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PARAMETER**

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In considering various classes of two-point boundary problems, whose boundary conditions involve the characteristic parameter linearly, both Bobonis [4] and the author [7] have imposed a condition on the coefficients of the boundary conditions in addition to requiring that the boundary conditions be linearly independent. If the boundary conditions are written in vector form

$$(1) \quad s[y; \lambda] \equiv (M_0 + \lambda M_1)y(a) + (N_0 + \lambda N_1)y(b) = 0,$$

where  $M_0, M_1, N_0, N_1$  are each  $n \times n$  constant matrices whose elements may be complex-valued, and  $y(a), y(b)$  denote the end-values of the  $n$ -dimensional vector  $y(x)$ ,  $a \leq x \leq b$ , and the  $n \times 2n$  matrix  $\begin{vmatrix} M_0 + \lambda M_1 & N_0 + \lambda N_1 \\ M_1 & N_1 \end{vmatrix}$  has rank  $n$  for every complex value of  $\lambda$ , the imposed assumption is

CONDITION (A). *There exist  $n \times n$  constant matrices  $M_2, N_2, P_2, Q_2$  and  $n \times n$  matrices  $P(\lambda), Q(\lambda)$  such that for all complex values of  $\lambda$  the  $2n \times 2n$  matrices*

$$\begin{vmatrix} M_0 + \lambda M_1 & N_0 + \lambda N_1 \\ M_2 & N_2 \end{vmatrix}, \quad \begin{vmatrix} -P_2 & -P(\lambda) \\ Q_2 & Q(\lambda) \end{vmatrix}$$

*are reciprocals.*

Theorem 2.1 of [7] established that *if Condition (A) holds then  $P(\lambda)$  and  $Q(\lambda)$  must necessarily be linear in  $\lambda$* . It is to be noted that the boundary conditions associated with the problems discussed in [1], [2], [5], and [6] are each linear in  $\lambda$  and their coefficients satisfy Condition (A).

While Theorem 2.1 of [7] affords a necessary and sufficient test for Condition (A) to hold for a given set of boundary conditions (1), we shall consider necessary and sufficient conditions for the existence of an  $n \times n$  matrix  $I(\lambda)$  which is nonsingular for all  $\lambda$  and such that the coefficients of the equivalent set of boundary conditions  $I(\lambda)s[y; \lambda] = 0$  satisfy (A). Therein we restrict our attention to those nonsingular matrices  $I(\lambda)$  for which the resulting product matrices  $I(\lambda)(M_0 + \lambda M_1)$  and  $I(\lambda)(N_0 + \lambda N_1)$  remain linear in  $\lambda$ . Theorem 1 establishes that such a  $I(\lambda)$  always exists

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for one subclass of boundary conditions; and, in general, a simple necessary and sufficient test for its existence, together with one such possible choice when it does exist, is afforded by Theorem 2. The principal result is Theorem 3, which establishes the existence of an  $n \times n$  matrix  $\Gamma(\lambda)$ , nonsingular for all  $\lambda$  and such that the boundary conditions of the equivalent set  $\Gamma(\lambda)s[y; \lambda] = 0$  remain linear in  $\lambda$  and have coefficients satisfying Condition (A), in case there exist adjoint boundary conditions linear in  $\lambda$ . Thus, *the linearity of the adjoint boundary conditions is both necessary and sufficient for the existence of a suitable  $\Gamma(\lambda)$* . As an immediate consequence, Theorem 4 then indicates that such a nonsingular  $\Gamma(\lambda)$  exists for boundary conditions (1) that are equivalent to their adjoint conditions under nonsingular transformations. In particular, for the classes of problems considered in [4] and [7] Condition (A) may be eliminated as a restrictive condition on the coefficients of the boundary conditions in that it can always be satisfied on replacing the boundary conditions by an equivalent set of linearly independent boundary conditions, which is also linear in  $\lambda$ .

Matrix notation will be employed throughout. The  $n \times n$  identity matrix will be designated by  $E$ , while  $e_j$ ,  $0 < j \leq n$ , will denote the  $j \times j$  identity matrix. In addition,  $M^*$  will denote the conjugate transpose of the matrix  $M$ , and vectors will be treated as  $n \times 1$  matrices.

For convenience, we state Theorem 2.1 of [7].

**THEOREM 2.1 OF [7].** *A necessary and sufficient condition that Condition (A) hold for a set of linearly independent boundary conditions (1) is that the  $2n \times 2n$  matrix*

$$(2) \quad \begin{vmatrix} M_0 & N_0 \\ M_1 & N_1 \end{vmatrix}$$

*have rank  $n + \rho$ , where  $\rho$  is the rank of the  $n \times 2n$  matrix  $\|M_1 N_1\|$ . Moreover, in this case  $P(\lambda)$  and  $Q(\lambda)$  must be linear in  $\lambda$ .*

First, let us consider a set of linearly independent boundary conditions (1) for which the matrix (2) has rank  $n$ . In this case there exists an  $n \times n$  constant matrix  $C$  such that  $\|M_1 N_1\| = C\|M_0 N_0\|$ ; and, thus, the boundary conditions (1) may be written as

$$s[y; \lambda] \equiv (E + \lambda C) \cdot (M_0 y(a) + N_0 y(b)) = 0.$$

Moreover, as the linear independence of the boundary conditions guarantees that the  $n \times 2n$  product matrix  $(E + \lambda C) \cdot \|M_0 N_0\|$  has rank  $n$  for every value of  $\lambda$ , the  $n \times n$  matrix  $(E + \lambda C)$  is nonsingular for each  $\lambda$ . Consequently, the boundary conditions  $\Gamma(\lambda)s[y; \lambda] \equiv M_0 y(a) + N_0 y(b) = 0$ , with  $\Gamma(\lambda) \equiv (E + \lambda C)^{-1}$ , are equivalent to (1); and, as Condition (A) is known

to hold, with  $P(\lambda)$  and  $Q(\lambda)$  constant matrices, for a set of linearly independent two-point boundary conditions not involving  $\lambda$  (see, for example, Bliss [3, p. 565]), we have the following result.

**THEOREM 1.** *If the matrix (2) has rank  $n$  for a set of  $n$  linearly independent boundary conditions  $s[y; \lambda] = 0$ , then  $s[y; 0] = 0$  is an equivalent set of boundary conditions, whose coefficients satisfy Condition (A) with  $P(\lambda)$  and  $Q(\lambda)$  independent of  $\lambda$ .*

In general, one has the following result first noted in the author's doctoral dissertation, written under the direction of Professor W. T. Reid.

**LEMMA 1.** *For a set of  $n$  linearly independent boundary conditions (1) there exists an  $n \times n$  matrix  $\Gamma(\lambda)$ , nonsingular for all  $\lambda$ , such that the equivalent set of boundary conditions  $\Gamma(\lambda)s[y; \lambda] = 0$  is linear in  $\lambda$  and has coefficients satisfying Condition (A) if and only if there exists an  $n \times n$  constant matrix  $G$  such that*

(a)  $G$  is nilpotent ,

(b)  $G(M_1 - GM_0) = G(N_1 - GN_0) = 0$  ,

(c) the  $n \times 2n$  matrix  $\|M_1 - GM_0 \ N_1 - GN_0\|$  has rank equal to the excess of the rank of (2) over  $n$ .

Moreover, in this event the matrix  $\Gamma(\lambda)$  can be chosen as  $\Gamma(\lambda) \equiv (E + \lambda G)^{-1}$ .

The necessary condition  $-\Gamma(\lambda)(M_0 + \lambda M_1)P_2 + \Gamma(\lambda)(N_0 + \lambda N_1)Q_2 = E$  implies that  $\Gamma^{-1}(\lambda)$  is linear in  $\lambda$ ; and, hence, without loss of generality, we may write  $\Gamma^{-1}(\lambda) \equiv E + \lambda G$ . In view of Theorem 2.1 of [7], conditions (a), (b) and (c) are then necessary and sufficient that  $\Gamma(\lambda)$  be nonsingular for each  $\lambda$ , that the product matrices  $\Gamma(\lambda)(M_0 + \lambda M_1)$ ,  $\Gamma(\lambda)(N_0 + \lambda N_1)$  remain linear in  $\lambda$ , and that Condition (A) hold.

Suppose, now, that for a set of linearly independent boundary conditions (1) the matrix (2) has rank  $n + \rho - r$ ,  $0 < r < \rho$ , where  $\rho$  denotes the rank of the  $n \times 2n$  matrix  $\|M_1 \ N_1\|$ . Then, by multiplying on the left by a product of suitable elementary constant matrices, we can transform the set (1) into the equivalent set of boundary conditions

$$\begin{aligned} \tilde{s}[y; \lambda] &\equiv (\tilde{M}_0 + \lambda \tilde{M}_1)y(a) + (\tilde{N}_0 + \lambda \tilde{N}_1)y(b) = 0 , \\ \tilde{M}_1 &\equiv \begin{Bmatrix} \tilde{m}_1^{(1)} \\ \tilde{m}_1^{(2)} \\ 0 \end{Bmatrix} , & \tilde{N}_1 &\equiv \begin{Bmatrix} \tilde{n}_1^{(1)} \\ \tilde{n}_1^{(2)} \\ 0 \end{Bmatrix} , \end{aligned}$$

where  $\tilde{m}_1^{(1)}$  and  $\tilde{n}_1^{(1)}$  are each  $(\rho - r) \times n$  matrices such that the  $(\rho - r) \times 2n$  matrix  $\|\tilde{m}_1^{(1)} \ \tilde{n}_1^{(1)}\|$  has rank  $\rho - r$  and its rows are linearly independent

of the rows of  $\|\tilde{M}_0 \tilde{N}_0\|$ , and  $\tilde{m}_1^{(2)}$  and  $\tilde{n}_1^{(2)}$  are each  $r \times n$  matrices such that the  $r \times 2n$  matrix  $\|\tilde{m}_1^{(2)} \tilde{n}_1^{(2)}\|$  has rank  $r$  and its rows are linearly independent of the  $\rho - r$  rows of  $\|\tilde{m}_1^{(1)} \tilde{n}_1^{(1)}\|$  and, furthermore, linearly dependent only on the rows of  $\|\tilde{M}_0 \tilde{N}_0\|$ . Thus, there exists an  $r \times n$  matrix  $\sigma$ , of rank  $r$ , such that

$$\|\tilde{m}_1^{(2)} \tilde{n}_1^{(2)}\| = \sigma \|\tilde{M}_0 \tilde{N}_0\|.$$

Now, if we let  $\sigma \equiv \|\sigma_1 \sigma_2 \sigma_3\|$  denote a partitioning of the matrix  $\sigma$  such that the submatrices  $\sigma_1$ ,  $\sigma_2$  and  $\sigma_3$  have dimensions  $r \times (\rho - r)$ ,  $r \times r$ , and  $r \times (n - \rho)$ , respectively, then multiplication of the boundary conditions  $\tilde{s}[y; \lambda] = 0$  on the left by

$$(3) \quad \Delta(\lambda) \equiv \begin{vmatrix} e_{\rho-r} & 0 & 0 \\ 0 & e_r & -\lambda \sigma_3 \\ 0 & 0 & e_{n-\rho} \end{vmatrix}$$

yields an equivalent set of boundary conditions that may be obtained from the set  $\tilde{s}[y; \lambda] = 0$  by replacing each element of the submatrix  $\sigma_3$  by zero. Consequently, if the matrix (2) has rank  $n + \rho - r$ ,  $0 < r < \rho$ , for a set of  $n$  linearly independent boundary conditions (1), then, by multiplication on the left by a suitable  $n \times n$  nonsingular matrix, the set (1) may be transformed into the equivalent set  $s^\Delta[y; \lambda] \equiv \Delta(\lambda)\tilde{s}[y; \lambda] = 0$ , with  $\tilde{s}[y; \lambda]$  as above and  $\Delta(\lambda)$  given by (3), of the form

$$(4) \quad s^\Delta[y; \lambda] \equiv (\tilde{M}_0 + \lambda M_1^\Delta)y(a) + (\tilde{N}_0 + \lambda N_1^\Delta)y(b) = 0,$$

$$M_1^\Delta \equiv \begin{vmatrix} \tilde{m}_1^{(1)} \\ \sigma^\Delta \tilde{M}_0 \\ 0 \end{vmatrix}, \quad N_1^\Delta \equiv \begin{vmatrix} \tilde{n}_1^{(1)} \\ \sigma^\Delta \tilde{N}_0 \\ 0 \end{vmatrix}, \quad \sigma^\Delta \equiv \|\sigma_1 \sigma_2 0\|,$$

where the  $n \times 2n$  matrix  $\|\tilde{M}_0 + \lambda M_1^\Delta \tilde{N}_0 + \lambda N_1^\Delta\|$  has rank  $n$  for all  $\lambda$ , the  $(\rho - r) \times 2n$  matrix  $\|\tilde{m}_1^{(1)} \tilde{n}_1^{(1)}\|$  has rank  $\rho - r$  and its rows are linearly independent of the rows of  $\|\tilde{M}_0 \tilde{N}_0\|$ , and  $\sigma^\Delta$  is an  $r \times n$  matrix with partitioning submatrices  $\sigma_1$  and  $\sigma_2$  of dimensions  $r \times (\rho - r)$  and  $r \times r$ , respectively.

**THEOREM 2.** *Suppose that, for a set of  $n$  linearly independent boundary conditions (1), the matrix (2) has rank  $n + \rho - r$ ,  $0 < r < \rho$ , where  $\rho$  denotes the rank of  $\|M_1 N_1\|$ . If the boundary conditions (1) are transformed into the equivalent set (4), then there exists an  $n \times n$  matrix  $\Gamma(\lambda)$ , nonsingular for all  $\lambda$ , such that the further equivalent set of boundary conditions  $\Gamma(\lambda)s^\Delta[y; \lambda] = 0$  is linear in  $\lambda$  and has coefficients satisfying Condition (A) if and only if each element of the  $r \times (\rho - r)$  submatrix  $\sigma_1$  is zero. Moreover, in this case the  $r \times r$  submatrix  $\sigma_2$  is nilpotent and  $\Gamma(\lambda)$  may be chosen as*

$$(5) \quad \Gamma(\lambda) \equiv \begin{vmatrix} e_{\rho-r} & 0 & 0 \\ 0 & (e_r + \lambda\sigma_2)^{-1} & 0 \\ 0 & 0 & e_{n-\rho} \end{vmatrix}.$$

If the matrix (2) has rank  $n + \rho - r$ ,  $0 < r < \rho$ , for a set of  $n$  linearly independent boundary conditions (1), then, clearly, the matrix corresponding to (2) for the transformed set (4),

$$\begin{vmatrix} \tilde{M}_0 & \tilde{N}_0 \\ M_1^\blacktriangle & N_1^\blacktriangle \end{vmatrix}$$

also has rank  $n + \rho - r$ . Applying Lemma 1 to the boundary conditions (4), we first note that if the matrix  $G$  is partitioned as

$$G \equiv \begin{vmatrix} g^{(1)} \\ g^{(2)} \\ g^{(3)} \end{vmatrix}$$

where the dimensions of  $g^{(1)}$ ,  $g^{(2)}$  and  $g^{(3)}$  are  $(\rho - r) \times n$ ,  $r \times n$  and  $(n - \rho) \times n$ , respectively, then condition (c) of Lemma 1 is the condition that the  $n \times 2n$  matrix

$$(6) \quad \begin{vmatrix} \tilde{m}_1^{(1)} - g^{(1)}\tilde{M}_0 & \tilde{n}_1^{(1)} - \tilde{N}g^{(1)}_0 \\ (\sigma^\blacktriangle - g^{(2)})\tilde{M}_0 & (\sigma^\blacktriangle - g^{(2)})\tilde{N}_0 \\ -g^{(3)}\tilde{M}_0 & -g^{(3)}\tilde{N}_0 \end{vmatrix}$$

have rank  $\rho - r$ . As the  $(\rho - r) \times 2n$  matrix  $\|\tilde{m}_1^{(1)} \tilde{n}_1^{(1)}\|$  has rank  $\rho - r$  and its rows are linearly independent of the rows of  $\|\tilde{M}_0 \tilde{N}_0\|$ , it follows that the top  $\rho - r$  rows of (6) are linearly independent, and that, moreover, condition (c) holds if and only if  $g^{(2)} = \sigma^\blacktriangle$  and each element of  $g^{(3)}$  is zero. Now, if the matrix  $g^{(1)}$  is further partitioned as  $g^{(1)} \equiv \|g_1^{(1)} g_2^{(1)} g_3^{(1)}\|$ , where the dimensions of  $g_1^{(1)}$ ,  $g_2^{(1)}$  and  $g_3^{(1)}$  are  $(\rho - r) \times (\rho - r)$ ,  $(\rho - r) \times r$  and  $(\rho - r) \times (n - \rho)$ , respectively, then, under condition (c), condition (b) of Lemma 1 reduces to the condition that the matrix product

$$\begin{vmatrix} g_1^{(1)} \\ \sigma_1 \end{vmatrix} \cdot \|\tilde{m}_1^{(1)} - g^{(1)}\tilde{M}_0 \tilde{n}_1^{(1)} - g^{(1)}\tilde{N}_0\| = 0.$$

Thus, under condition (c), condition (b) of Lemma 1 holds for the set (4) if and only if each element of each of the submatrices  $g_1^{(1)}$  and  $\sigma_1$  is zero. Finally, under conditions (b) and (c) of Lemma 1, condition (a) holds if and only if the  $r \times r$  submatrix  $\sigma_2$  is nilpotent. However, if each element of  $\sigma_1$  is zero then the  $\rho - r + 1, \dots, \rho$  rows of the  $n \times 2n$  matrix  $\|\tilde{M}_0 + \lambda M_1^\blacktriangle \tilde{N}_0 + \lambda N_1^\blacktriangle\|$  may be written as the matrix product

$$(7) \quad (e_r + \lambda\sigma_2) \cdot \|\tilde{m}_0^{(2)} \tilde{n}_0^{(2)}\|,$$

where  $\|\tilde{m}_0^{(2)} \tilde{n}_0^{(2)}\|$  denotes the  $r \times 2n$  submatrix consisting of the  $\rho - r + 1, \dots, \rho$  rows of  $\|\tilde{M}_0 \tilde{N}_0\|$ . The linear independence of the boundary conditions (4) then guarantees that the matrix product (7) has rank  $r$  for all  $\lambda$ , and, hence, the  $r \times r$  matrix  $e_r + \lambda\sigma_2$  must be nonsingular for every value of  $\lambda$ . Consequently, if each element of  $\sigma_1$  is zero for a set of transformed boundary conditions (4) then  $\sigma_2$  is nilpotent, and the remainder of the theorem follows at once.

It is to be noted that linearly independent boundary conditions of the form (1) exist for which the corresponding  $\sigma_1 \neq 0$  as, for example, the following choices, for  $n = 2$ , indicate:

$$M_0 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad N_0 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}, \quad M_1 = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, \quad N_1 = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}.$$

Boundary conditions adjoint to a set of linearly independent boundary conditions (1) may be defined independently of Condition (A). If  $n \times n$  matrices  $P(\lambda), Q(\lambda)$  are such that the  $2n \times n$  matrix  $\begin{vmatrix} P(\lambda) \\ Q(\lambda) \end{vmatrix}$  has rank  $n$  for each value of  $\lambda$ , and if

$$(8) \quad (M_0 + \lambda M_1)P(\lambda) \equiv (N_0 + \lambda N_1)Q(\lambda) \quad \text{for all } \lambda,$$

then the boundary conditions

$$(9) \quad [P(\bar{\lambda})]^*z(a) + [Q(\bar{\lambda})]^*z(b) = 0,$$

where  $z(a), z(b)$  denote the end-values of an  $n$ -dimensional vector  $z(x)$ ,  $a \leq x \leq b$ , will be termed *adjoint* to the conditions (1).

**THEOREM 3.** *Suppose that, for a set of linearly independent boundary conditions (1), adjoint boundary conditions (9) exist which are linear in  $\lambda$ . Then, Condition (A) is satisfied by the coefficients of the equivalent set of boundary conditions  $s^*[y; \lambda] \equiv \Gamma(\lambda)s^\Delta[y; \lambda] = 0$ , where  $s^\Delta[y; \lambda] = 0$  is the transformed set (4) and  $\Gamma(\lambda)$  is given by (5). Moreover, the boundary conditions (9) are also adjoint to the set  $s^*[y; \lambda] = 0$ .*

If there exist  $n \times n$  constant matrices  $P_0, P_1, Q_0$  and  $Q_1$  such that the  $2n \times n$  matrix  $\begin{vmatrix} P_0 + \lambda P_1 \\ Q_0 + \lambda Q_1 \end{vmatrix}$  is of rank  $n$  for all  $\lambda$  and relation (8) holds for  $P(\lambda) \equiv P_0 + \lambda P_1, Q(\lambda) \equiv Q_0 + \lambda Q_1$ , then  $P(\lambda), Q(\lambda)$  also satisfy relation (8) with the coefficient matrices of the transformed set (4), i.e.,

$$(10) \quad \tilde{M}_0 + \lambda M_1^\Delta (P_0 + \lambda P_1) \equiv (\tilde{N}_0 + \lambda N_1^\Delta) (Q_0 + \lambda Q_1) \quad \text{for all } \lambda.$$

Partitioning the matrices  $M_1^\Delta$  and  $N_1^\Delta$ , as indicated in (4), and substituting in (10), we have that

$$\begin{aligned}
 & \tilde{M}_0 P_0 = \tilde{N}_0 Q_0, \\
 & \tilde{m}_1^{(1)} P_1 = \tilde{n}_1^{(1)} Q_1, \\
 (11) \quad & \tilde{M}_0 P_1 - \tilde{N}_0 Q_1 = - \left\| \begin{array}{c} \tilde{m}_1^{(1)} P_0 - \tilde{n}_1^{(1)} Q_0 \\ 0 \\ 0 \end{array} \right\|, \\
 & \sigma^\Delta(\tilde{M}_0 P_1 - \tilde{N}_0 Q_1) = 0.
 \end{aligned}$$

Consequently,

$$\sigma_1 \tilde{m}_1^{(1)} P_0 - \sigma_1 \tilde{n}_1^{(1)} Q_0 = 0,$$

and, in view of the first relation of (11) and the fact that the  $n \times 2n$  matrix  $\|\tilde{M}_0 \tilde{N}_0\|$  has rank  $n$ , it follows that there exists an  $r \times n$  matrix  $\tau$  such that

$$\sigma_1 \|\tilde{m}_1^{(1)} \tilde{n}_1^{(1)}\| = \tau \|\tilde{M}_0 \tilde{N}_0\|.$$

However, as the rows of the  $(n + \rho - r) \times 2n$  matrix

$$\left\| \begin{array}{cc} \tilde{M}_0 & \tilde{N}_0 \\ \tilde{m}_1^{(1)} & \tilde{n}_1^{(1)} \end{array} \right\|$$

are linearly independent, every element of each of the matrices  $\sigma_1$  and  $\tau$  must be zero, and the desired result then follows from Theorem 2. The final remark of the theorem is evident from the fact that the equivalent set  $s^*[y; \lambda] = 0$  is obtained from the original set  $s[y; \lambda] = 0$  by a succession of multiplications on the left by  $n \times n$  nonsingular matrices.

Boundary conditions (1) will be said to be *equivalent* to their adjoint conditions (9) under a pair of  $n \times n$  transformation matrices  $[T_a, T_b]$  provided  $T_a$  and  $T_b$  are each  $n \times n$  nonsingular constant matrices such that the end-values  $y(a)$ ,  $y(b)$  satisfy conditions (1) if and only if the corresponding end-values  $z(a) = T_a y(a)$ ,  $z(b) = T_b y(b)$  satisfy conditions (9). Consequently, a necessary and sufficient condition that a set of linearly independent boundary conditions (1) be equivalent to its adjoint set (9) under a pair of  $n \times n$  nonsingular constant matrices  $[T_a, T_b]$  is that

$$(12) \quad (M_0 + \lambda M_1) T_a^{-1} (M_0^* + \lambda M_1^*) \equiv (N_0 + \lambda N_1) T_b^{-1} (N_0^* + \lambda N_1^*) \quad \text{for all } \lambda.$$

**THEOREM 4.** *If a set of linearly independent boundary conditions (1) is equivalent to its adjoint set (9) under a pair of  $n \times n$  nonsingular constant matrices  $[T_a, T_b]$ , then Condition (A) is satisfied by the coefficients of the equivalent set of boundary conditions  $s^*[y, \lambda] \equiv \Gamma(\lambda) s^*[y; \lambda] = 0$ , where  $s^*[y; \lambda]$  is the transformed set (4) and  $\Gamma(\lambda)$  is given by (5). Moreover, the set  $s^*[y; \lambda] = 0$  is also equivalent to the same adjoint set (9) under  $[T_a, T_b]$ .*



Clearly, for the choices  $P(\lambda) \equiv T_a^{-1}(M_0^* + \lambda M_1^*)$ ,  $Q(\lambda) \equiv T_b^{-1}(N_0^* + \lambda N_1^*)$ , the  $2n \times n$  matrix  $\begin{vmatrix} P(\lambda) \\ Q(\lambda) \end{vmatrix}$  has rank  $n$  for each  $\lambda$  and relation (8) holds. Hence, adjoint boundary conditions (9) exist which are linear in  $\lambda$ , and the result then follows immediately from Theorem 3.

Moreover, for symmetrizable boundary problems, defined in § 5 of [7], the further condition placed on the boundary conditions (1) is equivalent to the assumption that the  $2n \times 2n$  constant matrix

$$(13) \quad \begin{vmatrix} T_a^* P_2 M_1 & T_a^* P_2 N_1 \\ T_b^* Q_2 M_1 & T_b^* Q_2 N_1 \end{vmatrix}$$

be hermitian. For a given set of boundary conditions (1) equivalent to its adjoint set (9) under  $[T_a, T_b]$  Theorem 5.1 and the Corollary to Theorem 3.2 of [7] imply that a necessary and sufficient condition that the matrix corresponding to (13) for the transformed set  $s^*[y; \lambda] = 0$  be hermitian is that the matrix  $W^*$  for the transformed set, corresponding to

$$W \equiv M_0 T_a^{*-1} M_1^* - N_0 T_b^{*-1} N_1^* ,$$

be hermitian. Now, for boundary conditions (1) equivalent to their adjoint conditions (9) under  $[T_a, T_b]$  the transformed set has the form  $s^*[y; \lambda] \equiv \Gamma(\lambda) \Delta(\lambda) F \cdot s[y; \lambda] = 0$ , where  $\Delta(\lambda)$  and  $\Gamma(\lambda)$  are given by (3) and (5), and  $F$  is a product of suitable elementary constant  $n \times n$  matrices; and, hence,  $W^* \equiv F W F^*$ . As  $F$  is nonsingular,  $W^*$  is hermitian if and only if  $W$  is hermitian; and, therefore, the hermitian character of  $W$  would be preserved under the transformation.

For various classes of problems considered in [4] and [7] equivalence of a boundary problem with its adjoint under a nonsingular transformation matrix  $T(x)$  (see [7, §3]) clearly implies the equivalence of the related boundary conditions (1) with their adjoint set (9) under  $[T_a \equiv T(a), T_b \equiv T(b)]$ . Consequently, for these problems the replacement of their boundary conditions (1) by the set  $s^*[y; \lambda] = 0$  yields an equivalent problem for which Condition (A) is always satisfied and, thus, need not be postulated.

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