Pacific Journal of Mathematics

ON ALMOST-COMMUTING PERMUTATIONS

DANIEL E. GORENSTEIN, REUBEN SANDLER AND WILLIAM H. MILLS

Vol. 12, No. 3 March 1962

ON ALMOST-COMMUTING PERMUTATIONS

DANIEL GORENSTEIN, REUBEN SANDLER AND W. H. MILLS

Suppose A and B are two permutations on a finite set X which commute on almost all of the points of X. Under what circumstances can we conclude that B is approximately equal to a permutation which actually commutes with A? The answer to this question depends strongly upon the order of the centralizer, $C(A)$, of A in the symmetric group on X; and this varies greatly according to the cycle structure of A, being comparatively small when A is either a product of few disjoint cycles or a product or a large number of disjoint cycles of different lengths and being comparatively large when A is a product of many disjoint cycles, all of the same length. We shall show by example that when the order of $C(A)$ is small there may exist a permutation B which commutes with A "almost everywhere" yet is not approximated by any element of $C(A)$. On the other hand, when A is a product of many disjoint cycles of the same length, we shall see that for any such permutation B, there must exist a permutation in $C(A)$ which agrees closely with B.

It is clear that if B is a permutation leaving fixed almost all points of X, then no matter what permutation A is given, B will commute with A on almost all points of X, and at the same time B can be closely approximated by an element of $C(A)$—namely, the identity. However, the examples we shall give will show that only when all (or nearly all) of the cycles of A are of the same length can we hope to approximate *every* B which nearly commutes with A by an element in $C(A)$. Accordingly, the bulk of this paper will be taken up with the study of the case in which A is a product of many disjoint cycles, all of the same length.

1. In order to get a satisfactory notation and a more compact way of discussing the problem, we begin by making the symmetric group $S_N(X)$ on the space X into a metric space. Here N denotes the cardinality of X, and it is to be understood that N is finite. Define, for any A in $S_N(x)$,

$$\| A \| = \frac{N - f_A}{N} \tag{1}$$

where f_A is the number of fixed points of A on X. Now define the distance $d(A, B)$ between two elements A and B of $S_N(X)$ to be

$$d(A, B) = \| AB^{-1} \| . \tag{2}$$

Received December 2, 1961,

Under these definitions, the identity is the only permutation of norm 0, every permutation has norm $\leqq 1$, and a permutation has norm p if and only if it moves pN points of X. In particular, the permutations A and B commute if and only if $\| [A, B] \| = 0$, or equivalently, if and only if $d(AB, BA) = 0$.

In order to see that these definitions make $S_N(X)$ into a metric space, we need only verify the triangle inequality, since the other properties are trivial. But the points of X displaced by AB are clearly among those which are displaced by either A or B. Hence $N - f_{AB} \leqq (N - f_A) + (N - f_B)$ and consequently $\| AB \| \leqq \| A \| + \| B \|$. We thus have the following lemma.

LEMMA 1. *With the norm defined above, $S_N(X)$ forms a metric space.*

When no restriction is placed upon the cycle structure of A, we have the following result:

PROPOSITION 1. For any $\varepsilon > 0$, there exists an integer N and permutations A and B in $S_N(X)$ such that $\| [A, B] \| < \varepsilon$ and such that $d(B, D) = 1$ for every D in $C(A)$.

Proof. We shall give two examples of permutations A and B which satisfy the conditions of the proposition; in the first, A will be a product of cycles of relatively prime lengths, and in the second, a product of cycles of lengths n and $2n$.

EXAMPLE 1. Let $X = \{1, 2, \cdots, N\}$, where $N = 2n > 4/\varepsilon$. Let A be the permutation

$$(1\,2 \cdots n-1)(n)(n+1\; n+2 \cdots 2n)$$

and B the permutation $xB = x + n$ if $x \leqq n$, and $xB = x - n$ if $x > n$. By direct verification, we find that A and B commute except on the points $n-1, n, 2n-1, 2n$. Thus $f_{[A,B]} = N - 4$ and hence $\| [A, B] \| = 4/N < \varepsilon$.

On the other hand, any element D of $C(A)$ must map each cycle of A into itself, since these cycles are of different lengths. But, for any x in X, xB and x lie in distinct cycles of A. It follows that for any D in $C(A)$, BD^{-1} displaces every point of X and hence that $d(B, D) = 1$.

EXAMPLE 2. Let $X = \{1, 2, \cdots, N\}$, where $N = 4nm$ and $n > 1/\varepsilon$. Let A be the permutation with m cycles of length $2n$ and $2m$ cycles of length n, defined as follows:

$$(1\ 2 \cdots 2n)(2n+1 \cdots 4n) \cdots (2n(m-1)+1 \cdots 2nm)$$
$$(2nm+1 \cdots 2nm+n)(2nm+n+1 \cdots 2nm+2n) \cdots$$
$$(4nm-n+1 \cdots 4nm)\ .$$

Let B be the permutation $xB = x + 2nm$ if $x \leqq 2nm$, and $xB = x - 2nm$ if $x > 2nm$.

Again, by direct computation, we find that A and B commute on all points x of X except when $x \equiv 0 \pmod{n}$. Thus $f_{[A,B]} = 4nm - 4m$ and hence $\|[A, B]\| = 1/n < \varepsilon$. On the other hand, if $D \in C(A)$, D must permute the cycles of A of length n among themselves and must permute the cycles of A of length $2n$ among themselves. But if x is in a cycle of length n, then xB is in a cycle of length $2n$, and vice versa. It follows that BD^{-1} displaces every point of X and hence that $d(B, D) = 1$, for any D in $C(A)$.

2. The two examples given in Proposition 1 indicate that unless severe restrictions are placed on the cycle structure of A, the fact that B comes very close to commuting with A does not necessarily imply that B can be approximated by an element in $C(A)$. In fact, it seems that unless A consists almost entirely of cycles of the same length, little can be said in general of the relation between $\|[A, B]\|$ and the distance from B to $C(A)$.

In order to be able to make as exact statements as possible, we shall assume in the balance of the paper that A is the product of m disjoint cycles, each of length n. In this case our statements about the distance from B to $C(A)$ will depend only upon $\|[A, B]\|$ and n.

We may take $X = \{1, 2, \cdots, N\}$, where now $N = nm$. Let x, k be integers such that $1 \leqq x \leqq N$, $0 \leqq k \leqq n$, and write $x = in + r$, where $1 \leqq r \leqq n$. We shall adopt the following notation:

$$\overline{x+k} = in + s, \text{ where } 1 \leqq s \leqq n \text{ and } s \equiv r + k \pmod{n}\ . \tag{3}$$

Without loss of generality we may assume that A is the mapping

$$xA = \overline{x+1},\ x \in X\ . \tag{4}$$

We shall say that B in $S_N(X)$ *transforms* the cycle a of A into the cycle a' if, for some x in a, xB is in a' and

$$(\overline{x+k})B = \overline{xB+k}, \qquad k = 0, 1, \cdots, n-1\ . \tag{5}$$

We shall write $(a)B = a'$ if B transforms a into a'. We shall also say that B commutes with A on a cycle a if it commutes with A on each point of a.

LEMMA 2. (a) *A permutation B commutes with A on a cycle a if and only if B tranforms a into a cycle a'.*

(b) *if B commutes with A on $n-1$ points of a cycle a, then B commutes with A on a.*

(c) *If B transforms r cycles of A into cycles of A, there exists an element D in $C(A)$ which agrees with B on these r cycles.*

Proof. For A and B to commute on a point x of X we must have $xBA = xAB$, and hence

$$\overline{xB+1} = \overline{(x+1)}B . \tag{6}$$

Suppose $(a)B = a'$; then (6) follows at once from (5) for any x in a. Conversely if (6) holds for all x in a, (5) follows at once by induction on k.

To prove (b), suppose B and A commute on x, $\overline{x+1}, \cdots, \overline{x+n-2}$. Again by induction on k, (5) holds for $k = 0, 1, \cdots, n-2$. In particular, $\overline{(x+n-2)}B = \overline{xB+n-2}$. Now using (6) with x replaced by $\overline{x+n-2}$, we obtain

$$\begin{aligned}\overline{(x+n-1)}B &= \overline{\overline{(x+n-2)}B+1}\\ &= \overline{\overline{xB+n-2}+1} = \overline{xB+n-1} .\end{aligned}$$

Thus (5) holds for all k, and hence A and B commute on a by part (a).

Finally suppose B transforms the cycles $a_1, \cdots, a_r$ into the cycles $a'_1, \cdots, a'_r$. Denote by $a'_{r+1}, \cdots, a'_m$ the remaining cycles of A. Let D be a permutation which agrees with B on $a_1, \cdots, a_r$ and transforms a_i into a'_i, $i = r+1, \cdots, m$. By (a) D is in $C(A)$.

3. We shall now begin the analysis of the relationship between $\| [A, B] \|$ and the minimum distance from B to $C(A)$, under the assumption that A is the product of n-cycles. We shall denote this minimum distance by $d_A(B)$. Thus

$$d_A(B) = \min_{D \in C(A)} d(B, D) . \tag{7}$$

Then following estimate for $d_A(B)$ is easily obtained.

PROPOSITION 2. For any B in $S_N(X)$,

$$d_A(B) \leqq \frac{n \| [A, B] \|}{2} .$$

Proof. If $\| [A, B] \| \geqq 2/n$, the proposition is vacuously true since $d_A(B) \leqq 1$. Hence we may assume that $\| [A, B] \| < 2/n$.

Now $N = nm$, where m is the number of cycles in A. It suffices to show that B transforms at least

$$m - \frac{N \| [A, B] \|}{2}$$

cycles of A into cycles of A. For then by Lemma 2(c) we can find an element D in $C(A)$ which agrees with B on these cycles and hence on at least

$$N - \frac{nN}{2} \cdot \| [A, B] \|$$

points of X. It follows that

$$d(B, D) \leqq \frac{n \| [A, B] \|}{2} .$$

By the definition of $\| [A, B] \|$, $N \cdot \| [A, B] \|$ is the number of points displaced by $[A, B]$ and hence on which A and B do not commute. But by Lemma 2(b) any cycle of A which is not transformed by B into a cycle of A contains at least 2 points on which A and B do not commute. Thus there are at most

$$\frac{N \| [A, B] \|}{2}$$

cycles of A which are not transformed by B into cycles of A, and hence B transforms at least

$$m - \frac{N \| [A, B] \|}{2}$$

cycles of A into cycles of A.

Proposition 2 gives an upper bound for $d_A(B)$, which depends only upon $\| [A, B] \|$ (and n), but not upon the particular structure of B. Our main concern in the paper will be in improving this upper bound. The next proposition shows the limit to which this estimate can be improved.

PROPOSITION 3. *If A contains at least two distinct cycles, then there exists a permutation B in $S_N(X)$ such that*

$$d_A(B) = \frac{n \| [A, B] \|}{4}$$

when n is even, and such that

$$d_A(B) = \frac{n - 1}{4} \| [A, B] \|$$

when n is odd. Furthermore for any $\varepsilon > 0$, N and B can be chosen so that $\|[A, B]\| < \varepsilon$.

Proof. Assume first that n is even. Set $m = m_1 + m_2$, where $m_1 \geqq 0$ and $m_2 \geqq 2$. Define the permutation B as follows: $xB = x$ if $1 \leqq x \leqq nm_1$; if $x > nm_1$, write $x = in + k$ where $1 \leqq k \leqq n$, and define $xB = x$ if $k \leqq n/2$, $xB = x + n$ if $i \neq m - 1$ and $k > n/2$, and $xB = nm_1 + k$ if $i = m - 1$ and $k > n/2$.

Thus B leaves the first m_1 cycles of A pointwise fixed, one half of each of the remaining m_2 cycles pointwise fixed, and permutes the other halves of these m_2 cycles cyclically. From its definition, we see that B commutes with A except on the points $x > nm_1$ for which $x \equiv 0 \pmod{n/2}$. Thus

$$\|[A, B]\| = \frac{2m_2}{N} . \tag{8}$$

Since $N = n(m_1 + m_2)$, $2m_2/N$ can be made arbitrarily small by making m_1 sufficiently large. Thus, to prove the proposition, we have only to show that

$$d_A(B) = \frac{n \|[A, B]\|}{4} .$$

Observe, first of all, that the identity, I, is in $C(A)$ and agrees with B on

$$nm_1 + \frac{nm_2}{2}$$

points of X, whence

$$d(I, B) = \frac{N - nm_1 - \dfrac{nm_2}{2}}{N} = \frac{nm_2}{2N} = \frac{n}{4} \|[A, B]\| . \tag{9}$$

On the other hand, by Lemma 2, any D in $C(A)$ must transform each cycle a_i of A into some other cycle a_j. Since B transforms the two halves of the cycles a_i into distinct cycles of A, $m_1 \leqq i \leqq -1$, D and B can agree on at most half of the nm_2 points in these cycles. Hence DB^{-1} displaces at least $nm_2/2$ points of X, which implies that

$$d(D, B) \geqq \frac{nm_2}{2N} = \frac{n}{4} \|[A, B]\|$$

for any D in $C(A)$.

When n is odd, the construction of B is entirely analogous.

4. If we set

$$d_A = \max_{\substack{B \in S_N(X) \\ B \notin C(A)}} \frac{d_A(B)}{\| [A, B] \| n},$$

then d_A is a measure of the extent to which *every* permutation in $S_N(X)$ can be approximated by elements in $C(A)$. Propositions 2 and 3 show that

$$\frac{1}{4} \leqq d_A \leqq \frac{1}{2} \text{ or } \frac{n-1}{4n} \leqq d_A \leqq \frac{1}{2} \tag{10}$$

according as n is even or odd.

In the balance of the paper we shall sharpen these inequalities by lowering the upper bound for d_A. Our next result will show that in considering this problem, we may restrict our attention to those cycles of A on which B commutes with A on exactly $n, n-2$, or $n-3$ points. Let U_B, V_B, W_B be the set of points in those cycles of A on which B commutes with A on $n, n-2$, and $n-3$ points respectively; and let $u_B = | U_B |$, $v_B = | V_B |$, $w_B = | W_B |$.

THEOREM 1. *Suppose there exists an element D in $C(A)$ which agrees with B on at least $u_B + (1/2)v_B + (1/3)w_B$ points of X. Then*

$$d_A(B) \leqq \| [A, B] \| \frac{n}{4}.$$

Proof. For simplicity of notation, we drop the subscript B, and define

$$t = N - u - v - w. \tag{11}$$

Thus t is the number of points in those cycles of A on which A and B commute on no more than $n-4$ points. Then by definition of u, v, w, t, we have

$$u + \frac{n-2}{n}v + \frac{n-3}{n}w + \frac{n-4}{n}t \geqq f_{[A,B]}. \tag{12}$$

Now, by hypothesis,

$$d(B, D) \leqq \frac{N - \left(u + \frac{1}{2}v + \frac{1}{3}w\right)}{N} = \frac{\frac{1}{2}v + \frac{2}{3}w + t}{N}. \tag{13}$$

We must show that

$$\frac{\frac{1}{2}v + \frac{2}{3}w + t}{N} \leqq \frac{n}{4} \| [A, B] \|. \tag{14}$$

But using (1), we can rewrite (14) as:

$$f_{[A,B]} \leqq u + \frac{n-2}{n}v + \left(1 - \frac{8}{3n}\right)w + \left(\frac{n-4}{n}\right)t \,. \tag{15}$$

Since (15) is an immediate consequence of (12), the theorem follows.

5. In this section, we prove that $d_A \leqq 1/4$, by proving that for any B in $S_N(X)$, there exists a permutation D in $C(A)$ which satisfies the conditions of Theorem 1.

To treat our problem, we need an additional concept: By a *block* of a cycle a of A, we shall mean a maximal sequence $x, \overline{x+1}, \cdots, \overline{x+r-1}$ of points of a such that A and B commute on every point of the sequence except $\overline{x+r-1}$. The integer r will denote the *length* of the block. According to the definition, if A and B commute on every point of a then a contains no blocks. When B and A do not commute on every point of a, we have the following obvious lemma:

LEMMA 3. *If A and B commute on exactly $n-k$ points of a cycle a of A, $k > 0$, then A contains exactly k blocks, the sum of whose lengths is n.*

Thus when a cycle a of A lies in V_B, a consists of 2 blocks which we denote by p_1, p_2; and when a lies in W_B, a consists of 3 blocks which we denote by q_1, q_2, q_3. We define $|p_j|$, $|q_j|$ to be the lengths of p_j, q_j, respectively. Furthermore we order the blocks so that $|p_1| \geqq |p_2|$ and $|q_1| \geqq |q_2| \geqq |q_3|$. Since $|p_1| + |p_2| = n$,

$$|p_1| \geqq \frac{n}{2} \tag{16}$$

and likewise

$$|q_1| \geqq \frac{n}{3} \,. \tag{17}$$

Let $x, \overline{x+1}, \cdots, \overline{x+r-1}$ be a block contained in a cycle a. If $xB = y$, then, it follows from (6) as in the proof of Lemma 2, that

$$(\overline{x+k})B = \overline{y+k} \,, \qquad 0 \leqq k \leqq r-1 \,; \tag{18}$$

and

$$(\overline{x+r})B \neq \overline{y+r} \,. \tag{19}$$

Thus the image of the block is a consecutive sequence of points in a cycle a'. It follows that there exist permutations which transform a

into a' and agree with B on the block $b = \{x, \overline{x+1}, \cdots, \overline{x+r-1}\}$. In fact, any D in $C(A)$ for which $xD = y$ has this property. If D is such a permutation, we shall write simply $(a)D = a'$; $(b)D = (b)B$.

From this fact, we easily derive the following lemma:

LEMMA 4. *Let $a_1, \cdots, a_k$ be distinct cycles of A containing the blocks $b_1, \cdots, b_k$ respectively. If the images of b_i under B lie in distinct cycles a'_i of A, $i = 1, 2, \cdots, k$, then there exist permutations D in $C(A)$ such that $(a_i)D = a'_i$; $(b_i)D = (b_i)B$, $i = 1, 2, \cdots, k$.*

We are now in a position to prove the following result:

THEOREM 2. *Given any B in $S_N(X)$, there exists an element D in $C(A)$ which agrees with B on at least*

$$u_B + \frac{1}{2}v_B + \frac{1}{3}w_B$$

points of X.

Proof. Let $a_1, a_2, \cdots, a_m$ be the cycles of A. For any $i, j, 1 \leqq i, j \leqq m$, let b_{ij} be the maximal number of elements of a_i on which a permutation D in $C(A)$ mapping a_i into a_j can agree with B. Thus if B transforms a_i into a_j, $b_{ij} = n$. If $(a_i)B \cap a_j = \phi$, then $b_{ij} = 0$. Now, to any $m \times m$ permutation matrix (e_{ij}) there corresponds a permutation D in $C(A)$ which agrees with B on

$$\sum_{i,j} e_{ij}b_{ij} \tag{20}$$

points, where D is defined to transform a_i into a_j if $e_{ij} = 1$, and to map a_i so as to agree with B on b_{ij} points.

We wish to show

$$\max_{(e_{ij})} \sum e_{ij}b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w\ , \tag{21}$$

where (e_{ij}) ranges over all permutation matrices. To do this, consider the set of all real $m \times m$ matrices (x_{ij}) such that

$$x_{ij} \geqq 0\ ; \qquad 1 \leqq i, j \leqq m \tag{22}$$

$$\sum_i x_{ij} = 1\ ; \qquad 1 \leqq j \leqq m \tag{23}$$

$$\sum_j x_{ij} = 1\ ; \qquad 1 \leqq i \leqq m. \tag{24}$$

This is the set of doubly stochastic matrices and is a convex, bounded set whose vertices consist of exactly the permutation matrices (see [1], pp. 132–3).

The following lemma will be useful in proving the theorem.

LEMMA 5. *If* (x_{ij}) *is any doubly stochastic matrix, then there exists a permutation matrix* (e_{ij}) *such that*

$$\sum_{i,j} e_{ij}b_{ij} \geqq \sum_{i,j} x_{ij}b_{ij} \ . \tag{25}$$

Proof. See [1], p. 134.

If we can now demonstrate a doubly stochastic matrix such that

$$\sum_{i,j} x_{ij}b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w \ , \tag{26}$$

we will clearly be finished since, by Lemma 5, there must then be some permutation matrix (e_{ij}) such that

$$\sum_{i,j} e_{ij}b_{ij} \geqq u + \frac{1}{2}v + \frac{1}{3}w \ ,$$

and this permutation matrix will yield the desired mapping D.

To find a matrix satisfying (26), define

$$x_{ij} = \frac{n_{ij}}{n} \ , \tag{27}$$

where n_{ij} is the number of points of a_i which B maps into a_j. The matrix (x_{ij}) is clearly doubly stochastic, so we must show that (26) holds. But if $a_i \subseteqq U_B$, then

$$\sum_j x_{ij}b_{ij} = n \ ,$$

since $(a_i)B = a_{j_1}$ for some j_1. If $a_i \subseteqq V_B$, there exist indices j_1 and j_2 such that $(p_1)B \subset a_{j_1}$ and $(p_2)B \subset a_{j_2}$. Note that $j_1 \neq j_2$, or else a_i would be transformed by B into a_{j_1}. In this case, then,

$$\sum_j x_{ij}b_{ij} = \frac{|\,p_1\,|^2}{n} + \frac{|\,p_2\,|^2}{n} \geqq \frac{n}{2}$$
$$(\text{remember } |\,p_1\,| + |\,p_2\,| = n) \ .$$

Finally, when $a_i \subseteqq W_B$, one of three things can happen:

(a) q_1, q_2, q_3 can be mapped by B into three distinct cycles of A.

(b) q_1, q_2, q_3 can be mapped by B into only two cycles of A,

(c) q_1, q_2, q_3 can be mapped into one cycle of A.

In the first case,

$$\sum_j x_{ij}b_{ij} = \sum_{k=1}^{3} \frac{|\,q_k\,|^2}{n} \ .$$

In the second case,

$$\sum_j x_{ij} b_{ij} = \frac{|q_{k_1}|^2}{n} + \frac{(|q_{k_2}| + |q_{k_3}|)}{n} |q_{k_2}|$$
where $|q_{k_2}| \geqq |q_{k_3}|$.

Finally, in case c,

$$\sum_j x_{ij} b_{ij} = |q_1| \frac{(|q_1| + |q_2| + |q_3|)}{n},$$

where $|q_1| \geqq |q_2|, |q_3|$.

Since $|q_1| + |q_2| + |q_3| = n$, it follows at once in all three cases that

$$\sum_j x_{ij} b_{ij} \geqq \frac{n}{3}.$$

We have thus demonstrated the existence of a doubly stochastic matrix (x_{ij}) with the property

$$\sum_{i,j} x_{ij} b_{ij} \geqq u + \frac{1}{2} v + \frac{1}{3} w.$$

Together with Lemma 5, this completes the proof of the theorem.

As an immediate corollary of Theorems 1 and 2, we obtain our main result:

THEOREM 3. *Let A contain at least two distinct cycles. If n is even,* $d_A = 1/4$. *If n is odd,*

$$\frac{n-1}{4n} \leqq d_A \leqq \frac{1}{4}.$$

REFERENCE

1. S. Karlin, *Mathematical Methods and Theory in Games, Programming and Economics*, vol. I (1959).

CLARK UNIVERSITY
INSTITUTE FOR DEFENSE ANALYSES
YALE UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS
Stanford University
Stanford, California

M. G. ARSOVE
University of Washington
Seattle 5, Washington

A. L. WHITEMAN
University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE
University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY
D. DERRY
M. OHTSUKA
H. L. ROYDEN
E. SPANIER
E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA
STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON

* * *

AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 12, No. 3 March, 1962