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ON CERTAIN FINITE RINGS AND RING-LOGICS

ADIL MOHAMED YAQUB

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Introduction. Boolean rings $(B, \times, +)$ and Boolean logics (=Boolean algebras) $(B, \cap, *)$ though historically and conceptionally different, are equationally interdefinable in a familiar way [6]. With this equational interdefinability as motivation, Foster introduced and studied the theory of ring-logics. In this theory, a ring (or an algebra) R is studied modulo K , where K is an arbitrary transformation group in R . The Boolean theory results from the special choice, for K , of the "Boolean group", generated by $x^* = 1 - x$ (order 2, $x^{**} = x$). More generally, in a commutative ring $(R, \times, +)$ with identity 1, the *natural group* N , generated by $x^\wedge = 1 + x$ (with $x^\vee = x - 1$ as inverse) proved to be of particular interest. Thus, specialized to N , a commutative ring with identity $(R, \times, +)$ is called a *ring-logic*, mod N if (1) the $+$ of the ring is equationally definable in terms of its N -logic $(R, \times, \wedge, \vee)$, and (2) the $+$ of the ring is *fixed* by its N -logic. Several classes of ring-logics (modulo suitably chosen groups) are known [1; 2; 7], and the object of this manuscript is to extend further the class of ring-logics. Indeed, we shall prove the following:

THEOREM 1. *Let R be any finite commutative ring with zero radical. Then, R is a ring-logic, mod N .*

1. The finite field case. Let $(R, \times, +)$ be a commutative ring with identity 1. We denote the generator of the natural group by $x^\wedge = 1 + x$, with inverse $x^\vee = x - 1$. Following [1], we define $a \times \wedge b = (a^\wedge \times b^\wedge)^\vee$. It is readily verified that $a \times \wedge b = a + b + ab$.

Let $(F_{p^k}, \times, +)$ be a finite field with exactly p^k elements (p prime). We now have the following:

THEOREM 2. *$(F_{p^k}, \times, +)$ is a ring logic (mod N). Indeed, the ring (field) $+$ is given by the following N -logical formula:*

$$(1.1) \quad x + y = \{(x(yx^{p^k-2})^\wedge)\} \times \wedge \{y((x^{p^k-1})^\vee)^2\} .$$

Proof. It is well known that in the Galois field F_{p^k} , we have

$$(1.2) \quad a^{p^k-1} = 1, a \in F_{p^k}, a \neq 0 .$$

we now distinguish two cases:

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Case 1. Suppose $x \neq 0$. Then, by (1.2), the right-side of (1.1) reduces to $\{x(1 + yx^{p^k-2})\} \times \wedge 0 = x + yx^{p^k-1} = x + y$, since $((x^{p^k-1})^\wedge)^2 = (1^\wedge)^2 = 0; a \times \wedge 0 = a$. This proves (1.1).

Case 2. Suppose $x = 0$. Then, $x^\wedge = 1 + x = 1$. Hence, the right side of (1.1) reduces to $0 \times \wedge \{y((0^\wedge)^2)\} = y = 0 + y = x + y$, since $((x^{p^k-1})^\wedge)^2 = (0^\wedge)^2 = 1; 0 \times \wedge a = a$. Again, (1.1) is verified. Hence, $(F_p k, \times, +)$ is *equationally* definable in terms of its N -logic. Next, we show that $(F_p k, \times, +)$ is *fixed* by its N -logic. Suppose then that there exists another ring $(F_p k, \times, +')$, with the same class of elements $F_p k$ and the same “ \times ” as $(F_p k, \times, +)$ and which has the *same logic* as $(F_p k, \times, +)$. To prove that $+ ' = +$. Again, we distinguish two cases.

Case 1. Suppose $x \neq 0$. Then, using (1.2), we have $x + ' y = x(1 + ' yx^{p^k-2}) = x(yx^{p^k-2})^\wedge = x(1 + yx^{p^k-2}) = x + y$, since, by hypothesis, $x^\wedge = 1 + x = 1 + ' x$.

Case 2. Suppose $x = 0$. Then, $x + ' y = 0 + ' y = y = 0 + y = x + y$. Therefore, $+ ' = +$, and the theorem is proved.

COROLLARY. $(F_p, \times, +)$, the ring (field) of residues (mod p), p prime, is a ring-logic (mod N) the $+$ being given by setting $k = 1$ in (1.1):

$$(1.3) \quad x + y = \{(x(yx^{p-2})^\wedge)\} \times \wedge \{y((x^{p-1})^\wedge)^2\}.$$

2. The general case. In attempting to extend Theorem 2 to *any* finite commutative ring with zero radical, the following concept of independence, introduced by Foster [3], is needed.

DEFINITION. Let $\bar{A} = \{A_1, A_2, \dots, A_n\}$ be a finite set of algebras of the same species Sp . We say that the algebras A_1, A_2, \dots, A_n satisfy the *Chinese residue condition*, or are *independent*, if, corresponding to each set $\{\varphi_i\}$ of expressions of species Sp ($i = 1, \dots, n$), there exists at least on expression Ψ such that $\Psi = \varphi_i \pmod{A_i}$ ($i = 1, \dots, n$). By an *expression* we mean some composition of one or more indeterminate-symbols ξ, \dots , in terms of the primitive operations of A_1, A_2, \dots, A_n ; $\Psi = \varphi \pmod{A}$, also written $\Psi = \varphi(A)$, means that this is an identity of the Algebra A .

We shall now extend the concept of ring-logic to the direct sum of certain ring-logics. We shall denote the direct sum of the rings A_1 and A_2 by $A_1 \oplus A_2$. The direct power A^m will denote $A \oplus A \oplus \dots \oplus A$ (m summands).

THEOREM 3. *Let $(A_1, \times, +), \dots, (A_t, \times, +)$ be a finite set of ring-logics (mod N), and let the N -logics $(A_1, \times, \hat{\cdot}, \check{\cdot}), \dots, (A_t, \times, \hat{\cdot}, \check{\cdot})$ be independent. Then $A = A_1^{m_1} \oplus \dots \oplus A_t^{m_t}$ is also a ring-logic (mod N).*

Proof. Since A_i is a ring-logic (mod N), there exist an N -logical expression φ_i such that, for every $x_i, y_i \in A_i$ ($i = 1, \dots, t$),

$$x_i + y_i = \varphi_i = \varphi_i(x_i, y_i; \times, \hat{\cdot}, \check{\cdot}).$$

Since the N -logics are independent, there exists an expression X such that

$$X = \begin{cases} \varphi_1(\text{mod } A_1) \\ \vdots \\ \varphi_t(\text{mod } A_t) \end{cases}.$$

Therefore, for every $x_i, y_i \in A_i$ ($i = 1, \dots, t$),

$$x_i + y_i = \varphi_i = X = X(x_i, y_i; \times, \hat{\cdot}, \check{\cdot}).$$

Hence, the N -logical expression X represents the $+$ of each A_i . Since “ $+$ ” and “ \times ” are component-wise in A , therefore, for all $x, y \in A$,

$$x + y = X(x, y; \times, \hat{\cdot}, \check{\cdot}).$$

Hence, A is *equationally* definable in terms of its N -logic. Next, we show that A is *fixed* by its N -logic. Suppose there exists $a +'$ such that $(A, \times, +')$ is a ring, with the same class of elements A and the same “ \times ” as the ring $(A, \times, +)$, and which has the *same logic* $(A, \times, \hat{\cdot}, \check{\cdot})$ as the ring $(A, \times, +)$. To prove that $+ = +'$.

Now, let $a = (a_{11}, \dots, a_{1m_1}, a_{21}, \dots, a_{2m_2}, \dots, a_{t1}, \dots, a_{tm_t}) \in A$. A new $+'$ in A defines and is defined by new $+'_1$ in $A_1, +'_2$ in $A_2, \dots, +'_t$ in A_t , such that $(A_i, \times, +'_i)$ is a ring ($i = 1, \dots, t$); i.e., for $a, b \in A$,

$$\begin{aligned} (2.1) \quad a +' b &= (a_{11}, \dots, a_{21}, \dots, a_{t1}, \dots) +' (b_{11}, \dots, b_{21}, \dots, b_{t1}, \dots) \\ &= (a_{11} +'_1 b_{11}, \dots, a_{21} +'_2 b_{21}, \dots, a_{t1} +'_t b_{t1}, \dots). \end{aligned}$$

Furthermore, the assumption that $(A, \times, +')$ has the same logic as $(A, \times, +)$ is equivalent to the assumption that $(A_1, \times, +'_1)$ has the same logic as $(A_1, \times, +)$, and similarly for $(A_i, \times, +'_i)$ and $(A_i, \times, +)$ ($i = 2, \dots, t$). Since $(A_1, \times, +)$ is a ring-logic, and hence with its $+$ fixed, it follows that $+'_1 = +$; similarly $+'_2 = +, \dots, +'_t = +$. Hence, using (2.1), $+ = +'$, and the proof is complete.

A careful examination of the proof of Theorem 3 shows that the independence of the logics was *not* used in the “fixed” part of the proof. Hence, we have the following

COROLLARY. *Let $(A_1, \times, +), \dots, (A_t, \times, +)$ be a finite set of ring-*

logics (mod N). Then, $A_1^{m_1} \oplus \dots \oplus A_t^{m_t}$ is fixed by its N -logic.

We now examine the independence of the logics $(F_{p_i}^{m_i}k_i, \times, +)$ ($i = 1, \dots, t$).

THEOREM 4. *Let p_1, \dots, p_t be distinct primes, and let $F_{p_i}^{m_i}k_i$ be the m_i direct power of the Galois field $F_{p_i}k_i$ ($i = 1, \dots, t$). Then the logics $(F_{p_i}^{m_i}k_i, \times, \hat{}, \check{})$ ($i = 1, \dots, t$) are independent.*

Proof. Let $n_i = p_i^{k_i}$, and let $P(i) = \prod_{j=1}^t n_j, j \neq i$. Let $F_i = F_{p_i}k_i$ ($i = 1, \dots, t$). Clearly, $P(i)$ and n_i are relatively prime. Hence, there exist integers $r_i > 0, s_i > 0$ such that $r_i P(i) - s_i n_i = 1$. Define $\varepsilon(x)$ and $\delta(x)$ as follows:

$$\varepsilon(x) = x^{(n_1-1)(n_2-1)\dots(n_t-1)}; \delta(x) = \varepsilon(x) \times \wedge((\varepsilon(x))\check{})^2.$$

It is easily seen that $\delta(x) = 1, x \in F_i^{m_i}$ ($i = 1, \dots, t$). Let $x^{\wedge k} = (\dots((x^{\wedge})^{\wedge})^{\wedge}\dots)^{\wedge}, k$ iterations. Then one easily verifies that for $i \neq j$,

$$w_i = w_i(x) = (\delta(x))^{\wedge s_i n_i} = \begin{cases} 1 \pmod{F_i^{m_i}} \\ 0 \pmod{F_j^{m_j}} \end{cases}.$$

Now, to prove the independence of the logics $(F_i^{m_i}, \times, \hat{}, \check{})$ ($i = 1, \dots, t$), let $\{\delta'_i\}$ be any set of t expressions of species $\times, \hat{}, \check{}$; i.e., a primitive composition of indeterminate-symbols in terms of the operations $\times, \hat{}, \check{}$. Let $X = \delta'_1 w_1 \times \wedge \delta'_2 w_2 \times \wedge \dots \times \wedge \delta'_t w_t$. Then it is easily seen that $X = \delta'_i$ (mod $F_i^{m_i}$) ($i = 1, \dots, t$), since $a \times \wedge 0 = a = 0 \times \wedge a$, and the theorem is proved.

We are now in a position to prove the following theorem (see introduction).

THEOREM 5. *Any finite commutative ring R with zero radical is a ring-logic (mod N).*

Proof. First, if R consists of one element, then $R = \{0\}$. Clearly, R is a ring-logic (mod N) in this case, since $a + b = a \times b$, for example. Hence, assume that R has more than one element. It is well known (see [5]) that any finite commutative ring R with zero radical and with more than one element is isomorphic to the complete direct sum of a finite number of finite fields $F_{p_1}k_1, \dots, F_{p_t}k_t$: i.e., $R \cong F_{p_1}k_1 \oplus \dots \oplus F_{p_t}k_t$. Now, by Theorem 2, each $(F_{p_i}k_i, \times, +)$ is a ring-logic (mod N). Hence, by the corollary to Theorem 3, $F_{p_1}k_1 \oplus \dots \oplus F_{p_t}k_t$ is fixed by its N -logic. Therefore, by the above isomorphism, R , too, is fixed by its N -logic, and there only remains to show that the $+$ of R is equationally definable in terms of its N -logic. To this end, we distinguish two cases.

Case 1. Suppose p_1, \dots, p_t are all *distinct*. By Theorem 2, $(F_{p_i}k_i, \times, +)$ is a ring-logic (mod N) ($i = 1, \dots, t$). By Theorem 4 (with $m_1 = \dots = m_t = 1$), the N -logics $(F_{p_i}k_i, \times, \hat{}, \check{})$ are independent ($i = 1, \dots, t$). Therefore, by Theorem 3 (with $m_1 = \dots = m_t = 1$), the direct sum $F_{p_1}k_1 \oplus \dots \oplus F_{p_t}k_t$ (and hence R , by the above isomorphism) is a ring-logic (mod N). Hence, in particular, the $+$ of R is equationally definable in terms of its N -logic.

Case 2. Suppose p_1, \dots, p_t are *not* all distinct. Let q_1, \dots, q_r be the *distinct* primes in $\{p_1, \dots, p_t\}$. Since the Galois fields $F_{p_i}k_i$ and $F_{p_j}k_j$ are both subfields of $F_{p_i}k_i k_j$, it is easily seen that $F_{p_1}k_1 \oplus \dots \oplus F_{p_t}k_t$ is a *subring* of a direct sum of direct powers of $F_{q_i}h_i$ ($i = 1, \dots, r$); i.e., $F_{p_1}k_1 \oplus \dots \oplus F_{p_t}k_t$ is a subring of $F_{q_1}^{m_1}h_1 \oplus \dots \oplus F_{q_r}^{m_r}h_r$, for some positive integers $h_1, \dots, h_r, m_1, \dots, m_r$. Now, the rest of the proof is similar to that of *Case 1*. Thus, by Theorem 2, $(F_{q_i}h_i, \times, +)$ is a ring-logic (mod N) ($i = 1, \dots, r$). By Theorem 4, the N -logics $(F_{q_i}h_i, \times, \hat{}, \check{})$ are independent ($i = 1, \dots, r$). Hence, by Theorem 3, $F_{q_1}^{m_1}h_1 \oplus \dots \oplus F_{q_r}^{m_r}h_r$ is a ring-logic (mod N). Therefore, in particular, the $+$ of $F_{q_1}^{m_1}h_1 \oplus \dots \oplus F_{q_r}^{m_r}h_r$ is equationally definable in terms of its N -logic. Hence, afortiori, the $+$ of the *subring* $F_{p_1}k_1 \oplus \dots \oplus F_{p_t}k_t$ (and therefore the $+$ of R , by the above isomorphism) is equationally definable in terms of the N -logic of R . Therefore, R is a ring-logic (mod N), and the theorem is proved.

3. p -rings and p^k -rings. We shall now make an attempt to generalize Theorem 3, and apply this generalization to p -rings and p^k -rings. We first observe that the proof of Theorem 3 does *not* depend on the cardinality of the powers m_i . Furthermore, the proof still remains valid if one considers a *subdirect* sum of *subdirect* powers of A_i instead of the *complete* direct sum of direct powers of A_i ($i = 1, \dots, t$). In view of this, Theorem 3 can now be cast in the following more general form.

THEOREM 3'. *Let $(A_1, \times, +), \dots, (A_t, \times, +)$ be a finite set of ring-logics (mod N), and let the N -logics $(A_1, \times, \hat{}, \check{}), \dots, (A_t, \times, \hat{}, \check{})$ be independent. Let A be any subdirect sum with identity of (not necessarily finite) subdirect powers of A_i ($i = 1, \dots, t$). Then A is a ring-logic (mod N).*

Now, it is well known (see [2; 4]) that every p -ring (p prime) is isomorphic to a subdirect power of F_p , and every p^k -ring (p prime) is isomorphic to a subdirect power of F_{p^k} . Hence, by letting $t = 1$ and $A_1 = F_p$ (respectively, F_{p^k}) in Theorem 3', we obtain the following corollary (compare with [1; 2]).

COROLLARY. *Any p -ring with identity, as well as any p^k -ring with identity, is a ring-logic (mod N).*

In conclusion, I wish to express my gratitude to the referee for his valuable suggestions.

REFERENCES

1. A. L. Foster, *p-rings and ring-logics*, University California Publ., **1** (1951), 385-396.
2. ———, *p^k -rings and ring-logics*, Ann. Scu. Norm. Pisa, **5** (1951), 279-300.
3. ———, *Unique subdirect factorization within certain classes of universal algebras*, Math. Z., **62** (1955), 171-188.
4. N. H. McCoy and D. Montgomery, *A representation of generalized Boolean rings*, Duke Math. J., **3** (1937), 455-459.
5. N. H. McCoy, *Rings and Ideals*, Carus Math Monog., **8** (1947).
6. M. H. Stone, *The theory of representations of Boolean algebras*, Trans. Amer. Math. Soc., **40** (1936), 37-111.
7. A. Yaqub, *On the theory of ring-logics*, Can. J. Math., **8** (1956), 323-328.

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