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1. Introduction. Let A be a family of real valued upper semi-continuous functions defined on a compact Hausdorff space E.

A closed set $F \subset E$ is called *determining for A* if every function $f \in A$ attains its maximum on F. If for the space E there exists one and only one minimal determining $F = F(E, A)$ (i.e., a determining set such that no proper closed subset of it is determining), then F is called the *boundary of E with respect to the family A*.

A function $h \in A$ is called a *barrier-function of A* at a point $\overset{\circ}{x} \in F = F(E, A)$ if and only if $h(\overset{\circ}{x}) > h(x)$ for $x \neq \overset{\circ}{x}$, $x \in F$.

A point $\overset{\circ}{x} \in F$ for which there is a barrier-function of A is called a *semiregular boundary point of E* with respect to A. If for a point $\overset{\circ}{x} \in F$ there exists a continuous (at the point $\overset{\circ}{x}$) barrier-function, then $\overset{\circ}{x}$ will be called a *regular boundary point of E* with respect to A.

Let D be a set contained in a topological space and let $f(x)$ be a real function defined on D. Then the function f^* defined in the closure $\bar{D}$ of D by means of

$$(1) \qquad f^*(x) = \limsup_{x' \to x} f(x') , \qquad\qquad x' \in D, \; x \in \bar{D} ,$$

is called an *upper regularization of f*.

Let A_1 be a subfamily of A. Then the function

$$(2) \qquad \varphi(x) = \{\sup_{f \in A_1} f(x)\}^* , \qquad\qquad x \in E ,$$

is called the *upper envelope of A_1*.

Let f be an upper semicontinuous nonnegative function defined in a compact set E. We shall denote by $\|f\|_E$ the maximum of f on E, $\|f\|_E = \max_{x \in E} f(x)$.

We say that a family A of functions f defined on E is *separating* (or *A separates the points of E*) if for any two points $x_1 \neq x_2$ of E there is a function $f \in A$ such that $f(x_1) \neq f(x_2)$.

A well known theorem of Šilov [5] asserts: *If A is a family of absolute values of all functions of a separating algebra of complex continuous functions defined on a compact Hausdorff space E, then E has the boundary F with respect to the family A.*

This boundary is sometimes called a *Šilov boundary of E* (with respect to the given algebra).

E. Bishop [3] has recently proved that *if E is metrizable and A is a complete* (with respect to the uniform convergence) *Banach algebra*

of continuous function on E, then the Šilov boundary of E is the closure
of regular points of E with respect to A.

Let us mention that the papers of S. Bergman [1], [2] on the domains
with a distinguished boundary surface are the first to indicate the
significance of the boundary of a domain D with respect to the algebra
of holomorphic functions of several complex variables in D.

Recently it appeared that the notion of the boundary of a set with
respect to the family of functions, which do not necessarily form an
algebra, may be useful. For instance, Bremermann [4], considering a
generalized solution of the Dirichlet boundary value problem within the
family of pluri-subharmonic functions in a domain D of the space C^n of
n complex variables, had to consider the boundary of D with respect to
the family of pluri-subharmonic functions in D. The boundary values,
in the procedure described by Bremermann, could be given just on the
Šilov boundary of D and nowhere else. But the family of pluri-subharmonic
functions does not form any algebra. Also in the case of the first boundary
value problem for the heat conduction equation $u_{xx} - u_t = 0$ in a domain
D, the boundary values can be given only on a part of the boundary of
D. That part is a Šilov boundary of D with respect to the solutions of
the inequality $u_{xx} - u_t \geqq 0$. Those solutions do not form any algebra,
of course.

The aim of this paper is to prove the existence of the boundary
with respect to function families much more general than the algebras,
namely, for the families A which are closed only under the multiplication
or addition of functions of A.

This fact can be applied to a uniform treatment of a Perron procedure
of upper envelopes with respect to various function families having the
boundary. Suppose that for a function family A there exists a boundary
$F = F(E, A)$. Then, by means of reasoning classical in potential theory,
we have the following theorem:

If, along with f and g, the family A contains $\alpha f + \beta g$, where
$\alpha \geqq 0$ *and* $\beta \geqq 0$, *if A contains all real constants, and if* $\overset{\circ}{x} \in F$ *is a*
regular boundary point of E with respect to A then for any real function
$b(x)$ defined and continuous on F we have

$$b(\overset{\circ}{x}) = \lim_{x \to \overset{\circ}{x}} \varphi(x) \, ,$$

where $\varphi(x)$ denotes the upper envelope of all functions $f \in A$ such that
$f(x) \leqq b(x)$ for $x \in F$.

Let any point of F be regular. Then $\varphi(x) = b(x)$ for $x \in F$, and it
is quite natural to look at such an upper envelope $\varphi(x)$ as at a generalized
solution of the Dirichlet boundary value problem within the family A.
If the function family A is closed under the operation of taking the

upper envelope, then the generalized solution is a function of A.

There are well known examples of function families (which are not any algebra) within which the solution of the Dirichlet problem was found just by means of the Perron procedure [4], [6].

2. Some general function families with boundary. We shall need the following general

LEMMA 1 (Šilov). *For any family A of upper semicontinuous functions defined on a compact Hausdorff space E there exists a minimal determining set (one at least).*

This lemma can be proved by means of transfinite induction (see the proof by Šilov [5]).

THEOREM 1. *Let A be a family of nonnegative functions defined and upper semicontinuous on a compact Hausdorff space E. If the family A satisfies the following conditions:*

1° *If f and g are functions of A then the product $f \cdot g \in A$;*

2° *If $\overset{\circ}{x}$ is an arbitrary fixed point of E then for any neighborhood $U(\overset{\circ}{x})$ of $\overset{\circ}{x}$ and for any $\varepsilon > 0$ there exists a finite system of functions $f_1, f_2 \cdots, f_k \in A$ such that the set*

$$U^* = \{x \in E \,|\, f_\mu(x) < \varepsilon \,, \quad \mu = 1, 2, \cdots, k\}^1$$

is contained in $U(\overset{\circ}{x})$ and U^* contains a neighborhood $U'(x)$ of x; then the set E has a boundary with respect to A.[2]

Proof. Due to Lemma 1 it is sufficient to prove that E has only one minimal determining set with respect to A. The proof of the uniqueness may be given by a literal repetition of Šilov's proof in [5]. This repetition is possible because Šilov used only the assumptions formulated in Theorem 1.

REMARK. If c is a positive real number and $f(x)$ is any real function upper semicontinuous on a closed set E, then the functions $c \cdot f$ and f attain their maxima at the same points of E. Therefore, E has a boundary with respect to A if and only if E has a boundary with respect to $\tilde{A}$, where $\tilde{A}$ denotes the family of functions g which can be written in the form $g = c \cdot f$, $c > 0$, $f \in A$.

The function family A considered in Theorem 1 is closed under the operation of multiplication of functions of A. A similar theorem holds

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[1] The integer k may depend on $\overset{\circ}{x}$, $U(\overset{\circ}{x})$ or ε.

[2] The similar theorem has been proved in [7].

for function families closed with respect to additions of functions of A.

THEOREM 1'.   *Let A be a family of upper semicontinuous functions defined on a compact Hausdorff space E.   If A satisfies conditions:*
  1°   If $f, g \in A$, then $f + g \in A$;
  2°   If $\overset{\circ}{x} \in E$ and $U(\overset{\circ}{x})$ is a neighborhood of $\overset{\circ}{x}$ and $\varepsilon > 0$, then there exists a finite system of functions $f_1, f_2, \cdots, f_k \in A$ such that

$$U^* = \{x \mid e^{f_\mu(x)} < \varepsilon, \mu = 1, \cdots, k\}$$

is contained in $U(\overset{\circ}{x})$ and U^* contains a neighborhood $U'(\overset{\circ}{x})$ of $\overset{\circ}{x}$; then E has a boundary with respect to A.

Proof.   It is sufficient to observe that the family A_1 of functions $h = e^f$, where $f \in A$, satisfies the assumptions of Theorem 1.   Therefore, E has a boundary with respect to A_1.   But this is also a boundary of E with respect to A.

REMARK 1.   Any function family, which satisfies 2°, separates the points of E, but the converse statement is not true.   For instance, let E be the segment $[0, 1]$ and let A be the family of all powers x^μ, $\mu = 1, 2, \cdots$.   Then A satisfies 1° but it does not satisfy 2°, although A separates the points of E.   The boundary of E with respect to A is in this case the only point $x = 1$.
Now we shall prove the existence of the boundary for function families which are closed with respect to multiplication (or addition) and which only, instead of 2° in Theorem 1, separate the points of E.   But we now must assume that the space E is metric and the functions considered are continuous, while in Theorem 1 they could be only upper semicontinuous.

THEOREM 2.   *Let A be a family of nonnegative continuous functions defined on a compact metric space E.   If A satisfies the conditions:*
  1°   If $f, g \in A$, then $f \cdot g \in A$;
  2° A separates the points of E;
then E has a boundary with respect to A.[3]

Proof.   In virtue of Lemma 1 it is sufficient to prove only the uniqueness of the minimal determining set for A.   For the proof per reductio ad absurdum let us assume that there exist two different minimal determining sets F_1 and F_2 for A.   The set $F_1 \mid F_2$ is nonempty, since, otherwise, we would have $F_1 \subset F_2$ and F_2 would not be a minimal determining set.   Let $\overset{\circ}{x}_1 \in F_1 \mid F_2$ and let $U_1(\overset{\circ}{x}_1) = \{x \in E \mid \rho(x, \overset{\circ}{x}_1) < 1/2^{n_1}\}$ be

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[3] After submitting this paper for publication, the author discovered that H. Bauer, with different techniques, obtained more general results, see H. Bauer, Silovscher Rand und Dirichletsches Problem, Ann. L'Inst. Fourier XI (1961), 89–136.

a neighborhood of $\overset{\circ}{x}_1$, where n_1 is an integer so large that $U_1(\overset{\circ}{x}_1) \cap F_2 = \phi$. Since F_1 is a minimal determining set for A, there exists a function $f \in A$ which attains its maximum $m = \|f\|_{F_1}$ on F_1 in the set U_1, and such that $\max_{x \in U_1} f(x) = m > f(y)$, $y \in F_1 | U_1$. In virtue of 1° and the Remark on p. 377, we can assume that

$$\|f\|_{F_1} = \|f\|_{F_2} = \|f\|_E = 1 \,, \quad f(y) < \frac{1}{4} \quad \text{for } y \in F_1 | U_1 \,.$$

Since F_2 is determining and $U_1 \cap F_2 = \phi$, there must be a point $\overset{\circ}{y}_1 \in F_2$ such that $f(\overset{\circ}{y}_1) = 1$. The function $f(x)$ is continuous. Thus, there is a neighborhood $V_1(\overset{\circ}{y}_1) = \{y \mid \rho(y, \overset{\circ}{y}_1) < 1/2^{m_1}\}$ of $\overset{\circ}{y}_1$, where m_1 is so large that

$$f(x) > \frac{3}{4} \quad \text{for } x \in V_1 \quad \text{and} \quad \overline{V_1(\overset{\circ}{y}_1)} \cap \overline{U_1(\overset{\circ}{x}_1)} = \phi \,.$$

Since F_2 is minimal there is a function $g(x)$ such that $\|g\| = g(y_1) = 1$, y_1 being a point of V_1, and $g(x) < 1/4$ for $x \in F_2 | V_1$. Now we put $h(x) = f(x)g(x)$. We can easily verify that

$$h(y_1) \geqq \frac{3}{4} \,, \quad h(x) < \frac{1}{4} \quad \text{for } x \in F_1 | U_1 \quad \text{or} \quad x \in F_2 | V_1 \,.$$

Since $h(y_1) \geqq 3/4$, so $\max_{x \in U_1} h(x) = \|h\|_E \geqq 3/4$. Therefore, the function $h_1(x) = [h(x)/\|h\|_E]^k$, where k is a sufficiently large integer, satisfies the conditions,

$$\|h_1\|_E = 1 \,; \quad h_1(x) < \frac{1}{4} \quad \text{for } x \in F_1 | U_1 \quad \text{or} \quad x \in F_2 | V_1 \,,$$

and moreover there exists a point $x_1 \in U_1$ and a point $y_1 \in V_1$ for which $h_1(x_1) = h(y_1) = 1$.

This was the first step of our proof. To begin the next one, let us observe that one can find an integer $n_2 > n_1$ so large that

$$U_2(x_1) = \left\{ x \mid \rho(x, x_1) < \frac{1}{2^{n_2}} \right\} \subset U_1(\overset{\circ}{x}_1) \quad \text{and} \quad h_1(x) \geqq \frac{3}{4} \quad \text{for } x \in U_2 \,.$$

Since F_1 is a minimal determining set there exists $f_0 \in A$ such that $\|f_0\|_E = 1$, $f_0(x) < 1/4$ for $x \in F_1 | U_2$ and $f_0(\overset{\circ}{x}_2) = 1$ for a point $\overset{\circ}{x} \in U_2$. We define $f_1(x) = f_0(x)h_1(x)$. We have $f_1(x) < 1/4$ for $x \in F_1 | U_2$ or $x \in F_2 | V_2$ and $f_1(\overset{\circ}{x}_2) \geqq 3/4$. So $\|f_1\|_E \geqq 3/4$ and there is a point $y_2 \in V_1$ such that $f_1(\overset{\circ}{y}_2) = \|f_1\|_E$. Therefore, the function $f = (f_1/\|f_1\|_E)^k$, k being a suitable integer, satisfies the conditions

$$|f\| = f(y_2) = 1 \quad \text{and} \quad f(x) < \frac{1}{4} \quad \text{for } x \in F_1 | U_2 \quad \text{or} \quad x \in F_2 | V_1 \,.$$

The function $f(x)$ is continuous, so one can find an integer $m_2 > m_1$ so large that $V_2(y_2) = \{y \mid \rho(y_1, y_2) < 1/2^{m_2}\} \subset V_1$ and $f(y) \geqq 3/4$ for $y \in V_2$.

Since F_2 is minimal, there is a function $g \in A$ and a point $\overset{\smile}{y}_2 \in V_2(y_2)$ such that $\|g\|_E = g(\overset{\smile}{y}_2) = 1$ and $g(y) < 1/4$ for $F_2 \mid V_2$. Therefore, the function $h(x) = f(x)g(x)$ has the following properties: $h \in A$, $\|h\| \geqq 3/4$, $h(x) < 1/4$ for $x \in F_1 \mid U_2$ or $x \in F_2 \mid V_2$. Hence, the function $h_2(x) = (h(x)/\|h\|_E)^k$, where k is a sufficiently large integer, satisfies

$$\|h_2\|_E = 1 , \quad h_2(x) < \frac{1}{4} \quad \text{for } x \in F_1 \mid U_2 \text{ or } x \in F_2 \mid V_2 ,$$

and there is a point $x_2 \in U_2$ and a point $y_2 \in V_2$ for which $h_2(x_2) = h_2(y_2) = 1$.

Continuing this procedure, we construct two sequences of points $\{x_\nu\}$ and $\{y_\nu\}$, two descending sequences of neighborhoods $\{U_\nu(x_\nu)\}$ and $\{V_\nu(y_\nu)\}$, and a sequence of functions $\{h_\nu(x)\}$. By their construction these sequences satisfy the following conditions: The functions $h_\nu(x)$, $\nu = 1, 2, \cdots$ belong to A (in fact, we have only $h_\nu(x) = c_\nu h_\nu^*(x)$, where $h_\nu^* \in A$ and $c_\nu = \text{const} > 0$, but it does not matter because of the Remark on p. 377). The neighborhoods U_ν and V_ν converge to points $\overset{\circ}{x}$ and $\overset{\circ}{y}$, respectively. The points $\overset{\circ}{x}$ and $\overset{\circ}{y}$ are also the limits of $\{x_\nu\}$ and $\{y_\nu\}$, respectively. For any $\nu = 1, 2, \cdots$ we have

$$\|h_\nu\|_E = h_\nu(x_\nu) = h_\nu(y_\nu) = 1 , \quad h_\nu(x) < \frac{1}{4} \quad \text{for } x \in F_1 \mid U_\nu \text{ or } x \in F_2 \mid V_\nu .$$

Since $U_{\nu+1} \subset U_\nu$, $V_{\nu+1} \subset V_\nu$ and $U_\nu \cap V_\nu = \phi$, $\nu = 1, 2, \cdots$, we have $\overset{\circ}{x} \neq \overset{\circ}{y}$. The family A separates the points of E. Thus, there is a function $h \in A$ such that $h(\overset{\circ}{x}) \neq h(\overset{\circ}{y})$. Without any loss of generality we may assume that $h(\overset{\circ}{x}) < h(\overset{\circ}{y})$. Let $h(\overset{\circ}{y}) - h(\overset{\circ}{x}) = 3\varepsilon$. Since $h(x)$ is continuous, we may find two neighborhoods $U(\overset{\circ}{x})$ and $V(\overset{\circ}{y})$ such that

$$h(x) < h(\overset{\circ}{x}) + \varepsilon \quad \text{for } x \in U(\overset{\circ}{x}) \quad \text{and} \quad h(\overset{\circ}{y}) - \varepsilon < h(y) \quad \text{for } y \in V(\overset{\circ}{y}) .$$

Since U_ν and V_ν converge to $\overset{\circ}{x}$ and $\overset{\circ}{y}$, respectively, there is an integer ν_0 such that $U_{\nu_0} \subset U(\overset{\circ}{x})$ and $V_{\nu_0} \subset V(\overset{\circ}{y})$. Let $M = \|h\|_E$, and let m be so large that $M/2^m < \varepsilon$. Then the function $b(x) = h(x) [h_{\nu_0}(x)]^m$ satisfies the conditions:

(i) $b(x) \leqq \dfrac{M}{4^m} < \varepsilon$,   for $x \in F_1 \mid U_{\nu_0}$ or $x \in F_2 \mid V_{\nu_0}$;

(ii) $b(x) < h(\overset{\circ}{x}) + \varepsilon$,   for $x \in U_{\nu_0}$;

(iii) $b(x) \geqq h(\overset{\circ}{y}) - \varepsilon \; h(\overset{\circ}{x}) + \varepsilon$,   for $x \in V_{\nu_0}$.

Thus the function $b(x)$ attains its maximum $\|b\|_E$ on F_2 and $b(x) < \|b\|_E$ for $x \in F_1$. Therefore, F_1 is not a determining set. This contradiction completes our proof.

A simple consequence of Theorem 2 is the following

THEOREM 2'. *Let A be a separating family of real continuous functions defined on a compact metric space E, and let A be closed under the addition operation. Then E has a boundary with respect to A.*

3. Regular boundary points. The following theorem is a reformulation of the theorem by E. Bishop (see [3], p. 633) in a slightly more general form.

THEOREM 3. *If A is a separating family of nonnegative continuous functions defined on a compact metric Hausdorff space E and if*

1° *A contains positive constants,*

2° *A is closed under addition and multiplication of functions of A,*

3° *A contains limits of uniformly convergent sequences of functions of A;*

then E has a boundary F with respect to A, and F is the closure of regular boundary points of E with respect to A.

Proof. The boundary F exists by Theorem 2. Let x_0 be a fixed point of F and let $U_0 = U(\mathring{x})$ be a neighborhood of U_0. It will now be our task to find a regular point in the neighborhood U_0. By the definition of F and in virtue of assumptions 1° and 2° of the theorem, there is a function $f_0 \in A$ such that $\|f_0\|_E = 1$, $f_0(x_1) = 1$ for some point $x_1 \in U_0$, and $f_0(y) < 1/4$ for $y \in F|U_0$. Let $U_1 \subset U_0$ be a neighborhood of x_1 such that $f_0(x) > 3/4$ for $x \in U_1$. There is a function $f_1(x) \in A$ such that $\|f_1\|_E = 1$, $f_1(x_2) = 1$ for a point $x_2 \in U_1$, and $f_1(y) < 1/4$ for $x \in F|U_1$. Repeating this procedure, we can define:

1° a sequence of neighborhoods $\{U_\nu\}$, $U_{\nu+1} \subset U_\nu$, whose product contains a single point y_0,

2° a sequence of functions $f_\nu(x)$ such that

$$f_\nu(y_0) > \frac{3}{4}, \quad f_\nu(y) < \frac{1}{4} \quad \text{for } y \in F|U_\nu \quad \text{and} \quad \|f_\nu\|_E = 1, \quad \nu = 1, 2, \cdots.$$

Now we can define a function $g \in A$, which is a barrier function of A at the point y_0. Namely, in the same way as in the proof of Theorem 2 of [3], we at first construct, by induction, a sequence of functions $g_n \in A$ such that

(i) $\|g_{n+1} - g_n\|_E < 2^{-n+1}$

(ii) $\|g_n\|_E \leqq 3(1 - 2^{-n-1})$

(iii) $g_n(y_0) = 3(1 - 2^{-n})$

(iv) $|g_{n+1}(y) - g_n(y)| < 2^{-n-1}$ for $y \in F\backslash U_{\mu_{n+1}}$,

where $\{U_{\mu_{n+1}}\}$ is a suitably chosen subsequence of $\{U_n\}$.

We put $g_1(x) = 3/2\,[f_1(y_0)]^{-1} f_1(x)$ and check that g_1 satisfies (ii) and (iii). Assuming that $g_1, \cdots, g_k$ have been constructed, we define g_{k+1} in the following way. Since g_k is continuous and $g_k(y_0) = 3(1 - 2^{-k})$, we can find $\mu_{k+1} > \mu_k$ such that

$$g_k(x) < 3(1 - 2^{-k}) + 2^{-k-2} \quad \text{for } x \in U_{\mu_{k+1}}\,.$$

Now we define $g_{k+1}(x) = g_k(x) + 3 \cdot 2^{-k-1}\,[f_{\mu_{k+1}}(y_0)]^{-1} f_{\mu_{k+1}}(x)$ and check that $g_1, g_2, \cdots, g_k, g_{k+1}$ satisfy (i)–(iv) (for details see [3], p. 633). A barrier function of A at the point $\hat{y}$ is given by

$$g(x) = 3 \sum_{k=1}^{\infty} \frac{f_{\mu_k}(x)}{2^k f_{\mu_k}(y_0)} = \lim_{n \to \infty} g_n(x)\,, \qquad x \in E\,.$$

Namely, we have $g(y_0) = 3$, $\|g\|_E \leqq 3$. Since

$$g(x) = g_n(x) + 3 \sum_{k=n+1}^{\infty} \frac{f_{\mu_k}(x)}{2^k f_{\mu_k}(y_0)}$$

and since $f_{\mu_k}(x) < 1/4$ for $x \in F | U_{\mu_{k+1}}$, we have

$$g(x) < 3(1 - 2^{-n-1}) + 3 \sum_{k=n+1}^{\infty} \frac{1/4}{2^k \cdot 3/4} = 3 - 2^{-n-1} < 3 \quad \text{for } x \in F | U_{\mu_{n+1}}\,.$$

Hence, $g(x) < 3$ for $x \neq y_0$.

4. **Applications.** Let D be a bounded set in the Euclidean space R^n, $n \geqq 1$. Let $A = A(D)$ be a family of real functions defined on D. We denote by A^* the set of all the upper regularizations of functions of A. We shall call the boundary of $\bar{D}$ with respect to A^* also the boundary of D with respect to A.

By means of Theorems 1 and 2, we can easily check that the following function families have boundaries:

(1) The family of moduli of all polynomials of n complex variables for any bounded set $D \subset C^n$.

(2) The family of moduli of holomorphic functions of n complex variables for any bounded set $D \subset C^n$.

(3) The family of pluri-subharmonic functions for any bounded set $D \subset C^n$.

(4) The family of convex functions of n real variables for any bounded domain $D \subset R^n$, $n \geqq 1$.

(5) The family of solutions in a bounded domain $D \subset R^n$ (continuous in $\bar{D}$) of the system of differential inequalities (or equalities)

$$\mathscr{E}^{(i)}[u] = \sum a_{\mu_1}^{(i)}, \ldots, \mu_n \frac{\partial^\mu u}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} \geqq 0\,, \quad \begin{array}{l} k \geqq \mu_1 + \cdots + \mu_n = \mu \geqq 2 \\ i = 1, 2, \cdots, l \end{array}$$

(6)   The family of continuous solutions of the system of differential inequalities

$$P[u] = \sum a_{\mu_1}^{(i)}, \ldots, \mu_n \frac{\partial^\mu u}{\partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}} + \sum_{\nu=1}^{m} b_\nu \frac{\partial u}{\partial y_\nu} \geqq 0 \ ,$$

$$k \geqq \mu \geqq 2 \ , \quad b_\nu = \text{const} \qquad i = 1, 2, \cdots, l$$

in any bounded domain $D \subset R^{n+m}$ contained in the set

$$S = \left\{ (x_1, \cdots, x_n, y_1, \cdots, y_m) \middle| \begin{array}{cc} -\infty < x_i < \infty & y_j \in [0, \varepsilon_j \cdot \infty) \\ i = 1, 2, \cdots, n \ , & j = 1, 2, \cdots, m \end{array} \right\} ,$$

$$\varepsilon_j = \text{sgn } b_j \ .$$

The statements of (1) and (2) follow from Theorem 1 with $f_\mu(z) = a |z_\mu - z_\mu^0|$, $\mu = 1, 2, \cdots, n$ where $a = \text{const}$ is suitably chosen. (3) follows from Theorem 1' with $f_\mu(z) = \log |a(z_\mu - z_\mu^0)|$, $\mu = 1, \cdots, n$.

The families (4)–(6) are closed with respect to addition of their functions.   The function $f(x) = \sum_{k=1}^n (x_k - x_k^0)^2$ is a universal separating convex function.   The functions $f_\mu(x) = x_\mu - \mathring{x}_\mu$, $\mu = 1, 2, \cdots, n$ are separating for (4) and (5).   The functions $f_\mu(x) = x_\mu - \mathring{x}_\mu$, $\mu = 1, 2, \cdots, n$, $g_\mu(y) = \varepsilon_\mu(y_\mu - \mathring{y}_\mu)$, $\mu = 1, 2, \cdots, m$ are separating for (6).

Let us observe that the family (5) involves as a special case the family of double-harmonic functions.   It is well known [1] that the boundary of a bicylinder with respect to double-harmonic functions is equal to the boundary of the bicylinder with respect to holomorphic functions.   A similar situation holds for strictly pseudo-convex domains. But it is not known what is the situation for general domains.   The relation between the Šilov boundary of a domain $D \subset C^n$ with respect to holomorphic functions and with respect to pluri-subharmonic functions has been investigated in [4].

The family (6) involves as a special case the family of "subparabolic" functions (compare [6]).

Any linear function $f(x) = a_1 x_1 + a_2 x_2 + \cdots a_n x_n + b$, where a_k are real numbers, satisfies the system of inequalities $\mathscr{E}^{(i)}[f] \geqq 0$, $i = 1, 2, \cdots, l$.   Let D be a strictly convex domain in the space R^n. This means that for any point $\mathring{x} \in D^{\cdot}$, $D^{\cdot}$ being a topological boundary of D, there is a hyperplane $a_1 x_1 + \cdots + a_n x_n + b = 0$ which has no common points with $\bar{D}$, except the point $\mathring{x}$. Therefore, the function $f(x) = a_1 x_1 + \cdots + a_n x_n + b$ (multiplied by -1, if necessary) is a continuous barrier-function of family (5) at the point $\mathring{x}$. Hence, by the theorem on p. 376, we have

COROLLARY. *If D is a strictly convex domain, then for any continuous function $b(x)$ defined on $D^{\cdot}$ there is a generalized solution $\varphi(x)$ of the Dirichlet boundary value problem inside any family (5).*

Particularly, if D is strictly convex and b(x) is continuous on D·, then there is a convex function $\varphi(x)$, continuous in $\bar{D}$, such that $\varphi(x) = b(x)$ for $x \in D·$.

THEOREM. *If D is a bounded domain in C^n, then the boundary of D with respect to the family A of pluri-subharmonic functions in D, continuous in $\bar{D}$, is the closure of regular points of D with respect to A.*

Proof. Let A_1 denote the family of all functions g such that there exists a function $f \in A$ for which $g(x) = e^{f(x)}$. We can easily check that $A_1 \subset A$ and A_1 satisfies all the assumptions of Theorem 3. Indeed, if f and $g \in A_1$, then by a computation of the Hermitian form

$$\sum_{\mu, \nu = 1}^{n} \frac{\partial^2 \log (f + g)}{\partial z_\mu \, \partial \bar{z}_\nu} \lambda_\mu \overline{\lambda_\nu}$$

we check that $f + g \in A_1$ in the case of f and g being sufficiently regular. The general case is attained by approximation. The other assumptions follow directly from the known properties of pluri-subharmonic functions. By Theorem 1 domain D has a boundary with respect to A_1. By Theorem 3 this boundary is the closure of the regular boundary points with respect to A_1. Let us now observe that the boundary F of D with respect to A is the same as the boundary F_1 of D with respect to A_1. Namely, since $A_1 \subset A$, then $F_1 \subset F$. The function g and e^g attain their maxima in the same points of $\bar{D}$. If $g \in A$, then $e^g \in A_1$. Thus any function of A assumes its maximum on F_1, whence $F \subset F_1$. It follows that $F = F_1$, and the boundary of D with respect to A is the closure of regular boundary points of D with respect to A.

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