Pacific Journal of Mathematics

ON MEASURABILITY OF STOCHASTIC PROCESSES IN PRODUCTS SPACE

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Vol. 12, No. 1

January 1962

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1. Introduction. Let \mathscr{X} be a σ -algebra of subsets of X, and T a set. Let $\Omega = X^{T}$, and let \mathscr{C} be the σ -algebra of subsets of Ω generated by the finite cylinder sets, i.e., sets of the form $\Lambda = \{\omega \in \Omega \mid \omega(t_{1}) \in A_{1}, \cdots, \omega(t_{n}) \in A_{n}\}, A_{1}, \cdots, A_{n} \in \mathscr{X}$. Let P_{0} be a probability measure on \mathscr{C} . Thus the coordinate variables $x_{t}(\omega) = \omega(t), t \in T$, are the Kolmogorov version [5] of the stochastic process with joint distributions $F_{t_{1}}, \cdots, t_{1}(A_{1}, \cdots, A_{n}) = P_{0}\{\Lambda\}$. For various purposes, it is appropriate to enlarge this σ -algebra and extend the measure. In the present paper two methods of doing this will be mentioned, and one of the methods will be studied.

[A] Suppose X is a compact Hausdorff space and \mathscr{C} the Borel sets. Then Ω is a compact Hausdorff space in the product topology. A straightforward application of the Stone-Weierstrass theorem and the Riesz-Markov theorem shows that there is a unique regular measure on the Borel subsets \mathscr{B} of Ω which agrees with P_0 on \mathscr{C} , provided the finite-dimensional marginal measures are all regular. We call this measure P. This idea is due to S. Kakutani [3], and was discussed in detail by E. Nelson [8].

[B] By a condition is meant a set-valued function k from T to \mathscr{X} . For any condition k, we define

 $\Gamma(k) = \{ \omega \mid \omega(t) \in k(t) \text{ for all } t \in T \}$, and $\Gamma(S, k) = \{ \omega \mid \omega(t) \in k(t) \text{ for all } t \in S \}$,

S being a subset of T. It is possible to extend P_0 to a class of sets of the form $\Gamma(k)$, as follows.

The following lemma is a straightforward generalization of the separability lemma in [1], p. 56.

LEMMA 1.1. For any condition $k \ni a$ countable set $S \subset T$ such that $P_0\{\Gamma(S, k) - \Gamma(\{t\}, k)\} = 0$ for all $t \in T$.

The proof is a simple exhaustion argument. Such a countable subset S will be called *determining* for k.

Let \mathscr{K} be a family of sets with the properties (i) $X \in \mathscr{K}$

Received March 27, 1961. Written partly with support from Contract NONR 222 (60) and partly while an NSF fellow.

(ii) any countable subfamily of \mathscr{K} with the finite intersection property (F.I.P.) has nonnull intersection. Such a family will be called *countably compact*. If (ii) holds without the countability restriction, then \mathscr{K} is called *compact*. If a condition k has values in \mathscr{K} , it will be called a \mathscr{K} -condition.

The set of positive integers will be written I. Unions and intersections whose index set is I will be written simply \bigcup_{j} , etc. rather than $\bigcup_{j\in I}$, etc. The following result can then be proven. It is stated in [7].

LEMMA 1.2. Let S_n be a determining set for the \mathcal{K} -condition k_n , $n \in I$. Let $\Delta = \bigcup_n \{ \Gamma(S_n, k_n) - \Gamma(k_n) \}$. Then Δ has inner P_0 – measure 0.

 $\mathscr{C}_{\mathscr{H}}$ is now defined to be those subsets Γ of Ω such that $\exists \Gamma'$ in \mathscr{C} with $(\Gamma - \Gamma^{1}) \cup (I^{\prime 1} - \Gamma)$ subset of a set of the form of \varDelta in the above lemma. These sets Γ form a σ -algebra, and the assignment to Γ of the same measure as the P_{0} -measure of Γ^{1} determines unambiguously a measure $P_{\mathscr{H}}$ on $\mathscr{C}_{\mathscr{H}}$, which is an extension of P_{0} . This construction, based on ideas of Doob and Khintchine [4] is done by A. Mayer in [6], [7].

REMARK 1.1. Notice that $\mathscr{C}_{\mathscr{K}}$ contains all sets of the form $\Gamma(k)$, for any \mathscr{K} -condition k, assigning to such a set the measure $P_0\{\Gamma(S, k)\}$, S being any determining set for k.

REMARK 1.2. If X is compact Hausdorff, \mathscr{X} the Borel sets, \mathscr{K} the compact sets, and P_0 satisfies the regularity condition of [A], then $\mathscr{C}_{\mathscr{K}} \subset \mathscr{B}$, and $P | \mathscr{C}_{\mathscr{K}} = P_{\mathscr{K}}$. This is a consequence of the following (under the hypotheses of the last sentence):

LEMMA 1.3. If S is determining for the condition k, and k(t) is compact for all t, then $P\{\Gamma(k)\} = P\{\Gamma(S, k)\}.$

Proof. By Theorem 2.2 of [8] there is some countable subset S_1 of T such that $P\{\Gamma(S_1, k)\} = P\{\Gamma(k)\}$. Now, $\Gamma(S_1, k) \supset \Gamma(S \cup S_1, k) \supset \Gamma(k)$, so $P\{\Gamma(S \cup S_1, k)\} = P\{\Gamma(k)\}$. But

$$\Gamma(S, k) = \Gamma(S \cup S_1, k) \cap \bigcap_{s \in S} \{ \Gamma(S_1, k) = \Gamma(\{s\}, k) \}.$$

Thus $P\{\Gamma(S, k)\} = P\{\Gamma(S \cup S_1, k)\}.$

We will deal mainly with the situation where T is a topological space, and with a certain σ -subalgebra $\mathscr{D}_{\mathscr{H}}$ of $\mathscr{C}_{\mathscr{H}}$, where $\mathscr{D}_{\mathscr{H}}$ is defined like $\mathscr{C}_{\mathscr{H}}$, except that the only conditions k used for $\mathscr{D}_{\mathscr{H}}$ will be those of the form

$$k(t) = K \text{ for } t \in U$$
$$X \text{ for } t \notin U,$$

U being an open set in T, and $K \in \mathcal{H}$. For such a k, we write $\Gamma(k)$ as $\Delta(U, K)$. The restriction of $P_{\mathcal{H}}$ to $\mathcal{D}_{\mathcal{H}}$ will be called $Q_{\mathcal{H}}$.

If \mathscr{K} consists of closed sets in a metric space, T is locally compact, and τ is a regular measure on T, then $(\mathscr{D}_{\mathscr{H}}, Q_{\mathscr{H}})$ has the convenient property that whenever the map $t \to x_t$ (where $x_t(\omega) = \omega(t)$) is measurable in probability, i.e. is continuous in probability outside of some τ -null set, then the map $(\omega, t) \to \omega(t)$ can be made measurable the $\mu \times \tau$ -completion of $\mathscr{M} \times \mathscr{T}$, where \mathscr{T} is the Borel sets of T and (\mathscr{M}, μ) some extension of $(\mathscr{D}, \mathscr{H}, Q_{\mathscr{H}})$. (See [7], Theorem 2.) This says, in a sense, that $\mathscr{D}_{\mathscr{H}}$ is "not too large." On the other hand, it is "not too small," in the sense that it contains many natural subsets which are not in \mathscr{C} ; this will be shown.

In § 2 are given some examples and general remarks concerning compact and countably compact families.

In [8], with X and T compact metrizable spaces, various natural subsets of Ω and $\Omega \times T$ were shown to be in \mathcal{R} , $\overline{\mathcal{R}}$, or product σ -algebras derived from them (the bar over a σ -algebra signifies completion with respect to the measure being considered on it). In § 3 and 4 we show (in a somewhat more general context) that these subsets are in $\mathfrak{D}_{\mathcal{R}}, \overline{\mathfrak{D}}_{\mathcal{R}}$, or the corresponding product σ -algebrans, where \mathcal{K} is a countably compact family of closed subsets of X which contains a complete system of neighborhoods for each point of X (or, briefly, generates the topology of X).

2. Some topological considerations.

LEMMA 2.1. Let X be a 1-st countable Hausdorff space. Then any countable compact family \mathcal{K} of subsets of X which generates the topology of X consists of closed sets only.

Proof. Suppose $K \in \mathscr{K}$, and $x \notin K$. Choose a countable family $\{K_n \mid n \in I\}$ of neighborhoods of x in \mathscr{K} , with $\bigcap_n K_n = \{x\}$. If $x \in \overline{K}$, then $K \cap K_1 \cap \cdots \cap K_n$ is never empty. Thus, $K \cap \bigcap_n K_n$ is nonempty, so $x \in K$.

REMARK 2.1. If we assume that X actually has a countable base for its open sets, then clearly any intersection of sets of \mathscr{K} can be reduced to a countable intersection. In particular, it follows that \mathscr{K} is actually a *compact* family, not just countably compact.

LEMMA 2.2. (Alexander). Let \mathcal{K} be a compact family of subsets

of a set X. Let $\widetilde{\mathscr{K}}$ be the family of arbitrary intersections of finite unions of sets of then $\widetilde{\mathscr{K}}$ is closed under arbitrary intersections and finite unions, and is again a compact family.

Proof. See [9], p. 139.

COROLLARY 2.1. The most general compact family of sets on a set X arises by choosing a subfamily of the closed sets, for some compact topology on X.

Proof. Given a compact family \mathscr{K} on a set X, use $\widetilde{\mathscr{K}}$ as the family of closed sets for X; this gives a compact space.

REMARK 2.2. The property of *countable* compactness does *not* persist from \mathscr{K} to $\widetilde{\mathscr{K}}$. For example, let A be all ordinals up to and including the first uncountable ordinal α_0 . Let B be the rational numbers $\{0; 1, \frac{1}{2}, 1/3, \cdots\}$. Let $X = A \times B - \{(\alpha_0, 0)\}$. Let \mathscr{K} consist of all sets of the form $K_{\alpha,n} = \{(\alpha^1, x) | \alpha^1 > \alpha, x < 1/n\}$, where α is a countable ordinal and $n \in I$. Then *no* countable intersection of sets $K_{\alpha,n}$ is empty, so \mathscr{K} is countably compact. But let $L_n = \bigcap_{\alpha < \alpha 0} K_{\alpha,n} = \{(\alpha_0, x) | x < 1/n\}$. Then the L_n have the F.I.P., but $\bigcap_n L_n = \phi$.

In § 3 we shall be considering countably compact families \mathcal{K} on separable metrizable spaces X, \mathcal{K} generating the topology of X. Some examples follow.

(a) X a Banach space which is separable and a dual, \mathcal{K} the set of all closed spheres. This is mentioned in [6].

In this connection, however, notice that the separable Banach space C of all continuous functions on, say, the closed interval [-1, 1], is not a dual; and, in fact, the family of all closed spheres in this Banach space is *not* a countably compact family. To see this, let

$${f}_n(\lambda) = egin{cases} 1 ext{ if } -1 &\leq \lambda \leq 0 \ 1 - n\lambda ext{ if } 0 < \lambda < rac{1}{n} \ 0 ext{ if } rac{1}{n} &\leq \lambda \leq 1 \ , \end{cases}$$

and let $f'_n(\lambda) = -f_n(-\lambda)$. Let K_n be the closed sphere of radius 2 about $f'_n - 2$, and K'_n the closed sphere of radius 2 about $f'_n + 2$. Then

$$K_n\cap K'_n=\{g\,|\,f'_n\leq g\leq f_n\}
eq\phi$$
 .

Since $f_1 \ge f_2 \ge \cdots$ and $f'_1 \le f'_2 \le \cdots$, we have $K_1 \cap K'_1 \supset K_2 \cap K'_2 \supset \cdots$. Thus, the spheres $\{K_n, K'_n | n = 1, 2, \cdots\}$ have the F.I.P., but there is no continuous function in their intersection. The author does not know, however, whether some \mathscr{K} does not exist for C.

(b) An example where the metric space is not complete: let X be the nondyadic numbers in the unit interval. \mathscr{K} will be defined as follows. Let S_n be the set of dyadics of the form $k/2^n$, $k = 0, \dots, 2^n$. Then $X = [0, 1] - \bigcup_n S_n$. Let \mathscr{K}_n be the intersection with X of intervals [a, b], where $a = (k + 1/8)1/2^n$, $b = (k + 7/8)1/2^n$, $k = 0, 1, \dots, 2^n - 1$. Let $\mathscr{K} = \bigcup_n \mathscr{K}_n$.

To see that \mathscr{K} generates the topology of X, we must show that any $x \in X$ is an interior point of some interval in \mathscr{H}_n , for arbitrarily large n. But a nondyadic number x is characterized by the property that a zero followed by a one occurs arbitrarily far out in its dyadic expansion. Thus, for arbitrarily large n, we can get $k/2^n + 1/2^{n+2} < x < k/2^n + 1/2^{n+1}$, so that x is interior to an interval of \mathscr{H}_n .

To see that \mathscr{H} is countably compact, suppose we have a sequence K_1, K_2, \cdots with the F.I.P. Assume repetitions have been eliminated. Then no two can come from the same \mathscr{H}_n , since two members of \mathscr{H}_n are either identical or disjoint. Consider now the closed intervals \overline{K}_n in [0, 1]. These have the F.I.P., and are closed in [0, 1]. Thus their intersection is nonempty. Further, let $K_n \in \mathscr{H}_{i_n}$. Then $\overline{K}_n \cap S_{i_n} = \phi$, so $(\bigcap_n \overline{K}_n) \cap (\bigcup_m S_{i_m}) = \phi$. Since i_m does not repeat itself, and since $S_1 \subset S_2 \subset \cdots$, we have $\bigcup_m S_{i_m} = \bigcup_n S_n$. Thus, $(\bigcap_n \overline{K}_n) \cap X \neq \phi$. But this is the same as $\bigcap_n K_n$.

(c) A metric space for which *no* countably compact family can generate the topology: let X be the dyadic numbers in [0, 1]. Suppose, in fact, we had such a family \mathscr{H} . Let x_1, x_2, \cdots be an enumeration of X. Then one could choose a sequence K_j^n of neighborhoods of $x_j, K_j^n \in \mathscr{H}$, and with the length of K_j^n less than 1/n + j. Let U_j^n be the interior of $\overline{K_j^n}$. Then $x_j \in U_j^n$. Consider now the set $\bigcap_n \bigcup_j U_j^n$. This is a G_{δ} in the reals, and contains all the dyadics. Then it must contain some nondyadics, since the dyadics are not a G_{δ} . On the other hand, if ξ is a nondyadic in $\bigcap_n \bigcup_j U_j^n$, then ξ is in some $\bigcap_n U_{j_n}^n$. Thus $\{K_{j_n}^n \mid n \in I\}$ has the F.I.P. But $\bigcap_n \overline{K_{j_n}^n} = \{\xi\}$, since the lengths of the $K_{j_n}^n$ go to zero as $n \to \infty$. Thus $\bigcap_n K_{j_n}^n = \bigcap_n (\overline{K_{j_n}^n} \cap X) = \phi$.

The question remains open whether, for example, every complete separable metric space has a countably compact family which generates its topology.

3. Measurability of various classes of functions. Throughout this section, let X be a separable metric space; \mathscr{X} the Borel sets. Let \mathscr{K} be a collection of sets in \mathscr{X} such that

(a) \mathcal{K} is a countably compact family,

(b) \mathcal{K} generates the topology of X.

Let T be a compact metric space, and consider $\mathscr{D}_{\mathscr{H}}, Q_{\mathscr{H}}$, as defined in § 1. For brevity, we write simply \mathscr{D}, Q . We remark that the results of this section extend immediately to the case where T is locally compact metrizable, and separable, since the classes of functions discussed are defined by their local properties in T.

Let \mathscr{K}_0 be a countable subset of \mathscr{K} which still contains a complete system of neighborhoods at each point. Also, let $K_{\varepsilon,n}$ be an enumeration of the sets of \mathscr{K}_0 of diameter $\leq \varepsilon$. Let $\Lambda(\varepsilon, S) = \bigcap_{s \in S} \{\omega \mid \mathsf{E} \text{ some open}$ neighborhood U of s and some n such that ω sends U into $K_{\varepsilon,n}\}$. Finally, let $\varPhi(\varepsilon, S) = \{\omega \mid \exists \text{ some open } U \supset S \text{ and } n \text{ such that } \omega \text{ sends } U \text{ into } K_{\varepsilon,n}\}$.

LEMMA 3.1. $\Lambda(\varepsilon, S)$ and $\Phi(\varepsilon, S)$ are in \mathscr{D} for any closed set S and any $\varepsilon > 0$.

Proof. Let \mathscr{U} be a countable base for the open sets of T. Let $\mathscr{U}_{1}, \mathscr{U}_{2}, \cdots$ be an enumeration of the finite coverings of S by sets in \mathscr{U} . Then $\Lambda(\varepsilon, S) = \bigcup_{n} \bigcup_{m} \bigcap_{U \in \mathscr{U}_{n}} \mathcal{L}(U, K_{\varepsilon,m})$, and

$$\varPhi(\varepsilon, S) = \bigcup_m \bigcup_n \ \varDelta(\bigcap_{U \in \mathscr{U}_m} \ U, K_{\varepsilon, m}) .$$

THEOREM 3.1. The set of all functions which are continuous at all points of the closed set $S \subset T$ is in \mathscr{D} .

Proof. This set is precisely $\bigcap_m \Lambda(1/m, S)$.

THEOREM 3.2. For any regular measure ν on T, the set of ν -almost everywhere continuous functions is in \mathcal{D} .

Proof. Let $V_{n,m}$, $n, m \in I$, be an enumeration of those finite unions of sets \mathscr{U} such that $\nu(V_{n,m}) < 1/n$. A function ω is ν -almost everywhere continuous if and only if for arbitrary small $\varepsilon > 0$ there is a closed set S whose complement has arbitrarily small measure, such that $\omega \in \varDelta(\varepsilon, S)$. But $\omega \in \varDelta(\varepsilon, S) \Rightarrow \omega \in \varDelta(\varepsilon, \overline{U})$ for some open set $U \supset S$. New, S^{\perp} is a union of sets in \mathscr{U} . Since $S^{\perp} \supset U^{\perp}$, and U^{\perp} is compact, U^{\perp} is covered by a finite union of sets of \mathscr{U} which does not intersect S, and thus has ν -measure no greater than that of S. Hence, the set of ν -almost everywhere continuous functions is contained in $\bigcap_{J} \bigcap_{n} \bigcup_{m} \varDelta(1/j, V_{n,m})$. The converse inclusion is obvious.

THEOREM 3.3. The set of functions whose points of discontinuity form a first category set, is in \mathcal{D} ,

Proof. Let $O_{\varepsilon}(\omega) = \{s \mid \text{for every open } U \ni s \exists r, t \in U \text{ with } d(\omega(r), \omega(t)) > \varepsilon\}$. $O_{\varepsilon}(\omega)$ is a closed set, and increases as ε decreases. Thus, the set $\bigcup_{\varepsilon \to 0} O_{\varepsilon}(\omega)$ is of first category if and only if each $O_{\varepsilon}(\omega)$ is nowhere dense. Let D be a countable dense subset of T, and let $D_{n,m}$ be an enumeration of the finite 1/m-dense subsets of $D(\text{i.e. every point of } T \text{ is within } 1/m \text{ of some point of } D_{n,m}$, for every n, m). Then following Nelson in Theorem 3.3 of [8], $O_{\varepsilon}(\omega)$ is nowhere dense if and only if, for every $m \in I$, $O_{\varepsilon}(\omega) \subset \text{some } D_{n,m}^{\perp}$. Thus, ω has a first category set of discontinuities if and only if

$$\omega \in \bigcap_{j} \bigcap_{m} \bigcup_{n} \angle (\downarrow_{j}, D_{n,m})$$
.

THEOREM 3.4. Let T be a compact interval. Then the set of all ω with discontinuities of the first kind only, is in \mathcal{D} .

Proof. If ω has only discontinuities of the first kind, then for any $\varepsilon > 0$ one can choose, for each $t \in T$, an open interval R_t such that there are some fixed integers n_+ and n_- for which $\omega(s) \in K_{\varepsilon,n_+}$ for all s in $(R_t - \{t\})_+ \cap T$ and $\omega(s) \in K_{\varepsilon,n_-}$ for all s in $(R_t - \{t\})_- \cap T$. (Note: $(R_t - t)_{(\pm)}$ denotes the $\binom{\text{upper}}{\text{lower}}$ of the two intervals into which $R_t - \{t\}$ splits.)

Let S_t be a rational open interval with $t \in S_t \subset \overline{S}_t \subset R_t$, and, for given $\delta > 0$, let U_t be another rational interval, of length $< \delta$, with $t \in U_t \cup S_t$. Then $\omega \in \mathcal{O}(\varepsilon, (\overline{S}_t - U_t)_+ \cap T)$, and $\omega \in \mathcal{O}(\varepsilon, (\overline{S}_t - U_t) \cap T)$. Since T can be covered by finitely many of the S_t , we finally get the following: let $\mathscr{S}_1, \mathscr{S}_2, \cdots$ be an enumeration of the finite coverings of T by rational open intervals. For any rational open interval S, let $\mathscr{V}_k(S)$ be the set of all open rational subintervals of S having length < 1/k. Then if ω has only discontinuities of the first kind, we have $\omega \in \bigcap_n \bigcup_m \bigcap_k \bigcap_{s \in \mathscr{S}_m} \bigcup_{v \in \mathscr{U}_k(S)} \{ \mathscr{O}(1/m, (\overline{S} - U)_+ \cap T) \cap \mathscr{O}(1/n, (\overline{S} - U)_- \cap T) \}$. And conversely, if ω has a discontinuity of the second kind at t_0 , then there is some integer n such that no matter what open rational interval S one chooses about t_0, ω will oscillate by more that 1/n either in $(\overline{S} - U)_+ \cap T$ or $(\overline{S} - U)_- \cap T$, provided U is a sufficiently short interval. Thus, the inclusion is an equality.

THEOREM 3.5. The set Θ of pairs (ω, t) in $\Omega \times T$ such that ω is discontinuous at t, is in $\mathscr{D} \times \mathscr{T}_r(\mathscr{T}_r \text{ being the Borel sets in } T)$. The function $(\omega, t) \to \omega(t) \mathscr{C} \times \mathscr{T}_r | \Theta^{\perp}$ -measurable, and a fortiori $\mathscr{D} \times \mathscr{T}_r$ -measurable.

(Note: for a σ -algebra α on a set Z, and a set $Z_0 \subset Z$, we denote by $\mathscr{A} \mid Z_0$ the σ -algebra $\{A \cap Z_0 \mid A \in \mathscr{A}\}$. In case $Z_0 \in \mathscr{A}$, we get

$$\mathscr{A} \mid Z_{\scriptscriptstyle 0} = \{A \in \mathscr{A} \mid A \subset Z_{\scriptscriptstyle 0}\}$$
 .)

Proof of Theorem 3.5. \mathscr{U} is again a countable basis for the open sets of T. Then we have $\Theta^{\perp} = \bigcap_{n} \bigcup_{U \in \mathscr{U}} \bigcup_{m} [\mathcal{A}(U, K_{1/n,m}) \times U]$. As for measurability of the function $(\omega, t) \to \omega(t)$: let T_0 be a countable dense subset of T. Let \mathscr{V}_k be a finite covering of T by sets of diameter < 1/k. Let $\{g_{k,V} \mid V \in \mathscr{V}_k\}$ be a partition of unity for \mathscr{V}_k . Let f be a continuous function on X. Let $\tilde{f}_k(\omega, t) = \sum_{V \in \mathscr{U}_k} g_{k,V}(t) \sup_{s \in T_0 \cap V} f(\omega(s))$. Then \tilde{f}_k is $\mathscr{C} \times \mathscr{B}_T$ -measurable, and, for fixed ω , $f_k(t, \omega)$ is continuous in t. Furthermore, at all points (ω, t) in Θ^{\perp} , we have $\tilde{f}_k(\omega, t) \to f(\omega(t))$. Thus, $f(\omega(t))$ is $\mathscr{C} \times \mathscr{B}_T | \Theta^{\perp}$ -measurable for each continuous f. Now : for any closed set K in X there is a continuous function f_K which is 1 only on that set. Then $\{(\omega, t) \mid \omega(t) \in K\} \cap \Theta^{\perp} = \{(\omega, t) \mid f_K(\omega(t)) = 1\} \cap \Theta^{\perp}$, which is in $\mathscr{C} \times \mathscr{B}_T | \Theta^{\perp}$. This completes the proof.

The generalization of Theorem 4.1 of [8] now goes through exactly as done there, by applying Fubini's theorem. Namely, if ν is a regular measure on T, then $\{\omega \mid \omega \text{ continuous at } t\}$ has Q-measure 1 for ν -almost every $t \iff \{t \mid \omega \text{ continuous at } t\}$ has ν -measure 1 for Q-almost every $t \iff \Theta$ has $Q \times \nu$ -measure 0. Similarly, Theorem 4.2 of [8] generalizes to the present context: if $\{\omega \mid \omega \text{ continuous at } t\}$ has Q-measure 0 for each $t \in T$, then $\{\omega \mid \text{the discontinuities of } \omega \text{ form a cat } I \text{ set in } T\}$ has Q-measure 1. The proof is gotten in the same way, but substituting \tilde{f} of Theorem 3.5 above for Nelson's f^+ . The details will be omitted.

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

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