

# Pacific Journal of Mathematics

**BOUNDS IN THE NEUMANN PROBLEM FOR SECOND ORDER  
UNIFORMLY ELLIPTIC OPERATORS**

JAMES HENRY BRAMBLE AND LAWRENCE EDWARD PAYNE

# BOUNDS IN THE NEUMANN PROBLEM FOR SECOND ORDER UNIFORMLY ELLIPTIC OPERATORS

J. H. BRAMBLE AND L. E. PAYNE

**1. Introduction.** In this paper we derive certain a priori inequalities which are useful for obtaining bounds in the interior Neumann problem for second order elliptic partial differential equations. In establishing these inequalities by our methods it is necessary to obtain lower bounds for the inverse of the Poincaré constant ( $\mu_2$  of eq. 3.3) and the first nonzero Steklov eigenvalue ( $p_2$  of eq. 3.11). An optimal Poincaré inequality for convex domains in  $n$ -dimensions was given by Payne and Weinberger [5], and a method for obtaining lower bounds for  $p_2$  for  $n$ -dimensional star-shaped regions was indicated by Payne and Weinberger [3]. However, to the authors' knowledge, no explicit lower bounds for  $p_2$  and  $\mu_2$  for general  $n$ -dimensional regions have previously been given. Lower bounds for these constants which lead to the above mentioned inequalities in the Neumann problem are of interest in themselves and should prove useful in other applications.

For the special case of the Laplace equation other methods for deriving bounds for the Dirichlet integral in the Neumann problem appear in the literature (see [2], [6]). For starshaped regions a method similar to that proposed here was obtained in [3]. Bounds in exterior Neumann problems were given in [4].

**2. Preliminary inequalities.** Let  $R$  be a simply connected bounded region with boundary  $C$  in Euclidean  $n$ -space. In  $R$  we assume that the operator  $L$  given by

$$(2.1) \quad Lu \equiv (a^{ij}u_{,i})_{,j}$$

is a uniformly elliptic operator, defined for sufficiently smooth functions  $u$ . In (2.1)  $,i$  denotes partial differentiation with respect to the coordinate  $x_i$  and the summation convention is assumed. The coefficient matrix  $a^{ij}$  is symmetric and the condition of uniform ellipticity may be stated as follows: There exist positive constants  $a_0$  and  $a_1$  such that for every real vector  $(\xi_1, \dots, \xi_n)$ , the relation

$$(2.2) \quad a_0 \sum_{i=1}^n \xi_i \xi_i \leq a^{ij} \xi_i \xi_j \leq a_1 \sum_{i=1}^n \xi_i \xi_i$$

is valid uniformly in  $R$ .

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We consider now an arbitrary point of  $R$  which we choose to be the origin. Let  $S_a$  be the interior of a sphere of radius  $a$ , with center at the origin and such that  $S_a \subset R$ . The surface of the sphere will be called  $\Sigma_a$ . We denote by  $R_a$  the region  $R - \bar{S}_a$ , where  $\bar{S}_a$  is the closure of  $S_a$ .

Let  $u$  be any sufficiently smooth function in  $R + C$  and let  $f^i$  be a sufficiently smooth vector field defined in  $\bar{R}_a$ . Then, by the divergence theorem, we have

$$(2.3) \quad \oint_{\sigma} f^i n_i u^2 ds = - \oint_{\Sigma_a} f^i n_i u^2 ds + \int_{R_a} f^i_{,i} u^2 dv + 2 \int_{R_a} f^i u u_{,i} dv ,$$

where  $n_i$  is the component of the unit normal directed outward on  $C$ . An application of the arithmetic-geometric mean inequality applied to the last term on the right of (2.3) yields

$$(2.4) \quad \oint_{\sigma} f^i n_i u^2 ds \leq - \oint_{\Sigma_a} f^i n_i u^2 ds + \int_{R_a} \left( f^i_{,i} + \frac{1}{\alpha} f^i f^i \right) u^2 dv + \int_{R_a} \alpha u_{,i} u_{,i} dv$$

where  $\alpha$  is some positive function in  $R_a$ .

We assume now that  $f^i$  and  $\alpha$  have been chosen so that

$$(2.5) \quad \begin{aligned} t \equiv f^i n_i &\geq K_1 > 0 \text{ on } C \\ - f^i n_i &\leq K_2 \text{ on } \Sigma_a \\ f^i_{,i} + \frac{1}{\alpha} f^i f^i &\leq 0 \text{ in } R_a , \end{aligned}$$

where  $K_1$  and  $K_2$  are constants. (We shall in a subsequent section construct vectors  $f^i$  satisfying (2.5) for certain domains.) Using conditions (2.5) with (2.4) we have that

$$(2.6) \quad \oint_{\sigma} t u^2 ds \leq K_2 \oint_{\Sigma_a} u^2 ds + \bar{\alpha} \int_{R_a} u_{,i} u_{,i} dv ,$$

where  $\bar{\alpha}$  is an upper bound for  $\alpha$  in  $R_a$ . Suppose that  $u$  is normalized such that

$$(2.7) \quad \oint_{\Sigma_a} u ds = 0 .$$

Then

$$(2.8) \quad \oint_{\Sigma_a} u^2 ds \leq p_2 \int_{S_a} u_{,i} u_{,i} dv$$

where  $p_2$  is the first nonzero eigenvalue in the Steklov problem for the sphere  $S_a$ . That is

$$(2.9) \quad p_2 = \min \frac{\int_{S_a} \chi_i \chi_i dv}{\oint_{S_a} \chi^2 ds}$$

where the minimum is taken over all sufficiently smooth functions in  $S_a$  which satisfy (2.7). For the sphere of radius  $a$ ,  $p_2$  is explicitly given by

$$(2.10) \quad p_2 = 1/a .$$

Combining (2.6) and (2.8) it follows that

$$(2.11) \quad \oint_{\sigma} t u^2 ds \leq K_3 D(u, u) \equiv K_3 \int_R u_i u_i dv ,$$

where  $K_3 = \max (aK_2, \bar{\alpha})$ , or using (2.5)

$$(2.12) \quad \oint_{\sigma} u^2 ds \leq (K_3/K_1) D(u, u) .$$

Now from the divergence theorem

$$(2.13) \quad \oint_{\sigma} x^i n_i u^2 ds = n \int_R u^2 dv + 2 \int_R x^i u u_i dv .$$

Using the arithmetic-geometric mean inequality it follows easily that

$$(2.14) \quad \int_R u^2 dv \leq \frac{2r_M}{n} \oint_{\sigma} u^2 ds + \frac{4r_M^2}{n^2} D(u, u) ,$$

where  $r_M$  is the maximum distance from the origin to  $C$ . Inequality (2.12) with (2.14) yields

$$(2.15) \quad \int_R u^2 dv \leq K_4 D(u, u) ,$$

where

$$K_4 = \frac{2r_M}{n} \left[ K_3/K_1 + \frac{2r_M}{n} \right] .$$

The preceding inequalities depended entirely on the existence of a vector field  $f^i$  satisfying (2.5). In certain cases, as will be shown in a subsequent section, such a vector field can be explicitly constructed so as to yield explicit, easily computable constants  $K_1$ ,  $K_2$  and  $K_3$ .

In some cases it may be that for the region  $R$  the vector field  $f^i$  is quite difficult to construct. We can make use of an additional inequality to reduce the problem to that of obtaining an inequality of the form (2.12) for a subregion of  $R$ .

Let us divide the region  $R$  into two disjoint subregions  $R_1$  and  $R_2$ .

These regions are to be separated by a surface  $C'$ . The portion  $C$  which is part of the boundary of  $R_i$  will be denoted by  $C_i$ ,  $i = 1, 2$ . Thus the boundary of  $R_i$  will be  $C_i + C'$ . We further assume that the subdivision has been made in such a way that  $C_1$  is star shaped with respect to some point  $P$  not in  $R_1 + C'$ . We choose  $P$  to be the origin and apply the divergence theorem in  $R_1$  to obtain

$$(2.16) \quad \int_{C_1+C'} x^i r^{-(n+1)} n_i u^2 ds = - \int_{R_1} r^{-(n+1)} u^2 dv + 2 \int_{R_1} x^i r^{-(n+1)} u u_{,i} dv$$

for any function  $u$  sufficiently smooth in  $R$ . Defining  $\hat{i} = x_i n_i / r$  and using the arithmetic-geometric mean inequality we obtain

$$(2.17) \quad \oint_{C_1} u^2 ds \leq \frac{1}{\hat{t}_m} \left( \frac{r_M}{r_m} \right)^n \oint_{C'} u^2 ds + \frac{r_m}{\hat{t}_m} \left( \frac{r_M}{r_m} \right)^n D_1(u, u).$$

In (2.17)  $r_M$  and  $r_m$  denote upper and lower bounds for  $r$  in  $\bar{R}_1$  and  $\hat{t}_m$  a lower bound for  $\hat{t}$  on  $C_1$ .  $D_1(u, u)$  denotes the Dirichlet integral over  $R_1$ .

Now suppose that for  $R_2$  we could find a vector field  $f^i$  satisfying (2.5) relative to  $R_2$  and obtain the inequality

$$(2.18) \quad \oint_{C_2+C'} u^2 ds \leq K_2/K_1 D_2(u, u).$$

Then clearly (2.17), together with (2.18) would yield

$$(2.19) \quad \oint_C u^2 ds \leq K_5 D(u, u)$$

where of course  $u$  is assumed normalized with respect to  $S_a$  in  $R_2$ , and  $K_5$  is a constant.

It is now obvious that such a procedure could be repeated a finite number of times, finally reducing the region to one for which the inequality (2.12) may be more easily obtained. In particular if we iterate this procedure until the  $q$ th region  $R_q$  is star shaped, then, as we shall see in § 4, a vector field  $f^i$  for  $R_q$  is easily constructed.

**3. Lower bounds for eigenvalues.** The first nonzero eigenvalue  $\mu_2$  in the free membrane problem for  $R$  satisfies

$$(3.1) \quad \Delta \tilde{v} + \mu_2 \tilde{v} = 0 \text{ in } R$$

and

$$(3.2) \quad \frac{\partial \tilde{v}}{\partial n} = 0 \text{ on } C,$$

where  $\tilde{v}$  is the corresponding eigenfunction. It is well known that  $\mu_2$

may be characterized by the minimum principle

$$(3.3) \quad \mu_2 = \min \frac{D(\varphi, \varphi)}{\int_R \varphi^2 dv}$$

for sufficiently smooth functions  $\varphi$  satisfying

$$(3.4) \quad \int_R \varphi dv = 0 ,$$

and that  $\tilde{v}$  is the minimizing function. That is

$$(3.5) \quad \mu_2 = \frac{D(\tilde{v}, \tilde{v})}{\int_R \tilde{v}^2 dv} .$$

Now let  $u = \tilde{v} + c_1$  where

$$(3.6) \quad c_1 = -\frac{1}{\omega_n a^{n-1}} \int_{S_a} v ds$$

$\omega_n$  denoting the surface area of the  $n$ -dimensional unit sphere. Then  $u$  satisfies

$$(3.7) \quad \int_{S_a} u ds = 0 ,$$

and hence by (2.15)

$$(3.8) \quad \int_R u^2 dv \leq K_4 D(u, u) = K_4 D(\tilde{v}, \tilde{v}) .$$

But

$$(3.9) \quad \int_R u^2 dv = \int_R \tilde{v}^2 dv + c_1^2 \int_R dv \geq \int_R \tilde{v}^2 dv .$$

Thus

$$(3.10) \quad \frac{1}{K_4} \leq \frac{D(\tilde{v}, \tilde{v})}{\int_R \tilde{v}^2 dv} = \mu_2$$

or  $1/K_4$  is a lower bound for  $\mu_2$ .

A lower bound is also easily obtained for  $p_2$ , the first nonzero eigenvalue in the Steklov problem for  $R$ . Let  $w$  be the corresponding eigenfunction. Then we have that

$$(3.11) \quad p_2 = \frac{D(w, w)}{\oint_{\sigma} w^2 ds}$$

and

$$(3.12) \quad \oint_{\sigma} w ds = 0 .$$

Now let  $u = w + c_2$  where

$$(3.13) \quad c_2 = -\frac{1}{\omega_n a^{n-1}} \int_{\Sigma_a} w ds .$$

Then

$$(3.14) \quad \oint_{\Sigma_a} u ds = 0$$

and we may apply (2.12) to  $u$ . But by (3.14) we have that

$$(3.15) \quad \oint_{\sigma} u^2 ds = \oint_{\sigma} w^2 ds + c_2^2 \oint_{\sigma} ds \geq \oint_{\sigma} w^2 ds .$$

Thus from (2.12) and (3.15)

$$K_1/K_3 \leq \frac{D(w, w)}{\oint_{\sigma} w^2 ds} = p_2 ,$$

which gives the desired lower bound for  $p_2$ .

**4. Bounds in the Neumann problem for  $L$ .** We assume now that  $\psi$  is any sufficiently smooth function in  $R + C$ . We shall obtain bounds for the generalized Dirichlet integral,  $A(\psi, \psi)$ , given by

$$(4.1) \quad A(\psi, \psi) = \int_R a^{ij} \psi_{,i} \psi_{,j} dv$$

in terms of  $L\psi$  in  $R$  and

$$(4.2) \quad \frac{\partial \psi}{\partial \nu} \equiv a^{ij} n_i \psi_{,j} \text{ on } C .$$

We take  $u = \psi + c_3$  where

$$(4.3) \quad c_3 = -\frac{1}{\omega_n a^{n-1}} \int_{\Sigma_a} \psi ds .$$

As before

$$(4.4) \quad \oint_{\Sigma_a} u ds = 0 ,$$

Now by Green's identity

$$(4.5) \quad A(u, u) = \oint_{\sigma} u \frac{\partial \psi}{\partial \nu} ds - \int_R u L\psi dv .$$

We have used the fact that  $u$  and  $\psi$  differ only by a constant. By Schwarz's inequality we have that

$$(4.6) \quad A(u, u) \leq \left( \oint_{\sigma} tu^2 ds \right)^{1/2} \left( \oint_{\sigma} t^{-1} \left( \frac{\partial \psi}{\partial \nu} \right)^2 ds \right)^{1/2} + \left( \int_R u^2 dv \right)^{1/2} \left( \int_R (L\psi)^2 dv \right)^{1/2} .$$

Because of (4.4) inequalities (2.11) and (2.15) are applicable and we obtain that

$$(4.7) \quad A(\psi, \psi)^{1/2} \leq \left( \frac{K_3}{a_0} \right)^{1/2} \left( \oint_{\sigma} t^{-1} \left( \frac{\partial \psi}{\partial \nu} \right)^2 ds \right)^{1/2} + \left( \frac{K_4}{a_0} \right)^{1/2} \left( \int_R (L\psi)^2 dv \right)^{1/2}$$

since

$$D(u, u) \leq \frac{1}{a_0} A(u, u) = \frac{1}{a_0} A(\psi, \psi) .$$

The inequalities of this section and § 2 together with a mean value inequality given in [1] give immediately interior pointwise bounds for  $\psi + c_3$ .

As an application of the results of this section we note here that (4.7) may be used in conjunction with the Rayleigh-Ritz procedure to yield close bounds for the Dirichlet integral in a specific Neumann problem c.f. [1].

**5. Construction of the vector field.** We shall show in some cases how to construct a vector field satisfying (2.5).

(a) Star shaped regions.

We consider the case where  $C$  is star shaped with respect to some point. We choose this point to be the origin. Then if we take

$$(5.1) \quad f^i = x^i r^{-(n+1)}$$

and

$$(5.2) \quad \alpha = r^{-(n-1)} .$$

We have that

$$(5.3) \quad t = f^i n_i = x^i n_i r^{-(n+1)} \geq h_m r_M^{-(n+1)} \text{ on } C$$

where  $h(P)$  is the distance from the origin to the tangent plane at a point  $P$  on  $C$  and  $h_m$  is the minimum of this function. The condition



of star-shapedness insures that  $h_m > 0$ . Since  $n_i = -x^i/a$  on  $\Sigma_a$  it follows that

$$(5.4) \quad -f^i n_i = a^{-n} \text{ on } \Sigma_a$$

and

$$f^i_{,i} + \frac{1}{\alpha} f^i f^i = 0 \text{ in } R_a.$$

In this case, taking  $\alpha = r_m$ , we obtain

$$(5.5) \quad \mu_2 \geq \frac{n}{2r_m^2 \left[ \left( \frac{r_m}{r_M} \right)^{n-1} \frac{r_M}{h_m} + \frac{2}{n} \right]}$$

and

$$(5.6) \quad p_2 \geq \frac{1}{r_M} \left[ \left( \frac{r_m}{r_M} \right)^{n-1} \frac{h_m}{r_M} \right].$$

A different method for obtaining a lower bound for  $p_2$  for star shaped regions has been indicated by Payne and Weinberger [3]. For convex region Payne and Weinberger [5] also obtained the optimal lower bound  $\mu_2 \geq \pi^2 d^{-2}$  where  $d$  is the diameter of  $R$ .

(b) *Smooth boundaries.*

Let  $R$  be a region whose boundary  $C$  has continuous curvature. Call the largest principal curvature at a point  $P$  of  $C$ ,  $K_M(p)$ . Let  $\rho(p)$  be the radius of a sphere which is tangent to  $C$  at  $P$  and contained in  $R$ . In addition we require  $\rho(p)$  to be less than  $K_M(p)^{-1}$ . Denote by  $\bar{K}$  a bound for the maximum of  $\rho(p)^{-1}$  for  $P \in C$ . We consider the family of parallel surfaces

$$(5.7) \quad N(x) = N(x^1, \dots, x^n) = \text{constant}$$

with  $C$  given by

$$(5.8) \quad N(x) = 0$$

and

$$(5.9) \quad 0 \leq N(x) \leq 1/\bar{K}.$$

The outward normal vector  $n_i$  is defined in this strip and satisfies

$$(5.10) \quad n_{i,i}(x) = J(x)$$

where  $J(x)$  is the average curvature of the surface given by  $N(x)$  at the point  $x = (x^1, \dots, x^n)$  c.f. [7, p. 3]. We assume also that  $\bar{K}$  is chosen so that

$$(5.11) \quad J(x) \leq \bar{K}.$$

The above conditions and definitions involve the smoothness of  $C$  and essentially the thickness of  $R$ . We impose a further condition on the shape of  $R$ .

We assume that there is a point, which we choose to be the origin, such that

$$(5.12) \quad \frac{x^i n_i}{r} \geq -p + \beta > -p > -1$$

for some constants  $p$  and  $\beta > 0$  in the strip  $0 \leq N(x) \leq 1/\bar{K}$ . In this case  $f^i$  may be chosen as

$$(5.13) \quad f^i = \begin{cases} [pn_i(1 - \bar{K}N(x)) + x^i/r]r^{-q}, & 0 \leq N(x) \leq 1/\bar{K} \\ \frac{x^i}{r}r^{-q}, & \text{otherwise} \end{cases}$$

with  $q$  to be determined. Condition (5.12) means that there is an open subset  $\Omega$  of  $R$  which has the property that no ray from the boundary in the direction of the outward normal intersects  $\Omega$ .

Let  $a$  now be chosen so that  $S_a$  does not intersect the boundary strip.

Now on  $C$

$$(5.14) \quad f^i n_i = \left[ p + \frac{x^i n_i}{r} \right] r^{-q} \geq \beta r_M^{-q}.$$

For  $0 \leq N(x) \leq 1/\bar{K}$

$$(5.15) \quad f^i_{,i} = \left\{ p[J(1 - \bar{K}N(x)) + \bar{K}]r + n - 1 - q \right. \\ \left. \left[ p \frac{x_i n_i}{r} (1 - \bar{K}N(x)) + 1 \right] \right\} r^{-(q+1)},$$

since  $n_i(\partial/\partial x_i)N = -1$ . Because of (5.12) and the fact that  $0 \leq 1 - \bar{K}N(x) \leq 1$  we have that

$$(5.16) \quad f^i_{,i} \leq \{2\bar{K}r_M + n - 1 - q(1 - p^2)\}r^{-(q+1)}.$$

Now if we choose

$$(5.17) \quad q = \frac{2\bar{K}r_M + n + 3}{1 - p^2}$$

it follows that

$$(5.18) \quad f^i_{,i} \leq -4r^{-(q+1)} \leq -4r_M^{-(q+1)}$$

in the boundary strip. In the remaining part of  $R_a$  we have, since

$$q \geq n + 3$$

$$(5.19) \quad f^i_{,i} = [n - (q + 1)]r^{-(q+1)} \leq -4r_M^{-(q+1)} .$$

Now choose  $\alpha = r^{1-q}$ . Then

$$(5.20) \quad f^i_{,i} + \frac{1}{\alpha} f^i f^i \leq 0 \text{ in } R_a .$$

On  $\Sigma_a$

$$(5.21) \quad -f^i n_i = \alpha^{-q} .$$

In this case we have

$$(5.22) \quad \mu_2 \geq \frac{n}{2r_M^2 \left[ \frac{1}{\beta} \left( \frac{r_M}{\alpha} \right)^{q-1} + \frac{2}{n} \right]}$$

and

$$(5.23) \quad p_2 \geq \beta \left( \frac{\alpha}{r_M} \right)^{q-1} 1/r_M .$$

(c) *Boundaries with star-shaped irregularities.*

Suppose now that the boundary  $C$  consists of two parts  $C_1$  and  $C_2$  where  $C_1$  is smooth and  $C_2$  is star-shaped with respect to the chosen origin. We assume that the closure of the interior of  $C_2$  contains  $C_2$ . For example, in two dimensions, the components of  $C_2$  cannot be isolated points. Let  $\bar{K}$  now be defined relative to  $C_1$ . Denote by  $R_1$  the region consisting of  $R$  minus the strip adjacent to  $C_1$ . We suppose that  $\bar{K}$  is large enough (the strip small enough) to make  $R_1$  connected.

We assume that  $\beta$  is such that on  $C_2$

$$(5.24) \quad \frac{x^i n_i}{r} \geq \beta .$$

Then in place of (5.13) we have

$$(5.25) \quad f^i = \begin{cases} \left[ p n_i (1 - \bar{K} N(x)) + \frac{x^i}{r} \right] r^{-q}, & \text{in } R - R_1 \\ \frac{x^i}{r} r^{-q}, & \text{in } R_1 . \end{cases}$$

Since in the identity (2.3) it is only necessary that  $f^i$  have a continuous normal component on the boundaries of subregions of  $R_a$  this definition of  $f^i$  has sufficient smoothness properties.

In this case we again have the inequalities (5.22) and (5.23).

## BIBLIOGRAPHY

1. J. H. Bramble and L. E. Payne, *Bounds for solutions of second order elliptic partial differential equations*, Contributions to Diff. Eqns. (to appear).
2. J. B. Diaz and A. Weinstein, *Schwarz' inequality and the methods of Rayleigh-Ritz and Trefftz*, J. Math. and Physics, **26** (1947), 133-136; see also J. B., Diaz, *Upper and lower bounds for quadratic integrals*, Collectanea Mathematica, **4** (1951), 3-50.
3. L. E. Payne and H. F. Weinberger, *New bounds in harmonic and biharmonic problems*, J. Math. Phys. (1955), 291-307.
4. ———, *New bounds for solutions of second order elliptic partial differential equations*, Pacific. J. Math., **8** (1958), 551-573.
5. ———, *An optimal Poincaré inequality for convex domains*, Arch. Rat. Mech. Anal., **5** (1960), 280-292.
6. J. L. Synge, *The method of hypercircle in function-space for boundary value problems*, Proc. Roy. Soc. London, A **191** (1947), 447-467; see also J. L. Synge, *The hypercircle in mathematical physics*, Cambridge Univ. Press (1957).
7. C. E. Weatherburn, *Differential geometry in three dimensions*, Cambridge Univ. Press, vol. II (1930).

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