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ANTONI A. KOSINSKI

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A. KOSINSKI

In the first part of this note we discuss a group of theorems dealing with geometric configurations arising when we assign in continuous way compact, acyclic sets to k-planes in the Euclidean n-dimensional space E_n. A fairly representative example of those theorems is as follows:

Suppose that to every (unoriented) k-plane H through a point a of E_n there is upper semi-continuously assigned a compact and acyclic set $\Phi(H) \subset H$. Then for some plane H_0, $a \in \Phi(H_0)$.

In fact, we will prove a much more general theorem of which the above is one of the consequences.

In the second part of this note we give various applications of the above theorems. They are related to the theory of convex sets (§ 2.1–2.4), mappings of manifolds (§ 2.6), and to some relations between vector fields and involutions on S_n (§ 2.5).

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1. Families of compact sets over Grassmannians.

1.1 $H_n(X)$ will denote the nth Čech homology group of the space X with the group Z_2 of integers mod 2 as the group of coefficients. We will say that X is acyclic if X is connected and $H_n(X) = 0$, $n = 1, 2, \cdots$,.

Let X be a compact metric space and let $\Phi : X \rightarrow 2^Y$ be an upper semi-continuous mapping of X into the space 2^Y of all nonempty compact subsets of a space Y. The triple $\mathscr{F} = \{X, Y, \Phi\}$ will be called a family [3]. The set X will be called the basis of $\mathscr{F}$, the sets $\Phi(x)$—the elements of $\mathscr{F}$, the set $\bigcup_{x \in X} \Phi(x) \subset Y$ — the field of $\mathscr{F}$. The field will be also denoted $\Phi(X)$. A family $\mathscr{F}$ is said to be acyclic if all its elements are acyclic.

If $\mathscr{F} = \{X, Y, \Phi\}$ is a family then the subset $M = \{(x, y) \mid y \in \Phi(x)\}$ of the cartesian product $X \times Y$ is called the graph of $\mathscr{F}$.

M is a closed subset of $X \times Y$ (and, hence, compact) because of the upper semi-continuity of Φ, and this is the only reason for requiring the upper semi-continuity of Φ.

$G_{p,q}$ will denote the Grassmannian of (unoriented) q-planes through

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the origin of the p-dimensional Euclidean space E_p.

Let T be a fixed k-plane in E_n and let $r > k$. Let E_{n-k} be the Euclidean space orthogonal to T in E_n and for $x \in G_{n-k,r-k}$ let $H(x)$ be the r-plane in E_n spanned by x and T. The correspondence $x \to H(x)$ is obviously one-to-one correspondence between $G_{n-k,r-k}$ and the set of all r-planes in E_n containing T. Henceforth we will say that $H(x)$ is the r-plane corresponding to x.

1.2. THEOREM. *Let T be a fixed k-plane in E_n. For every $x \in G_{n-k,r-k}$ let $H(x)$ be the r-plane in E_n containing T and corresponding to x.*

If $\mathscr{F} = \{G_{n-k,r-k}, E_n, \Phi\}$ is an acyclic family then there exists an $x \in G_{n-k,r-k}$ such that $H(x) \cap \Phi(x) \neq 0$.

Proof. Let us remark first that it is enough to prove the theorem in case $r = k + 1$. For in a general case we may always choose an $(r-1)$-plane T_0 containing T and the Grassmannian of all r-planes containing T_0 is a subset of $G_{n-k,r-k}$.

Therefore we will assume that $r = k + 1$, i.e., the basis of $\mathscr{F}$ is the set of all $(k+1)$-planes containing a given k-plane T. Then

$$G_{n-k,r-k} = G_{n-k,1} = P_{n-k-1}$$

and we may write $\mathscr{F} = \{P_{n-k-1}, E_n, \Phi\}$, where P_{n-k-1} denotes the $(n-k-1)$-dimensional projective space.

Let S be an $(n-1)$-sphere in E_n with center o in T and containing the field of $\mathscr{F}$ in the interior. Since the field of $\mathscr{F}$ is compact such a sphere exists. Let $\Phi_1(x) = H(x) \cap S$ for every $x \in P_{n-k-1}$. Thus $\Phi_1(x)$ is a k-dimensional great circle in S and $\mathscr{F}_1 = \{P_{n-k-1}, E_n, \Phi_1\}$ is a family.

Let $J \subset P_{n-k-1} \times E_n \times I \times E_n$, $I = \langle 0, 1 \rangle$, be the union of all sets of the form

$$(x) \times \Phi(x) \times I \times \Phi_1(x), x \in P_{n-k-1} ;$$

i.e., $(x, y, t, y_1) \in J$ if and only if $y \in \Phi(x)$, $y_1 \in \Phi_1(x)$, $t \in \langle 0, 1 \rangle$.

We shall identify in J

$$(x, y, 1, y_1) \quad \text{with} \quad (\bar{x}, \bar{y}, 1, \bar{y}_1) \quad \text{if} \quad x = \bar{x}, y = \bar{y}$$

and

$$(x, y, 0, y_1) \quad \text{with} \quad (\bar{x}, \bar{y}, 0, \bar{y}_1) \quad \text{if} \quad x = \bar{x}, y_1 = \bar{y}_1 .$$

Denote the set obtained from J by these identifications by M. Then M is the union of sets of the form $(x) \times \Phi(x) * \Phi_1(x)$ where $x \in P_{n-k-1}$

and $A*B$ is the notation for the join[1] of A with B. It is easy to see that because of the upper semi-continuity of Φ M is compact.

Let $M_0 \subset M$ be composed of points with $t = 0$; more precisely, M_0 is the image under the identification mapping $J \rightarrow M$ of the subset of J composed of points of the form $(x, y, 0, y_1)$. Let $q\colon M_0 \rightarrow S$ and $p\colon M_0 \rightarrow P_{n-k-1}$ be defined by $q(x, y, 0, y_1) = y_1$, $p(x, y, 0, y_1) = x$. Then

(i) *M_0 is an $(n-1)$-manifold, and q is a mapping of degree* 1. *If $k = 0$ then p is a covering map. In particular, for every $k \geqq 0$ $p_* H_{n-1}(M_0) = 0$ and $q_* H_{n-1}(M_0) = H_{n-1}(S)$.*

For let M_1 be the graph of Φ_1, i.e. the set of points $(x, z) \in P_{n-k-1} \times E_n$ satisfying $z \in \Phi_1(x)$. Let $p_1\colon M_1 \rightarrow P_{n-k-1}$ and $q_1\colon M_1 \rightarrow S$ be defined by $p_1(x, z) = x$, $q_1(x, z) = z$ and let $h\colon M_0 \rightarrow M_1$ be defined by $h(x, y, 0, y_1) = (x, y_1)$. Then the diagram

$$\begin{array}{ccc} M_0 & \xrightarrow{p} & P_{n-k-1} \\ {\scriptstyle q}\downarrow & \searrow^{h} & \uparrow{\scriptstyle p_1} \\ S & \xleftarrow[q_1]{} & M_1 \end{array}$$

is commutative. Moreover, because of identifications, h is a homeomorphism. But it is easy to see that M_1 is a fibre space over P_{n-k-1} with the fibre S_k and p_1 as the fibering map. This proves that M_0 is an $(n-1)$-manifold. Now, since

$$\Phi_1(x) \cap \Phi_1(y) \subset T \cap S$$

if $x \neq y$ it follows that q_1 maps $q_1^{-1}(S - T)$ homeomorphically onto $S - T$. This proves that q is of degree 1. If $k = 0$ then q_1 is a homeomorphism and p_1 is a covering map. This proves (i).

Now, let us consider the diagram

$$\begin{array}{ccccc} M_0 & \xrightarrow{i} & M & \xrightarrow{\bar{p}} & P_{n-k-1} \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle g} & & \\ S & \xrightarrow{j} & G & & \end{array}$$

where i and j are inclusion maps, $\bar{p}\colon M \rightarrow P_{n-k-1}$ is defined by $\bar{p}(x, y, t, y_1) = x$ and $g(x, y, t, y_1) =$ point in E_n dividing the segment $\overline{y_1 y}$ in the ratio $t/(1-t)$; $G = g(M)$. Since

$$g(x, y, 0, y_1) = y_1 = q(x, y, 0, y_1)$$

the diagram is commutative.

[1] i.e., the set obtained from $A \times I \times B$ by identifying $A \times < 0 > \times B$ with A and $A \times < 1 > \times B$ with B by means of the mappings $(x, 0, y) \rightarrow x$ and $(x, 1, y) = y$.

Observe that

(ii) *$\bar{p}^{-1}(x)$ is acyclic and nonempty for every $x \in P_{n-k-1}$.*

For $\bar{p}^{-1}(x)$ is homeomorphic with $\Phi(x)*\Phi_1(x)$ and join of an acyclic set with a compact set is acyclic.[2]

Therefore it follows from the Vietoris Mapping Theorem [2] that in the diagram

$$\begin{array}{ccccc} H_{n-1}(M_0) & \xrightarrow{i_*} & H_{n-1}(M) & \xrightarrow{\bar{p}_*} & H_{n-1}(P_{n-k-1}) \\ \downarrow q_* & & \downarrow g_* & & \\ H_{n-1}(S) & \xrightarrow[j_*]{} & H_{n-1}(G) & & \end{array}$$

$\bar{p}_*$ is an isomorphism onto. But $\bar{p}i = p$ and it follows from (i) that p_* is trivial for every $k \geqq 0$. Therefore

(iii) *i_* is trivial .*

Now, again by (i), q_* is onto Thus (iii) implies that j_* is trivial, i.e., that S bounds in G. Therefore G must contain the ball bounded by S and we conclude that the center o of S is in G. This means that for some $x \in P_{n-k-1}$ there exist points $y \in \Phi(x)$, $y_1 \in \Phi_1(x)$ such that the segment joining y_1 with y contains o. Since $y_1 \in H(x)$ and $o \in H(x)$ it follows that $y \in H(x)$, i.e., $y \in \Phi(x) \cap H(x)$. This proves the theorem.

1.3. COROLLARY. *Let $G_{n-k,n-r}$ be the Grassmannian of all $(n-r)$-planes contained in an $(n-k)$-plane T. For every $x \in G_{n-k,n-r}$ let $H^*(x)$ be the orthogonal complement in E_n of the plane representing x. Let $\mathscr{F} = \{G_{n-k,n-r}, E_n, \Phi\}$ be an acyclic family. Then there exists an $x \in G_{n-k,n-r}$ such that*

$$\Phi(x) \cap H^*(x) \neq 0 \, .$$

Proof. Let T^* be the orthogonal complement of T in E_n and $G_{n-k,r-k}$ the Grassmannian of all r-planes in E_n containing T^*. For every $y \in G_{n-k,r-k}$ let $G(y)$ be the plane representing y and $G^*(y)$ its orthogonal complement. Then $G^*(y) \cap T$ represents an element $x = f(y)$ of $G_{n-k,n-r}$. Moreover it is easily seen that

$$\text{(i)} \qquad G(y) = H^*(f(y)) \, .$$

Let $\mathscr{F}_1 = \{G_{n-k,r-k}, E_n, \Phi(f(y))\}$. Then $\mathscr{F}_1$ is an acyclic family and by Theorem 1.2 there exists an $y \in G_{n-k,r-k}$ such that $\Phi(f(y)) \cap G(y) \neq 0$. Compared with (i) this gives

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[2] *Short proof.* Let $A*B$ be the join of A with B. Suppose that A is acyclic and let $f: A*B \to A*B$ be defined by $f(x,t,y) = (x_0, t, y)$ where x_0 is a fixed point of A. Then $f^{-1}(x_0, t, y)$ is homeomorphic with A if $t \neq 1$ and is a point if $t = 1$. In both cases $f^{-1}(x_0, t, y)$ is acyclic and, by Vietoris mapping theorem [2], f induces an isomorphism of $H_k(A*B)$ onto $H_k(f(A*B))$, $k = 0, 1, 2, \cdots,$. But $f(A*B) = (x_0)*B$ is acyclic, therefore $A*B$ is also.

$$\Phi(x) \cap H^*(x) \neq 0 \qquad \text{Q.E.D.}$$

In the following two corollaries we suppose that a point $p \in E_n$ is given and a natural one-to-one correspondence $x \to H(x)$ between the set of all k-planes $H(x)$ through p is fixed once and for all.

1.4. COROLLARY. *Let* $\mathscr{F} = \{G_{n,k}, E_n, \Phi\}$ *be an acyclic family. Then for some* $x_0 \in G_{n,k}$, $\Phi(x_0) \cap H(x_0) \neq 0$ *and for some* $x_1 \in G_{n,k}$, $\Phi(x_1) \cap H^*(x_1) \neq 0$.

Proof. (The second part of this corollary was first proved in another way by Dr. J.W. Jaworowski.) Existence of x_0 follows from the Theorem 1.2 with $k = 0$; existence of x_1 follows from the Corollary 1.3 with $r = n - k$.

1.5. COROLLARY. *Let* $\mathscr{F} = \{G_{n,k}, E_n, \Phi\}$ *be an acyclic family satisfying the condition* $\Phi(x) \subset H(x)$ *where* $H(x)$ *is the plane through* p *representing* x. *Then for some* $x, p \in \Phi(x)$.

Proof. By Corollary 1.4, for some x $\Phi(x) \cap H^*(x) \neq 0$. Since $\Phi(x) \subset H(x)$ this implies $\Phi(x) \cap H(x) \cap H^*(x) \neq 0$ and in view of $p = H(x) \cap H^*(x)$ this proves the corollary.

2. Applications.

2.1. We start with a simple application of 1.5. Let $A \subset E_n$ be a compact subset of E_n and k a fixed integer, $1 \leqq k \leqq n - 1$. Let A_k be the set of such points $p \in E_n$ that every k-plane H through p intersects A in an nonempty acyclic set. Applying 1.5 to the family $\mathscr{F} = \{G_{n,k}, E_n, H \cap A)$ we infer that $p \in A$, i.e. $A_k \subset A$.

In a subsequent paper we will prove that A is star-shaped with respect to every point of A_k, i.e. if $x \in A$ and $p \in A_k$ then the segment $\overline{xp} \subset A$. In particular, it follows $A_k = A_{k+1}, k = 1, \cdots, n-2$.

2.2. The following is a generalization of a theorem of H. Steinhaus [8] (see also S.K. Stein [7]).

THEOREM. *Let* A *be a convex subset of* E_n *and* $p \in \operatorname{Int} A$. *Then there exists a sequence of planes* $H_1, H_2, \cdots, H_{n-1}$ *such that* H_i *is an* i*-plane,* $H_i \subset H_{i+1}$ $i = 1, \cdots, n-2$, *and* p *is the center of gravity of* $A \cap H_i, i = 1, 2, \cdots, n-1$.

Proof. For every k-plane H through p let $x(H)$ be the center of gravity of $H \cap A$. Then our assumptions about A insure that $\mathscr{F} = \{G_{n,k}, E_n, x(H)\}$ is a family. Therefore it follows from 1.5 that $x(H) = p$ for some H. Thus to prove the theorem we start with $k = n - 1$ and find the $(n-1)$-plane H_{n-1} such that $x(H_{n-1}) = p$. Then in $A \cap H_{n-1}$, which is again convex and $p \in \operatorname{Int} A \cap H_{n-1}$, we find H_{n-2}, and so we continue until the sequence is complete.

We may remark that the assumption that A is convex and $p \in \operatorname{Int} A$ are needed only to insure that $x(H)$ is a continuous function of H. Also we may suppose that a continuous positive density is given on A. The theorem may be correspondingly generalized without a change in proof.

2.3. The following theorem, first proved by G. Aumann [1], is an easy consequence of 1.5.

THEOREM. *Let M be a compact subset of E_n and suppose that for some $k \geqq 1$ intersection of every k-plane with M is either empty or acyclic. Then M is convex.*

This is a consequence of the following lemma which we will prove first:

LEMMA. *Let M be a compact subset of E_n and suppose that some k-plane H intersects the convex hull of M but does not intersect M. Then there exists a k-plane H_0 such that $M \cap H_0$ is not connected.*

Proof. Let $p, q \in M$, let r be a point of the segment $\overline{pq}$ and let H be a k-plane in E_n such that $H \cap \overline{pq} = r$ and $H \cap M = 0$. Let H' be the $(k+1)$-dimensional plane in E_n containing H and the line through p, q. Then H disconnects H' between p and q. Thus if H_0 is a k-plane in H' containing p and q then $H_0 \cap M$ is not connected. This proves the lemma.

Proof of the theorem. Let M satisfy the conditions of the theorem and let $C(M)$ denote the convex hull of M. Let $p \in C(M)$. By the lemma every k-plane through p intersects M. Since the intersection is acyclic it follows from Corollary 1.5 that $p \in M$, i.e. $M = C(M)$. This proves the theorem.

2.4. The following theorem gives homological conditions insuring that a set $M \subset E_n$ shall be a convex sphere S_{n-1}. (For the case $n = 3$ see J. Schreier [6]).

THEOREM. *Let M be a compact subset of E_n. We suppose that*

(a) *$H_{n-1}(M) = Z_2$ and M is an irreducible carrier of the nonzero element α of $H_{n-1}(M)$.*[3]

(b) *There exists an integer $k, 0 < k < n-1$, such that for every k-plane T intersecting M*

$$H_i(T \cap M) = \begin{cases} 0 & \text{for } i < k-1 \\ 0 \text{ or } Z_2 & \text{for } i = k-1 . \end{cases}$$

Then M is a convex S_{n-1}.

Proof. It is well known that (a) implies that M disconnects E_n into two sets and is the boundary of each. Let G be the bounded component of $E_n - M$ and let $M^* = M \cup G$. Then

(i) *M^* is compact and $M = Fr\, M^*$*

Let $p \in G$ and let T be a k-plane through p. We will prove

(ii) *$T \cap M^*$ is acyclic.*

Since $T \cap M^* = (T \cap M) \cup (T \cap G)$ and $T \cap G \neq 0$ it follows that $T \cap M$ disconnects T. Therefore by (b)

(iii) $$H_i(T \cap M) = \begin{cases} 0 & \text{for } i < k-1 \\ Z_2 & \text{for } i = k-1 . \end{cases}$$

It follows that $T - M$ has exactly two components; let the bounded component be H. Then $p \in H$ and we have $T \cap G = H$. Thus we have proved

(iv) *$T \cap M^*$ is the union of $T \cap M$ and the bounded component H of $T - T \cap M$.*

Now, this implies that $Fr(T \cap M^*) \subset T \cap M$ and we may consider the sequence

$$H_l(Fr(T \cap M^*)) \xrightarrow{i_*} H_l(T \cap M) \xrightarrow{j_*} H_l(T \cap M^*)$$

where i_*, j_* are induced by inclusions. By [4] $j_* i_*$ is onto, thus such is also j_* and by (iii) this implies $H_l(T \cap M^*) = 0$ for $l < k-1$. Since (iv) implies that $T \cap M^*$ does not disconnect T, also $H_{k-1}(T \cap M^*) = 0$, which completes the proof of (ii).

Now, by the theorem mentioned in 2.1, (ii) implies that M^* is star-shaped with respect to every point of G. Let now $p \in M$ and $x \in M^*$. Let $p_n \in G, p_n \to p$, by the remark above segments $\overline{p_n x}$ are in M^*. Since $\overline{p_n x} \to \overline{px}$ it follows that $\overline{px} \subset M^*$ and thus M^* is star-shaped with respect to every point. Thus M^* is convex. Together with (i) this proves the theorem.

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[3] i.e., for every proper compact subset $A \subset M$ α is not in the image $H_{n-1}(A) \to H_{n-1}(M)$ (inclusion homomorphism).

2.5. It is interesting to note that the following theorems, connected with the Borsuk-Ulam theorem (see e.g. [5]), follow easily from the results in § 1.

S_n will stand for the unit n-sphere in E_{n+1}, $\alpha: S_n \to S_n$ will denote the antipodal involution on S_n.

2.51. THEOREM. *Let $f: S_n \to S_n$ be a continuous mapping and suppose that $f(x) \neq \alpha(x)$ for every $x \in S_n$. Then for some $x_0 \in S_n$ $\alpha f(x_0) = f\alpha(x_0)$.*

Proof. For every $x \in S_n$ let $a(x) = x + f(x)$ and let $H(x, \alpha(x))$ be the n-plane through 0 and perpendicular to the line through x and $\alpha(x)$. Then $a(x)$ lies on a sphere of radius 1 and with center at x and it follows from the assumption $f(x) \neq \alpha(x)$ that the points $a(x)$ and $a(\alpha(x))$ are distinct and lie in distinct components of $E_{n+1} - H(x, \alpha(x))$. Therefore the intersection $\Phi(x, \alpha(x))$ of the segment $\overline{a(x)a(\alpha(x))}$ with $H(x, \alpha(x))$ is nonempty and $\mathscr{F} = \{P_n, E_n, \Phi\}$ is an acyclic family. Thus, by 1.3, for some x $\Phi(x, \alpha(x))$ intersects also the orthogonal complement of $H(x, \alpha(x))$. It follows that for some x, $\Phi(x, \alpha(x)) = 0$. Then $0, a(x)$, $a(\alpha(x))$ are distinct and collinear. It is easy to see that this implies $\alpha a(x) = a(\alpha(x))$, i.e., $\alpha f(x) = f\alpha(x)$. Q.E.D.

By similar method one can obtain the following generalizations of 2.51 (proofs are omitted):

2.52. THEOREM. *Let $F: S_n \to S_n$ be an acyclic upper semicontinuous map and suppose that $F(x) \cap (\alpha(x)) = 0$. Then for some x_0, $\alpha(F(x_0)) \cap F(\alpha(x_0)) \neq 0$.*

2.53. THEOREM. *Let β be the antipodal involution in the bundle B of unit vectors tangent to S_n. Let $F: S_n \to B$ be a multivalent acyclic cross-section. Then for some $x_0 \in S_n$, $\beta F(x_0) \cap F(\alpha(x_0)) \neq 0$.*

2.6. In the following two theorems M will denote a differentiable $(n-1)$-manifold in E_n. If $p \in M$ then $T(p)$ will denote the tangent plane to M at p. $T \| T'$ will mean that T is parallel to T'.

2.61. THEOREM. *Let $f: M \to E_n$ be a continuous mapping such that if $T(p)$ is parallel to $T(p')$ then the vector $\overrightarrow{pf(p)}$ is not zero and parallel to $\overrightarrow{p'f(p')}$. Then for some $p_0 \in M$ $\overrightarrow{p_0 f(p_0)}$ is perpendicular to $T(p_0)$.*

Proof. It is known that for every $(n-1)$-plane H through the origin there exists a $p \in M$ such that $H \| T(p)$. Let then $\Phi(H)$ be the endpoint of the unit vector parallel to $\overrightarrow{pf(p)}$. By our assumption about f, the family $\mathscr{F} = \{P_{n-1}, E_n, \Phi\}$ is well-defined and, obviously, acyclic. The theorem follows then from 1.4.

Theorem 2.61 can be again generalized to acyclic multivalent maps $f: M \to E_n$. Instead, we prove:

2.62. THEOREM. *Let $f: M \to E_n$ be an immersion such that if $T(p) \| T(p')$ then $T(f(p)) \| T(f(p'))$. Then for some $p_0 \in M$, $T(p_0) \| T(f(p_0))$.*

Proof. We suppose M oriented and a field of normal unit vectors given on M. This defines a field of unit vectors normal to $f(M)$ and we define the family $\mathscr{F} = \{P_n, E_n, \Phi\}$ as follows: Let $H(x)$ represent a point $x \in P_n$, there exists a point $p \in M$ such that $T(p) \| H(x)$. We define $\Phi(x) =$ the endpoint of the unit vector normal to $f(M)$ at $f(p)$. By our assumption about f the family $\mathscr{F}$ is well-defined. Then the theorem follows from 1.4.

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