD/\mathscr{L}D,\ C \sqsubseteq D \in \mathfrak{D}^*,\ \delta TD \leqq \varepsilon$$

(If A is a subset of R^n, δA denotes the diameter of A).

$$\bar{d}(C, \mathscr{L}T) = \lim_{\varepsilon \to 0+} \bar{d}(C, \mathscr{L}T, \varepsilon)$$

and

$$\underline{d}(C, \mathscr{L}T) = \lim_{\varepsilon \to 0+} \underline{d}(C, \mathscr{L}T, \varepsilon).$$

If $\bar{d}(C, \mathscr{L}T)$ and $\underline{d}(C, \mathscr{L}T)$ are finite and equal, their common value is denoted by $M(C, T)$ and is termed the T magnification at C.

Lemma 1. *Let p be a positive number and let A be a T set with the following properties*:

(i) *If $u \in A$, then there is a set $C \in \mathfrak{C}_i(T, \boldsymbol{D})$ such that $u \in C$ and $\underline{d}(C, \mathscr{L}T) > p$.*

(ii) *If $C \in \mathfrak{C}_i(T, \boldsymbol{D})$ and $C \sqsubseteq A$, then for every domain G in R^n which contains TC and has a sufficiently small diameter it is true that $T^{-1}G$ possesses exactly one component D which meets A. Note that D must contain C and (provided only that the diameter of G is sufficiently small) be a* m.i.d. T ([4], 4 *and* 5, *Lemma* 2).

Then $\mathscr{L}A \leqq 1/p\, \mathscr{L}TA$.

Proof. Let η be any positive number. The proof will be completed by showing that $\mathscr{L}A \leqq 1/p\,\mathscr{L}TA + \eta$.

Let $x \in TA$ (the inequality is trivial if A is empty) and let $u \in A$ such the $Tu = x$. By (i) there is a set $C \in \mathfrak{C}_i(T, \boldsymbol{D})$ such that $u \in C$ and $\underline{d}(C, \mathscr{L}T) > p$. Thus there is an $\varepsilon > 0$ such that $\underline{d}(C, \mathscr{L}T, \varepsilon) > p$ and so

(1) $\qquad$ If $C \sqsubseteq D \in \mathfrak{D}^*$ and $\delta\, TD \leqq \varepsilon$, then $\mathscr{L}TD/\mathscr{L}D > p$

Since A is a T set, $C \sqsubseteq A$ and, by (ii), there exists a positive number r such that for every domain G in R^n which contains $TC(= x)$ and for which $\delta G \leqq r$ it is true that $T^{-1}G$ possesses exactly one component which meets A and, moreover, this component is a m.i.d. T containing C. For every positive integer i let Q_i be the open oriented n-cube with center at x and diameter equal to the smaller of ε, r and $1/i$. Then $T^{-1}Q_i$ possesses exactly one component D_i which meets A and D_i is a m.i.d. T containing C. By the Lemma in [4], 4, $TD_i = Q_i$ and $\mathscr{K}D_i \sqsubseteq D$. By definition, $D_i \in \mathfrak{D}^*$ and so, with the aid of (1), $\mathscr{L}D_i < 1/p\,\mathscr{L}TD_i$. Thus

(2) For every point x in TA there is associated a sequence of open oriented n-cubes Q_i with centers at x and a corresponding sequence of domains D_i such that, for all i, $\delta Q_i \leqq 1/i$, $\mathscr{L}D_i, < 1/p\,\mathscr{L}Q_i, D_i$ is a component of $T^{-1}Q_i$ and the only component of $T^{-1}Q_i$ which meets A.

Let $\mathfrak{Q}$ be the class of all n-cubes associated with points of TA in this manner. $\mathscr{L}TA$ is finite since T is bounded, and by a theorem of Rademacher ([2], p. 190) there is a $\mathfrak{Q}^*$, countable subclass of $\mathfrak{Q}$, such that

(3) $$TA \sqsubseteq \bigcup Q^*, Q^* \in \mathfrak{Q}^*$$

and

(4) $$\Sigma \mathscr{L}Q^* \leqq \mathscr{L}TA + \eta p.$$

(Rademacher's theorem is stated in terms of a covering made up of open n-spheres, but the corresponding theorem for a covering of open n-cubes is readily obtained from it). Let Q^* be an element of $\mathfrak{Q}^*$. By (2) there is a corresponding domain D^*, D^* a component $T^{-1}Q^*$ such that $\mathscr{L}D^* < 1/p\,\mathscr{L}Q^*$ and D^* is the only component of $T^{-1}Q^*$ which meets A. In this way exactly one domain D^* is associated with each $Q^* \in \mathfrak{Q}^*$. The class of domains D^* is countable and

(5) $$\Sigma \mathscr{L}D^* \leqq 1/p\, \Sigma \mathscr{L}Q^*.$$

Let $u \in A$. Then $Tu \in TA$ and by (3) there is a $Q^* \in \mathfrak{Q}^*$ such that $Tu \in Q^*$. Since the corresponding D^* is the only component of $T^{-1}Q^*$

which meets A it must contain u. Thus $A \subseteq \cup D^*$ and

$$(6) \qquad \mathscr{L}A \leqq \Sigma \mathscr{L} D^*.$$

By (4), (5) and (6), $\mathscr{L}A \leqq 1/p \mathscr{L} TA + \eta$. Since η is any positive number, the conclusion of the lemma is established.

LEMMA 2. *Let $\mathfrak{H}$ be a subclass of $\mathfrak{C}_i(T, \boldsymbol{D})$ such that if $C \in \mathfrak{H}$ then $\underline{d}(C, \mathscr{L}T) > 0$. Put $H = \cup C, C \in \mathfrak{H}$. If $\mathscr{L}TH = 0$, then $\mathscr{L}H = 0$.*

Proof. If H is not empty (the equality is trivial otherwise) then $\mathfrak{C}_i(T, \boldsymbol{D})$ is not empty and hence, by the Lemma in [4], 14, the set E_i can be expressed as the union of a countably infinite sequence of T sets U_k with the following property:

(1) If $C \in \mathfrak{C}_i$ and $U_k \supseteq C$, then for every domain G in R^n which contains TC and has a sufficiently small diameter it is true that $T^{-1}G$ possesses exactly one component D which meets U_k.

For every positive integer n let $\mathfrak{H}_n$ be the subclass of $\mathfrak{H}$ consisting of those elements C for which $\underline{d}(C, \mathscr{L}T) > 1/n$. Put $H_n = \cup C, C \in \mathfrak{H}_n$ and let $H_{nk} = H_n \cap U_k$. Then $H = \cup H_n$ and, for each n, $H_n = \cup H_{nk}$. The proof will be completed by showing that $\mathscr{L}H_{nk} = 0$ for arbitrary n and k. Since H_n and U_k are T sets,

(2) H_{nk} is a T set.

Clearly

(3) If $u \in H_{nk}$, then there is a set $C \in \mathfrak{C}_i$ such that $u \in C$ and $\underline{d}(C, \mathscr{L}T) > 1/n$.

By (1) and the definition of H_{nk},

(4) If $C \in \mathfrak{C}_i$ and $C \subseteq H_{nk}$, then for every domain G in R^n which contains TC and has a sufficiently small diameter it is true that $T^{-1}G$ possesses exactly one component D which meets H_{nk}.

(2), (3), (4) and Lemma 1 imply that $\mathscr{L}H_{nk} \leqq n\mathscr{L}TH_{nk}$. Since $TH_{nk} \subseteq TH$ and $\mathscr{L}TH = 0$, $\mathscr{L}TH_{nk} = 0$ and consequently $\mathscr{L}H_{nk} = 0$. Since n and k are arbitrary, it follows that $\mathscr{L}H = 0$.

5. THEOREM 3. *Let $\boldsymbol{D}$ be a bounded domain in Euclidean n-space R^n and let T be a bounded, continuous, single-valued transformation from $\boldsymbol{D}$ into R^n. For every positive integer j let $\boldsymbol{D}_j$ be a bounded domain in R^n and let T_j be a bounded, countinuous, single-valued transformation from $\boldsymbol{D}_j$ into R^n. Let $\mathfrak{B}$ be the subclass of $\mathfrak{C}_i(T, \boldsymbol{D})$*

consisting of those elements C for each of which C $M(C, T)$ exists and is positive and C contains more than a single point. Put $B = \bigcup C$, $C \in \mathfrak{B}$. If

(i) *The mappings T_j are UEAC.*

(ii) *The mappings T_j converge to T uniformly on compact subsets $\boldsymbol{D}$ and*

(iii) *T is eBV in $\boldsymbol{D}$*

then the following statements are equivalent:

(iv) *T satisfies the condition (N) on B,*

(iv)′ *$\mathscr{L}TB = 0$ and*

(iv)″ *$\mathscr{L}B = 0$*

and (i), (ii) *and* (iii) *together with* (iv) *or* (iv)′ *or* (iv)″ *imply that T is eAC in $\boldsymbol{D}$.*

Proof. First it will be shown that (i), (ii), (iii) and (iv) imply that T is eAC in $\boldsymbol{D}$. By the Theorem in [4], 16, there exist T sets V' and V'' contained in $\boldsymbol{D}$ such that $\mathscr{L}V' = 0$, $\mathscr{L}TV'' = 0$ and if $C \in \mathfrak{C}_i(T, \boldsymbol{D})$ and C does not meet $V' \cup V''$, then $M(C, T)$ exists and is positive. In view of (iii), in order to conclude that T is eAC in $\boldsymbol{D}$ it is sufficient to prove that T satisfies the condition (N) on $E = E(T, \boldsymbol{D})$. Clearly it is sufficient to show that T satisfies the condition (N) on each of the following sets whose union is E: $S_1 = E - E_i$, $S_2 = E_i^p$, $S_3 = (E_i - E_i^p) \cap V'$, $S_4 = (E_i - E_i^p) \cap V''$ and $S_5 = (E_i - E_i^p) - (V' \cup V'')$. Since T is eBV in $\boldsymbol{D}$, $\mathscr{L}TS_1 = 0$ (this is proved in the first step in the proof of the theorem in [4], 18) and so T satisfies the condition (N) on S_1. Any subset of S_2 is a T set contained in E and it follows by Theorem 2 that T satisfies the condition (N) on S_2. Again by Theorem 2, $\mathscr{L}TS_3 = 0$ and so T satisfies the condition (N) on S_3. $\mathscr{L}TS_4 \leqq \mathscr{L}TV'' = 0$ and so T satisfies the condition (N) on S_4. S_5 is a subset of B and so (iv) implies that T satisfies condition (N) on S_5.

If (i), (ii), (iii) and (iv) are satisfied, then it has just been shown that T satisfies the condition (N) on $E(T, \boldsymbol{D})$. Hence, by Lemma 4 of [3], IV. 4.2, $\mathscr{L}T(E - E_i^p) = 0$. Since B is a subset of $E - E_i^p$, (iv)′ must be satisfied. On the other hand, (iv)′ clearly implies (iv). Thus if (i), (ii) and (iii) are satisfied, (iv) and (iv)′ are equivalent.

By Lemma 2, $\mathscr{L}B = 0$ if $\mathscr{L}TB = 0$. On the other hand, since B is a T set contained in $E(T, \boldsymbol{D})$, (i) and (ii) imply, by Theorem 2, that $\mathscr{L}TB = 0$ if $\mathscr{L}B = 0$. Hence if (i) and (ii) are satisfied, then (iv)′ and (iv)″ are equivalent.

6. It is reasonable to inquire whether (i), (ii) and (iii) in Theorem 3 are sufficient to conclude that T is eAC in $\boldsymbol{D}$. After all, each of the sets C in $\mathfrak{B}$ is a non-point continuum for which the T magnification is positive and yet whose image under T is a single point in R^n. Might not (i), (ii) and (iii) imply, say, (iv)′ (or equivalently (iv) or (iv)″)? Since the class $\mathfrak{B}$ is clearly countable when T is a transformation into R^1, TB is then a countable set. Thus (iv)′ is always satisfied when T is a transformation into R^1. However, the author has constructed an example in R^2 for which (i), (ii) and (iii) are satisfied and for which the limit transformation is not eAC ([6]). In the example the limit transformation T is modeled on an example by Cesari ([1], IV. 13.1, Example A). The transformation that Cesari defined provides an example of a plane mapping that is eBV but not eAC. The example in [6] is somewhat more complicated by the need for (i) and (ii) to be satisfied.

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