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1. The following problem was raised by V. Klee in a seminar on convex sets :

How are the assumptions of Helly's [5] theorem on intersections of convex sets to be modified in order to guarantee that the intersection of all the members of a finite family of convex sets in E^n be of dimension at least k ?

In this note we shall prove the following variant of Helly's theorem :

THEOREM. *Let k and n, $0 \leqq k \leqq n$, be integers, and let $h(n, k)$ be the least integer rendering true the following statement* :

For every finite family $\mathscr{K}$ of convex sets in E^n, containing at least $h(n, k)$ members, the intersection $\Pi\mathscr{K}$ of all members of $\mathscr{K}$ has dimension dim $\Pi\mathscr{K}$ at least k provided the intersection of every $h(n, k)$ members of $\mathscr{K}$ is of dimension k at least.

Then

(i) $h(n, 0) = n + 1$

(ii) $h(n, 1) = 2n$

(iii) $h(n, k) = 2n - k$ for $1 < k < n$

(iv) $h(n, n) = n + 1$.

Proof. Essentially, only the assertion (iii) is new. The first statement, $h(n, 0) = n + 1$, is Helly's well-known theorem [5]. The assertion $h(n, n) = n + 1$ is a theorem due to Vincensini [14], generalized by the following result of Klee [9] (which, in turn, follows easily from Helly's theorem): If $\mathscr{K}$ is a finite family of convex sets (or an infinite family of compact convex sets) in E^n and if C is a convex set such that for every $n + 1$ members of $\mathscr{K}$ there exists a translate of C contained in their intersection, then a suitable translate of C is contained in $\Pi\mathscr{K}$.

The statement $h(n, 1) = 2n$, or facts equivalent to it, have been proved many times in a different terminology (Steinitz's [13] "irreducible all-sided families of rays", Dines-McCoy's [2] theorem on intersections of half-spaces through the origin (see also [11]) ; Gustin's [4] variant of Carathéodory's theorem ; Robinson's [12] theorem on intersections of spherical convex sets). To derive (ii) from, e.g., the theorem of Dines-McCoy ("A family of closed half-spaces in E^n, each containing the origin in its boundary, has a common point different from 0 provided this is true for every $2n$-membered subfamily") we observe first that it

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is sufficient to prove the theorem for families of polyhedra. (In each intersection of $h(n, k)$ members of the given family we choose an arbitrary point; each of the original sets is replaced by the convex hull of the (finitely many) chosen points belonging to it. See [11] for a more elaborate discussion of this procedure.) Next, assuming that the intersection of all the polyhedra $K \in \mathscr{K}$ (nonempty by Helly's theorem) consists of one point only (which we take as the origin 0), we may assume (because of (iv)) that 0 belongs to the boundary of each $K \in \mathscr{K}$. Replacing $\mathscr{K}$ by the family $\mathscr{H}$ of all closed half-spaces which contain some $K \in \mathscr{K}$ and have 0 in their boundary, it follows from the theorem of Dines-McCoy that the intersection of some $2n$ members of $\mathscr{H}$ is the single point 0; the intersection of the $2n$ or less members of $\mathscr{K}$ contained respectively in these members of $\mathscr{H}$ is then a fortiori reduced to 0.

Before proceeding to prove (iii), we shall establish a few auxiliary results, which are also of some independent interest.

2. A subset of the $n-1$ dimensional sphere $S^{n-1} = \{x \in E^n : \|x\| = 1\}$ is called *convex* if and only if it is the intersection of closed hemispheres. (In particular, a pair of antipodal points, or any "great S^k" for $k \leqq n-1$, is a convex set). Sets "convex" in this sense have been studied, e.g., by Favard [3] and Robinson [12]; in some connections other definitions of spherically convex sets seem to be more appropriate (see, e.g. Horn [7]).

LEMMA 1. *If $\mathscr{C}$ is a finite family of convex subsets of S^n such that the intersection of each $n+1$ members of $\mathscr{C}$ has a nonempty interior and such that $\Pi\mathscr{C} \neq \emptyset$, then Int $\Pi\mathscr{C} \neq \emptyset$.*

Proof. Let $P \in \Pi\mathscr{C}$ and let H be the *open* hemisphere of $S^n \subset E^{n+1}$ centered at P. Project H centrally onto the E^n tangent to S^n at P. Let $C' \subset E^n$ be the projection of $H \cap C$ for $C \in \mathscr{C}$. Then the intersection of each $n+1$ of the sets C has a nonempty interior and therefore, by the theorem of Vincensini [14] mentioned before, Int $\cap\, C' \neq \emptyset$. This obviously implies the assertion of the lemma, Int $\Pi\mathscr{C} \neq \emptyset$.

LEMMA 2. *If $C \subset S^n$ is convex, Int $C \neq \emptyset$ and $C \neq S^n$, then C is contractible (to any point in its interior).*

Proof. If $P_0 \in$ Int C, each point P of C is at a spherical distance $< \pi$ from P_0; therefore the smaller arc of the great circle joining P_0 and P is contained in C.

REMARK. In the terminology of Helly [6], each convex set $C \subset S^n$,

with nonempty interior and different from S^n, is a "cell". In the proof of the next lemma we shall use the following topological theorem of Helly [6] (see [1] for a similar result): "If a family of cells is given in E^n, such that the intersection of each n or less of them is a cell, and the intersection of each $n+1$ nonempty, then the intersection of all the members of the family is nonempty."

LEMMA 3. *Let $\mathscr{C}$ be a finite family of convex subsets of S^n, such that the intersection of every $n+1$ members of $\mathscr{C}$ has interior points, but $\Pi\mathscr{C} = \varnothing$. Then the intersection of some $n+2$ members of $\mathscr{C}$ is empty.*

Proof. The lemma is trivially true for families consisting of $n+2$ sets. Assume, by induction, that $k \geqq n+2$, that the lemma is proved for all families consisting of k sets, and that $\mathscr{C} = \{C_i : 0 \leqq i \leqq k\}$ satisfies $\Pi\mathscr{C} = \varnothing$. If the intersection of some $n+2$ or less members of $\mathscr{C}$ is empty, there is nothing to be proved; thus we may assume that all such intersections are nonempty. Consider the k-membered family $\mathscr{C}_0 = \{C_i' = C_0 \cup C_i : 1 \leqq i \leqq k\}$. Because of Lemmas 1 and 2 the family $\mathscr{C}_0$ satisfies the assumptions of the present lemma. Since $\Pi\mathscr{C}_0 = \varnothing$, the intersection of some $n+2$-membered subfamily of $\mathscr{C}_0$ is empty; assume $\bigcap_{i=1}^{n+2} C_i' = \varnothing$. This is a contradiction to Helly's theorem on intersections of cells: on applying a stereographic projection of S^n onto E^n, with center at a point of S^n not in C_0, the sets C_i', $1 \leqq i \leqq n+2$ yield a family of $n+2$ cells in E^n which has an empty intersection although the intersection of any $n+1$ or less of them is, by Lemmas 1 and 2, a cell and nonempty.

REMARK. Lemma 3 is easily seen to yield the finite case of Theorem 4 of Karlin and Shapley [8] which, in turn, implies results of Vincensini [14] and Molnár [10] for subsets of S^n convex in yet another sense. (In [14] and [10] no precise definition of the term "convex" is given, but the results are valid only if "convex" means that the set is the intersection of *open* hemispheres.) The above proof of Lemma 3 is an adaptation of the reasoning in Molnár [10]. It would be nice to have an elementary proof (avoiding the use of Helly's topological theorem) for Lemma 3 and the theorems on convex sets or systems of inequalities related to it.

LEMMA 4. *Let $\mathscr{K}$ be a finite family of convex polyhedra in E^n and let $1 \leqq k \leqq n$. If every intersection of k members of $\mathscr{K}$ is of dimension n, but $\dim \Pi\mathscr{K} = n-k$, then there exist $k+1$ members of $\mathscr{K}$ such that their intersection is $(n-k)$-dimensional.*

Proof. Let E^{n-k} be the subspace of E^n generated by $\Pi\mathscr{K}$, and let E^k be an orthogonal complement of E^{n-k} in E^n. Let $\mathscr{P}$ be the projection of E^n onto E^k along E^{n-k}. As easily seen, the family $\mathscr{K}' = \{K' = \mathscr{P}(K) : K \in \mathscr{K}\}$ has the following properties: (a) each K' is a k-dimensional polyhedron containing the origin 0; (b) the intersection of every k sets K' is of dimension k; (c) $\Pi\mathscr{K}'$ is the single point 0. The subfamily $\mathscr{K}'' \subset \mathscr{K}'$ consisting of those members of $\mathscr{K}'$ which contain 0 in their boundary, has the same properties. For each $K_i'' \in \mathscr{K}''$ let $C_i \subset S^{k-1}$ be the set of all points $P \in S^{k-1}$ for which the open half-line through P with end-point 0 intersects K_i''. Then the family $\mathscr{C} = \{C_i\}$ consists of convex sets, such that the intersection of every k of them has interior points while $\Pi\mathscr{C} = \emptyset$. By Lemma 3, with $n = k - 1$, it follows that there exist $k + 1$ sets C_i with an empty intersection. The intersection of the corresponding sets K_i'' is then reduced to 0, and that of the corresponding original sets K_i is $(n-k)$-dimensional.

3. We now return to the proof of assertion (iii) of the Theorem. We first prove $h(n, n-1) = n + 1$ for $n \geqq 3$. Since obviously $h(n, n-1) \geqq n + 1$, only the opposite inequality has to be proved. Let $\mathscr{K}$ be a finite family of convex polyhedra in E^n, such that each $n + 1$ of them have an intersection of dimension $\geqq n - 1$. If each such intersection has dimension n, then $\dim \Pi\mathscr{K} = n$ since $h(n, n) = n + 1$. In the other case, let $\mathscr{K}' \subset \mathscr{K}$ be a minimal subfamily of $\mathscr{K}$ whose members have an intersection of dimension less than n. The family contains at most $n + 1$ members and thus $\dim \Pi\mathscr{K}' = n - 1$. By Lemma 4, there exist $K_1, K_2 \in \mathscr{K}'$ such that $\dim(K_1 \cap K_2) = n - 1$. (The case that $\mathscr{K}'$ contains a member of dimension $n - 1$ is trivially reducible to $h(n-1, n-1) = n$.) Let E^{n-1} be the subspace of E^n spanned by $K_1 \cap K_2$, and let $\mathscr{K}^* = \{K_i^* = K_i \cap E^{n-1} : K_i \in \mathscr{K}\}$. If every n members of $\mathscr{K}^*$ have an $(n-1)$-dimensional intersection, we are through since $h(n-1, n-1) = n$. Assuming the other case, let $\mathscr{K}_0^*$ be a minimal subfamily of $\mathscr{K}^*$ such that its members have an intersection of dimension less than $n - 1$. Let $\mathscr{K}_0$ be the subfamily of $\mathscr{K}$ consisting of those sets whose intersections with E^{n-1} constitute $\mathscr{K}_0^*$. Then $\mathscr{K}_0$ (and $\mathscr{K}_0^*$) must contain exactly n members, since otherwise $\mathscr{K}_0 \cup \{K_1, K_2\}$ would have at most $n + 1$ members, with an intersection of dimension less than $n - 1$. If $\dim(\Pi\mathscr{K}_0 \cap E^{n-1}) = n - 2$ a contradiction results: by Lemma 4 it follows that the intersection of some two members of $\mathscr{K}_0^*$ has dimension $n - 2$, while the minimality of $\mathscr{K}_0^*$ then implies $n = 2$; but $n \geqq 3$ was assumed. Thus $\dim(\Pi_0 \cap E^{n-1}) \leqq n - 3$, and again a contradiction results: The sets K_1 and K_2, and hence the sets $K_1 \cap \Pi\mathscr{K}_0$ and $K_2 \cap \Pi\mathscr{K}_0$, are separated by E^{n-1}, and $\dim(K_i \cap \Pi\mathscr{K}_0) \geqq n - 1$. Therefore E^{n-1} has to intersect $\Pi\mathscr{K}_0$ in a set of dimension

$n-2$ at least. Thus a contradiction is reached in this case, too, and $h(n, n-1) = n+1$ is proved.

Next, we shall establish $h(n, k) \leqq 2n-k$ for $1 < k \leqq n-1$, using induction on n. In view of the above, we may assume $n \geqq k+2$. Starting, without loss of generality, with a family $\mathscr{K}$ of convex polyhedra of dimension n, each $2n-k$ of which have an intersection of dimension $\geqq k$, the assertion is again trivial if the intersection of every $2n-k$ members of $\mathscr{K}$ is n-dimensional. Let therefore $\mathscr{K}'$ be a minimal subfamily of whose members have an intersection of dimension less than n. If $\mathscr{K}'$ contains m members then, by Lemma 4, $r = \dim \Pi\mathscr{K}' \leqq n-m+1$ with $r \geqq k$. Applying the inductive assumption to the family $\mathscr{K}^* = \{K^* = K \cap E^r : K \in \mathscr{K}\}$, where E^r is the subspace of E^n spanned by $\Pi\mathscr{K}'$, the assertion $\dim \Pi\mathscr{K} \geqq k$ follows: if $r > k$, then $h(r, k) \leqq 2r-k$ and thus $h(r, k) + m \leqq 2(n-m+1) - k + m \leqq 2n-k$; therefore the intersection of every $h(r, k)$ members of $\mathscr{K}^*$ contains the at least k-dimensional intersection of some $2n-k$ members of $\mathscr{K}$; but $\Pi\mathscr{K} \supset \Pi\mathscr{K}^*$. On the other hand, if $r = k$ then $h(k, k) = k+1$ and $h(k, k) + m \leqq n+2 \leqq 2n-k$; since $k \leqq n-2$ was assumed, the result follows in this case as well.

Thus $h(n, k) \leqq 2n-k$ is proved for all $n > k > 1$. The converse inequality, and with it the theorem, may be established by easy examples.

REMARKS. It seems to be hard to extend the results of the present note to infinite families $\mathscr{K}$. (Vincensini's assertion in [14] that $h(n, n) = n+1$ holds for infinite families is incorrect.)

Professor F. A. Valentine has found alternative proofs for some of the results of the present paper, as well as related results.

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