

# Pacific Journal of Mathematics

## **FURTHER RESULTS ON $p$ -AUTOMORPHIC $p$ -GROUPS**

JAMES ROBERT BOEN, OSCAR S. ROTH AUS AND JOHN GRIGGS THOMPSON

# FURTHER RESULTS ON $p$ -AUTOMORPHIC $p$ -GROUPS

J. BOËN, O. ROTH AUS, AND J. THOMPSON

Graham Higman [3] has shown that a finite  $p$ -group,  $p$  an odd prime, with an automorphism permuting the subgroups of order  $p$  cyclically is abelian. In [1] a  $p$ -group was defined to be  $p$ -automorphic if its automorphism group is transitive on the elements of order  $p$ . It was conjectured that a  $p$ -automorphic  $p$ -group ( $p \neq 2$ ) is abelian and proved that a counterexample must be generated by at least four elements. In this present paper we prove that a counterexample generated by  $n$  elements must be such that  $n > 5$  and, if  $n \neq 6$ , then  $p < n3^{n^3}$  (Theorem 3). We also show that the existence of a counterexample implies the existence of a certain algebraic configuration (Theorem 1). All groups considered are finite.

Notation.  $\Phi(P)$  is the Frattini subgroup of the  $p$ -group  $P$  and  $P'$  is its commutator subgroup.  $\Omega_i(P)$  is the subgroup generated by the elements of  $P$  whose orders do not exceed  $p^i$ .  $Z(P)$  is the center of  $P$ .  $F(m, n, p)$  denotes the set of  $p$ -automorphic  $p$ -groups  $P$  which enjoy the additional properties:

1.  $P' = \Omega_1(P)$  is elementary abelian of order  $p^n$ .
2.  $\Phi(P) = Z(P) = \Omega_m(P)$  is the direct product of  $n$  cyclic groups of order  $p^m$ .
3.  $|P: \Phi(P)| = p^n$ .

In [1] it was shown that a counterexample generated by  $n$  elements has a quotient group in  $F(m, n, p)$ . Hence, in arguing by contradiction, we may assume that a counterexample  $P$  is in  $F(m, n, p)$ .

Let  $A = A(P) = \text{Aut } P$  and let  $A_0 = \ker(\text{Aut } P \rightarrow \text{Aut } P/\Phi(P))$ . Thus  $A/A_0 = B$  is faithfully represented as linear transformations of  $V = P/\Phi(P)$ , considered as a vector space over  $GF(p)$ .

Since  $p$  is odd and  $cl(P) = 2$ , the mapping  $\eta: x \rightarrow x^{p^m}$  is an endomorphism of  $P$  which commutes with each  $\sigma$  of  $\text{Aut } P$ . Since  $\Omega_m(P) = \Phi(P)$ ,  $\ker \eta = \Phi(P)$ , so  $\eta$  induces an isomorphism of  $V$  into  $W = P'$ . Since  $\dim V = \dim W$ ,  $\eta$  is onto.

The commutator function induces a skew-symmetric bilinear map of  $V \times V$  onto  $W$ , (onto since  $P$  is  $p$ -automorphic) and since  $\Phi(P) = Z(P)$ ,  $(, )$  is nondegenerate. Associated with  $(, )$  is a nonassociative product  $\circ$ , defined as follows: If  $\alpha, \beta \in V$ , say  $\alpha = x\Phi(P)$ ,  $\beta = y\Phi(P)$ , then  $[x, y]$  is an element of  $W$  which depends only on  $\alpha, \beta$ , and so  $[x, y] = z^{p^m}$  where the coset  $\gamma = z\Phi(P)$  depends only on  $\alpha, \beta$ . We write  $\alpha \circ \beta = \gamma$ . An immediate consequence of this condition is the statement that  $\alpha \rightarrow \alpha \circ \beta$

---

Received December 6, 1961. Boen's work on this paper was partly supported by N.S.F Grant G-9504.

is a linear map  $\phi_\beta$  of  $V$  into  $V$ . Thus,  $\circ$  induces a map  $\theta$  of  $V$  into  $\text{End } V$ , the ring of linear transformations of  $V$  to  $V$ .

If  $\bar{\sigma}$  is the inner automorphism of  $\text{End } V$  induced by  $\sigma \in B$ , then the diagram

$$\begin{array}{ccc} V & \xrightarrow{\theta} & \text{End } V \\ \sigma \downarrow & & \downarrow \bar{\sigma} \\ V & \xrightarrow{\theta} & \text{End } V \end{array}$$

commutes, that is  $\phi_{\beta\sigma} = \sigma^{-1}\phi_\beta\sigma$ . Since  $P$  is  $p$ -automorphic, if  $\alpha, \beta$  are nonzero elements of  $V$ , then  $\alpha = \beta^\sigma$  for suitable  $\sigma \in B$ , so that  $\phi_\alpha = \sigma^{-1}\phi_\beta\sigma$ .

**THEOREM 1.** *If  $\alpha \in V$ , then  $\phi_\alpha$  is nilpotent.*

*Proof.* We can suppose  $\alpha \neq 0$ . Since  $\alpha \circ \alpha = 0$ ,  $\phi_\alpha$  is singular. Let  $f(x) = x^n + c_1x^{n-1} + c_2x^{n-2} + \dots$  be the characteristic equation of  $\phi_\alpha$ .  $f(x)$  is independent of the nonzero element  $\alpha$  of  $V$ , and  $c_n = 0$  since  $\phi_\alpha$  is singular.

Let  $\alpha_1, \dots, \alpha_n$  be a basis for  $V$ , and identify  $\phi_\alpha$  with the matrix which is associated with  $\phi_\alpha$  and the basis  $\alpha_1, \dots, \alpha_n$ . Then  $c_i$  is the sum of all  $i$  by  $i$  principal minors of  $\phi_\alpha$ , so if  $\alpha = \lambda_1\alpha_1 + \dots + \lambda_n\alpha_n$ ,  $c_i$  is a homogeneous polynomial of degree  $i$  ( $\leq n - 1$ ) in the  $n$  variables  $\lambda_1, \dots, \lambda_n$ . By a Theorem of Chevalley [2], there are values  $\lambda_1, \dots, \lambda_n$  of  $GF(p)$  which are not all zero, such that  $c_i = 0$ . Since  $c_i$  is independent of the non-zero tuple  $(\lambda_1, \dots, \lambda_n)$ , it follows that  $c_i = 0$  so  $\phi_\alpha$  is nilpotent.

Theorem 1 states that  $\theta(V)$  is a linear variety of  $\text{End}(V)$  consisting only of nilpotent matrices such that any two nonzero  $x, y \in \theta(V)$  are similar. If one could show that the algebra generated by  $\theta(V)$  were nilpotent, an easy argument would show that all  $p$ -automorphic  $p$ -groups ( $p$  odd) are abelian.

**THEOREM 2.** *Let  $r$  be the rank of  $\phi_\alpha$ . If  $n > 3$ , then  $2 < r < n - 1$ .*

*Proof.* We assume  $n > 3$  because  $n \leq 3$  was treated in [1]. Clearly  $r \neq 0$  because  $P$  is non-abelian and  $r \neq n$  by Theorem 1.

*Case I.*  $r \neq n - 1$ . Suppose  $r = n - 1$ . Then, for  $\alpha \neq 0$ ,  $\beta \circ \alpha = \beta\phi_\alpha = 0$  implies that  $\beta \in \{\alpha\}$  where  $\{\alpha\}$  is the subspace of  $V$  spanned by  $\alpha$ . If  $\gamma\phi_\alpha^2 = (\gamma\phi_\alpha)\phi_\alpha = 0$ , then  $\gamma\phi_\alpha \in \{\alpha\}$ , say  $\gamma\phi_\alpha = k\alpha$ . But  $\gamma\phi_\alpha + \alpha\phi_\gamma = 0$  by the skew-symmetry of  $\circ$ , so  $\alpha\phi_\gamma = -k\alpha$ . By Theorem 1,  $k = 0$  and thus  $\gamma \in \{\alpha\}$ . Hence  $\text{rank } \phi_\alpha^2 = \text{rank } \phi_\alpha$ , a contradiction to Theorem 1.

*Case II.*  $r \neq 1$ . Choose a basis of  $V$ , say  $\alpha_1, \dots, \alpha_n$ , and suppose

that  $\phi_\alpha = (a_{ij})$  with respect to this basis;  $\text{End}(V)$  has the obvious matrix representation with  $\phi_\alpha \in \theta(V) \subset \text{End}(V)$ . Recall that  $\theta(V)$  becomes an  $n$ -space of  $n$  by  $n$  nilpotent matrices over  $GF(p)$  in which any two nonzero matrices are similar. If  $r = 1$ , then we may assume without loss of generality that  $\phi_\alpha$  has a 1 in the (1, 2) position and zeros elsewhere.

If every  $(x_{ij}) = X \in \theta(V)$  satisfies  $x_{ij} = 0$  for  $i > 1$ , then we are done because the nilpotency of  $X$  implies that  $x_{11} = 0$  for every  $X \in \theta(V)$ , which implies that  $\dim \theta(V) < n$ . If, on the other hand, there exists  $X \in \theta(V)$  with a nonzero entry below the first row, then we may use the fact that every 2 by 2 subdeterminant of every element of  $\theta(V)$  vanishes to show that every  $X$  has its nonzero elements in the second column only. But the nilpotency of  $X$  implies that  $x_{22} = 0$ . Hence  $\dim \theta(V) < n$ , a contradiction.

*Case III.  $r \neq 2$ .* If  $r = 2$ , we may assume without loss of generality that

- (a)  $\phi_\alpha$  has 1's in the (1, 2), (2, 3) positions and zeros elsewhere or else
- (b)  $\phi_\alpha$  has 1's in the (1, 2), (3, 4) positions and zeros elsewhere.

First consider (a).

If every  $(x_{ij}) = X \in \theta(V)$  satisfies  $x_{ij} = 0$  for  $i > 2$ , then  $Z(P) \cong \mathcal{O}(P)$ , a contradiction. If every  $X \in \theta(V)$  satisfies  $x_{ij} = 0$  for  $j \neq 2, 3$ , then  $x_{32} = 0$  because  $X + k\phi_\alpha$  is nilpotent for every  $k \in GF(p)$  and  $p > 2$ . But then  $\dim \theta(V) < n$ , a contradiction. Hence we need consider only the subcase of (a) in which some  $X \in \theta(V)$  has a nonzero entry below the third row and a nonzero entry that is not in columns two or three. Consider such an  $X$ . Unless  $x_{ij} = 0$  when  $i \neq 1, 2$  and  $j \neq 2, 3$ , it is easy to see that there exists a nonzero 3 by 3 determinant in  $X + k\phi_\alpha$  for some  $k$ . It is also easy to see that any two rows of  $X$  below the second row are dependent, and that any two columns other than the second and third are dependent. Using the fact that every 3 by 3 subdeterminant of every element of  $\theta(V)$  is zero, it is straightforward to show that there exist nonsingular matrices  $R$  and  $S$  such that  $RXS$  has 1's in the (1, 4), (3, 2) positions and zeroes elsewhere and  $R\phi_\alpha S$  has 1's in the (1, 3), (2, 2) positions and zeroes elsewhere.

Set  $X' = RXS$ ,  $\phi'_\alpha = R\phi_\alpha S$ . It is now straightforward to show that that if  $Y = (y_{ij}) \in R\theta(V)S$  is linearly independent from  $\{X', \phi'_\alpha\}$ , then  $y_{ij} = 0$  for  $i \neq 1$  and  $j \neq 2$ . This implies that  $\dim R\theta(V)S < n$ , a contradiction, since  $\dim R\theta(V)S = \dim \theta(V) = n$ .

Subcase (b), in which  $\phi_\alpha^2 = 0$ , is handled in a similar fashion except that we exclude the case in which every  $X \in \theta(V)$  satisfies  $x_{ij} = 0$ ,  $j \neq 2, 4$ , by noting the following: In such a case  $(X + k\phi_\alpha)^2 = 0$  for every  $k$  implies that  $x_{22} = 0$ , which in turn implies that  $\dim \theta(V) < n$ .

COROLLARY.  $F(m, n, p)$  is empty for all  $m$  and odd  $p$  unless  $n > 5$ .

*Proof.* Theorem 2 implies that  $n > 4$  and that if  $n = 5$ , then  $\text{rank } \phi_\alpha = 3$ . Let  $S_n$  denote the projective  $(n - 1)$ -space whose points are the 1-subspaces of  $V$ . If  $n = 5$  and  $\text{rank } \phi_\alpha = 3$ , then it follows that  $S_5$  is partitioned into lines according to the rule that  $\{\alpha\}, \{\beta\}$  ( $0 \neq \alpha, \beta \in V$ ) lie on the same line if and only if  $\alpha \circ \beta = 0$ . But  $S_5$  has  $p^4 + p^3 + p^2 + p + 1$  points and cannot be partitioned into disjoint subsets of  $p + 1$  points each.

THEOREM 3. If  $p \geq 3n^2$  and  $n \neq 6$ , then  $F(m, n, p)$  is empty for all positive integers  $m$ .

*Proof.* If  $GL(n, p)$  denotes the invertible elements of  $\text{End } V$ , then

$$|GL(n, p)| = p^{n(n-1)/2} \cdot k(n, p), \text{ where } k(n, p) = (p^n - 1)(p^{n-1} - 1) \cdots (p - 1).$$

If we consider  $GF(p^n)$  as a vector space over  $GF(p)$ , the right-regular representation shows that  $GL(n, p)$  contains a cyclic group of order  $p^n - 1$ .

Let  $\Phi_d(x)$  be the monic polynomial whose complex roots are the primitive  $d$ th roots of unity. Then  $p^n - 1 = \prod_{d|n} \Phi_d(p)$ . By an elementary number-theoretic theorem [4],  $\Phi_n(p)$  and  $k(n, p)/\Phi_n(p)$  are relatively prime, or their greatest common divisor is  $q$  where  $q$  is the largest prime divisor of  $n$ , in which case  $\Phi_n(p)/q$  is relatively prime to  $k(n, p)/\Phi_n(p)$ . Thus, we determine  $\varepsilon = 0$  or  $1$  so that  $\Phi_n(p)/q^\varepsilon$  is relatively prime to  $k(p, n)/\Phi_n(p)$ .

Let  $p \in F(m, n, p)$ . Since  $P$  is  $p$ -automorphic,  $|B|$  is divisible by  $p^n - 1$  and in particular is divisible by  $\Phi_n(p)/q^\varepsilon$ . Let  $r^\alpha$  be the largest power of the prime  $r$  which divides  $\Phi_n(p)/q^\varepsilon$ ,  $\alpha \geq 1$ , and let  $S_r$  be a Sylow  $r$ -subgroup of  $B$ . By Sylow's theorem and the preceding paragraph,  $S_r$  is cyclic with generator  $\sigma_r$ .

Since  $P$  belongs to the exponent  $n$  modulo  $r$ , it follows that  $\lambda, \lambda^p, \dots, \lambda^{p^{n-1}}$  are the characteristic roots of  $\sigma_r$ ,  $\lambda$  being a primitive  $r^\alpha$ th root of unity in  $GF(p^n)$ .

Since  $\eta$  commutes with  $\sigma_r$ ,  $\lambda$  is also a characteristic root of  $\sigma_r$  on  $W$ . Since  $(\alpha, \beta)^\sigma = (\alpha^\sigma, \beta^\sigma)$ , the characteristic roots of  $\sigma_r$  on  $W$  are to be found among the  $\lambda^{p^i + p^j}$ ,  $0 \leq i < j \leq n - 1$ , as can be seen by diagonalizing  $\sigma_r$  over  $V \otimes GF(p^n)$ . Hence,  $\lambda = \lambda^{p^i + p^j}$  for suitable  $i, j$  and so

$$(1) \quad p^i + p^j - 1 \equiv 0 \pmod{r^\alpha}.$$

Since  $r$  was any prime divisor of  $\Phi_n(p)/q^\varepsilon$ , we have

$$(2) \quad \prod_{0 \leq i < j \leq n-1} (p^i + p^j - 1) \equiv 0 \pmod{\Phi_n(p)/q^\varepsilon}.$$

The polynomials  $\Phi_n(x)$ ,  $n \neq 6$ , and  $x^i + x^j - 1$  are relatively prime, a fact

which can be seen geometrically, as pointed out by G. Higman. Namely, if  $\varepsilon, \varepsilon'$  are complex numbers of absolute value one, and  $\varepsilon + \varepsilon' = 1$ , then the points  $0, 1, \varepsilon$  are the vertices of an equilateral triangle, so that  $\varepsilon$  is a primitive sixth root of unity. Since  $n \neq 6$ , we can therefore find integral polynomials  $f(x), g(x)$ , such that

$$(3) \quad f(x)\Phi_n(x) + g(x) \prod_{0 \leq i < j \leq n-1} (x^i + x^j - 1) = |N|,$$

where

$$(4) \quad \begin{aligned} N &= \prod_{\zeta} \prod_{i,j} (\zeta^i + \zeta^j - 1) \\ \Phi_n(\zeta) &= 0 \end{aligned}$$

is the resultant of  $\Phi_n(x)$  and  $\prod (x^i + x^j - 1)$ .

From (4) we see that  $N \leq 3^{\phi(n)n^2}$ , since there are at most  $\phi(n)n^2$  triples  $(\zeta, i, j)$ . Now (2) and (3), the fact that  $\Phi_n(p)/q^e$  divides  $|N|$ , imply that

$$(5) \quad \Phi_n(p)/q^e \leq 3^{\phi(n)n^2}.$$

One sees geometrically that  $\Phi_n(p) \geq (p-1)^{\phi(n)}$ , so with (5) and  $q^e \leq n$  we find

$$(6) \quad p \leq 1 + n^{1/\phi(n)} 3^{n^2} < n 3^{n^2}.$$

REMARK. Theorem 3 of [3] provides a certain motivation for the detailed examination of  $\Phi_n(p)$  in the preceding theorem.

### BIBLIOGRAPHY

1. J. Boen, *On  $p$ -Automorphic  $p$ -Groups*, (to appear in Pacific Journal of Mathematics).
2. C. Chevalley, *Demonstration d'une hypothese de M. Artin*, Abh. Math. Seminar U. Hamburg, **11** (1936).
3. G. Higman, *Suzuki 2-groups*, (to appear).
4. T. Nagell, *Introduction to Elementary Number Theory*, Wiley (1951).

UNIVERSITY OF CHICAGO AND UNIVERSITY OF MICHIGAN  
 INSTITUTE FOR DEFENSE ANALYSES  
 UNIVERSITY OF CHICAGO



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RALPH S. PHILLIPS

Stanford University  
Stanford, California

M. G. ARSOVE

University of Washington  
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California  
Los Angeles 7, California

LOWELL J. PAIGE

University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



Alfred Aeppli, <i>Some exact sequences in cohomology theory for Kähler manifolds</i> .....	791
Paul Richard Beesack, <i>On the Green's function of an <math>N</math>-point boundary value problem</i> .....	801
James Robert Boen, <i>On <math>p</math>-automorphic <math>p</math>-groups</i> .....	813
James Robert Boen, Oscar S. Rothaus and John Griggs Thompson, <i>Further results on <math>p</math>-automorphic <math>p</math>-groups</i> .....	817
James Henry Bramble and Lawrence Edward Payne, <i>Bounds in the Neumann problem for second order uniformly elliptic operators</i> .....	823
Chen Chung Chang and H. Jerome (Howard) Keisler, <i>Applications of ultraproducts of pairs of cardinals to the theory of models</i> .....	835
Stephen Urban Chase, <i>On direct sums and products of modules</i> .....	847
Paul Civin, <i>Annihilators in the second conjugate algebra of a group algebra</i> .....	855
J. H. Curtiss, <i>Polynomial interpolation in points equidistributed on the unit circle</i> .....	863
Marion K. Fort, Jr., <i>Homogeneity of infinite products of manifolds with boundary</i> .....	879
James G. Glimm, <i>Families of induced representations</i> .....	885
Daniel E. Gorenstein, Reuben Sandler and William H. Mills, <i>On almost-commuting permutations</i> .....	913
Vincent C. Harris and M. V. Subba Rao, <i>Congruence properties of <math>\sigma_r(N)</math></i> .....	925
Harry Hochstadt, <i>Fourier series with linearly dependent coefficients</i> .....	929
Kenneth Myron Hoffman and John Wermer, <i>A characterization of <math>C(X)</math></i> .....	941
Robert Weldon Hunt, <i>The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order</i> .....	945
Edward Takashi Kobayashi, <i>A remark on the Nijenhuis tensor</i> .....	963
David London, <i>On the zeros of the solutions of <math>w''(z) + p(z)w(z) = 0</math></i> .....	979
Gerald R. Mac Lane and Frank Beall Ryan, <i>On the radial limits of Blaschke products</i> .....	993
T. M. MacRobert, <i>Evaluation of an <math>E</math>-function when three of its upper parameters differ by integral values</i> .....	999
Robert W. McKelvey, <i>The spectra of minimal self-adjoint extensions of a symmetric operator</i> .....	1003
Adegoke Olubummo, <i>Operators of finite rank in a reflexive Banach space</i> .....	1023
David Alexander Pope, <i>On the approximation of function spaces in the calculus of variations</i> .....	1029
Bernard W. Roos and Ward C. Sangren, <i>Three spectral theorems for a pair of singular first-order differential equations</i> .....	1047
Arthur Argyle Sagle, <i>Simple Malcev algebras over fields of characteristic zero</i> .....	1057
Leo Sario, <i>Meromorphic functions and conformal metrics on Riemann surfaces</i> .....	1079
Richard Gordon Swan, <i>Factorization of polynomials over finite fields</i> .....	1099
S. C. Tang, <i>Some theorems on the ratio of empirical distribution to the theoretical distribution</i> .....	1107
Robert Charles Thompson, <i>Normal matrices and the normal basis in abelian number fields</i> .....	1115
Howard Gregory Tucker, <i>Absolute continuity of infinitely divisible distributions</i> .....	1125
Elliot Carl Weinberg, <i>Completely distributed lattice-ordered groups</i> .....	1131
James Howard Wells, <i>A note on the primes in a Banach algebra of measures</i> .....	1139
Horace C. Wiser, <i>Decomposition and homogeneity of continua on a 2-manifold</i> .....	1145