

Pacific Journal of Mathematics

ON THE ZEROS OF THE SOLUTIONS OF $w''(z) + p(z)w(z) = 0$

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1. Introduction. Let $f(z)$ be a meromorphic function with, at most, simple poles in a simply-connected domain D , such that $f'(z) \neq 0$ for $z \in D$. Let

$$(1.1) \quad p(z) = \frac{1}{2}\{f(z), z\},$$

where

$$\{f(z), z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of $f(z)$ with respect to z . In connection with the function $f(z)$, we consider the differential equation

$$(1.2) \quad w''(z) + p(z)w(z) = 0.$$

The function $f(z)$ may be written in the form

$$f(z) = \frac{w_1(z)}{w_2(z)},$$

where $w_1(z)$ and $w_2(z)$ are linearly independent solutions of (1.2). The nontrivial solution

$$w(z) = Aw_1(z) + Bw_2(z)$$

vanishes at z_1, z_2, \dots, z_n if, and only if, $f(z)$ takes at these points the value $-BA^{-1}$. Hence, it follows that $f(z)$ takes some value in D n times if, and only if, there exists a nontrivial solution of (1.2) having n zeros in D .

This connection was pointed out by Nehari in [3].

DEFINITION 1. *The equation (1.2) is disconjugate in D if every solution of (1.2) vanishes in D not more than once.*

Hence, (1.2) is disconjugate in D if, and only if, $f(z)$ is univalent in D .

DEFINITION 2. *The equation (1.2) is nonoscillatory in D if every*

Received November 9, 1961. This paper represents part of a thesis submitted to the Senate of the Technion-Israel Institute of Technology in partial fulfillment of the requirements for the degree of Master of Science. The author wishes to thank Professor B. Schwarz for his guidance in the preparation of this paper. Acknowledgment is also due to Professor E. Netanyahu and Professor E. Jabotinsky for their help.

solution of (1.2) vanishes in D at most a finite number of times. Hence, (1.2) is nonoscillatory in D if, and only if, $f(z)$ is finitely-valent in D .

By imposing restrictions on $p(z)$, disconjugacy and nonoscillation theorems can be obtained. By the above connection, these theorems are equivalent to theorems about the distribution of the values of $f(z)$. In this paper these theorems will be formulated as disconjugacy and nonoscillation theorems only, and not as theorems regarding the values of $f(z)$.

2. **A lemma.** The lemma to be proved in this paragraph yields an upper bound for $|p(z)|$ on $|z| = \rho$, $\rho < 1$. This bound is connected with the area integral $\iint |p(z)| dx dy$.

LEMMA 1. Let $p(z)$ be analytic in $|z| < 1$. Then

$$(2.1) \quad |p(z')| \leq \frac{\iint_{|z| < 1} |p(z)| dx dy}{\pi(1 - |z'|^2)^2}, \quad |z'| < 1, \quad (z = x + iy).$$

Proof. Let

$$p(z) = \sum_{k=0}^{\infty} a_k z^k.$$

Then

$$\int_0^{2\pi} p(\rho e^{i\theta}) d\theta = 2\pi a_0, \quad 0 \leq \rho < 1,$$

and therefore

$$2\pi |p(0)| \leq \int_0^{2\pi} |p(\rho e^{i\theta})| d\theta.$$

Multiplying by $\rho d\rho$ and integrating, we obtain

$$(2.2) \quad |p(0)| \leq \frac{\iint_{|z| < 1} |p(z)| dx dy}{\pi},$$

which proves (2.1) for $z' = 0$. The transformation

$$z = \frac{\zeta + z'}{1 + \zeta \bar{z}'}, \quad (\zeta = \xi + i\eta)$$

maps $|\zeta| < 1$ onto $|z| < 1$. Let

$$(2.3) \quad p_1(\zeta) = p[z(\zeta)] \left(\frac{dz}{d\zeta}\right)^2 = p(z) \left[\frac{1 - |z'|^2}{(1 + \zeta \bar{z}')^2}\right]^2.$$

We have

$$(2.4) \quad \iint_{|z| < 1} |p(z)| dx dy = \iint_{|\zeta| < 1} |p[z(\zeta)]| \left| \frac{dz}{d\zeta} \right|^2 d\xi d\eta = \iint_{|\zeta| < 1} |p_1(\zeta)| d\xi d\eta,$$

and

$$(2.5) \quad p_1(0) = p(z')(1 - |z'|^2).$$

Applying (2.2) to the function $p_1(\zeta)$, and using (2.4) and (2.5) we obtain the required result (2.1).

REMARK 1. For the special case when the function $p(z)$ is a square of a function analytic in $|z| < 1$, Lemma 1 can be obtained using the Bergman kernel function of $|z| < 1$. (see [4], p. 261, ex. 4).

REMARK 2. (2.1) is a sharp inequality. It is easily proved that the sign of equality in (2.1) at a point $z' = z_0$, $|z_0| < 1$, occurs if (and only if) $p(z)$ is of the form

$$p(z) = c \left[\frac{1 - |z_0|^2}{(1 - z\bar{z}_0)^2} \right]^2.$$

3. A sufficient condition for disconjugacy (I). In [3], Nehari proved the following theorem (Th. 1, [3], sufficient condition): *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be disconjugate in $|z| < 1$ is:*

$$(3.1) \quad |p(z)| \leq \frac{1}{(1 - |z|^2)^2}, \quad |z| < 1.$$

This theorem is sharp, as is shown by an example due to E. Hille [2].

From Lemma 1 and from Nehari's theorem, we obtain a disconjugacy theorem, in which the restriction on $p(z)$ is given by a condition on the area integral.

THEOREM 1. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be disconjugate in $|z| < 1$ is:*

$$(3.2) \quad \iint_{|z| < 1} |p(z)| dx dy \leq \pi.$$

Proof. From (2.1) and (3.2) it follows that

$$|p(z')| \leq \frac{\iint_{|z| < 1} |p(z)| dx dy}{\pi(1 - |z'|^2)^2} \leq \frac{1}{(1 - |z'|^2)^2}, \quad |z'| < 1.$$

The assumption of Nehari's theorem is satisfied, and therefore (1.2) is

disconjugate in $|z| < 1$.

REMARK 1. The question of the sharpness of Theorem 1 is still open. Although both, inequality (2.1) and Nehari's theorem, are sharp, it does not follow that Theorem 1, which is deduced from them, is sharp too.

REMARK 2. In [7] the following theorem (Th. 4, [7]) is proved: *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be disconjugate in $|z| < 1$ is*

$$(3.3) \quad \int_0^{2\pi} |p(e^{i\theta})| d\theta \leq 4.$$

The integral on the left-hand side of (3.3) is defined as the limit for $\rho \rightarrow 1$, of the nondecreasing function

$$A(\rho) = \int_0^{2\pi} |\rho p(\rho e^{i\theta})| d\theta, \quad \rho < 1.$$

From our Theorem 1, it follows that the constant 4 in (3.3) can be improved to 2π . Indeed, if

$$\int_0^{2\pi} |p(e^{i\theta})| d\theta \leq 2\pi,$$

then

$$\int_0^{2\pi} |\rho p(\rho e^{i\theta})| d\theta \leq 2\pi, \quad \rho < 1.$$

This implies now the validity of (3.2), and therefore, by Theorem 1, (1.2) is disconjugate in $|z| < 1$.

The constant 2π is, however, not the best possible. In Theorem 6 it will be improved to 4π .

4. Invariance of the area integral. We shall prove that the area integral is invariant under the transformation of (1.2), resulting from a linear mapping of the variable z .

Let

$$(4.1) \quad \zeta = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \quad (\zeta = \xi + i\eta),$$

be a linear transformation analytic in the simply-connected domain D . D is mapped by (4.1) onto D' . By this mapping (1.2) will be transformed into an equation of the form

$$(4.2) \quad W''(\zeta) + P(\zeta)W'(\zeta) + Q(\zeta)W(\zeta) = 0,$$

where

$$W(\zeta) = w[z(\zeta)] .$$

By the further substitution

$$(4.3) \quad W(\zeta) = \frac{w_1(\zeta)}{a - \zeta c} ,$$

equation (4.2) takes the form:

$$(4.4) \quad w_1''(\zeta) + p_1(\zeta)w_1(\zeta) = 0 .$$

The solutions $w(z)$ and $w_1(\zeta)$ vanish in D and D' respectively at points z and ζ connected by (4.1). By a simple calculation, or from the properties of the Schwarzian derivative, it follows that

$$(4.5) \quad p_1(\zeta) = p[z(\zeta)] \left(\frac{dz}{d\zeta} \right)^2 ,$$

and hence

$$\iint_D |p(z)| dx dy = \iint_{D'} |p[z(\zeta)]| \left| \frac{dz}{d\zeta} \right|^2 d\xi d\eta = \iint_{D'} |p_1(\zeta)| d\xi d\eta .$$

We have thus proved that

$$(4.6) \quad \iint_D |p(z)| dx dy = \iint_{D'} |p_1(\zeta)| d\xi d\eta .$$

The property expressed by (4.6) is the *invariance of the area integral*.

The invariance of the area integral yields the following generalization of Theorem 1:

THEOREM 1'. *Let $p(z)$ be analytic in D , where D is a circle or a half plane. A sufficient condition for (1.2) to be disconjugate in D is*

$$(3.2)' \quad \iint_D |p(z)| dx dy \leq \pi .$$

Proof. By a suitable linear transformation, D will be mapped onto the unit circle. From the invariance of the area integral, from Theorem 1, and from the fact that the solutions of (1.2) and (4.4) vanish at corresponding points, the desired result follows.

5. A theorem about zeros on the boundary of a certain domain.

Using the invariance of the area integral and a theorem of Grunsky, we obtain a result regarding the zeros of the solutions of (1.2) on the boundary of a domain bounded by two orthogonal circular arcs.

THEOREM 2. *Let $p(z)$ be analytic in \bar{D} , where D is a domain bounded by two orthogonal circular arcs. If*

$$(5.1) \quad \iint_D |p(z)| dx dy \leq 1 ,$$

then no nontrivial solution of (1.2) vanishes twice on one of the arcs bounding D .

Proof. Let Γ be one of the two orthogonal arcs bounding D . Assume that there exists a nontrivial solution $w(z)$ of (1.2) and $z_1, z_2 \in \Gamma$, such that:

$$w(z_1) = w(z_2) = 0 .$$

Let D_1 be the domain bounded by the arc $z_1 z_2$ of Γ , and by the arc passing through z_1 and z_2 , orthogonal to Γ and lying within D . Let D_2 be the upper half of the unit circle. A suitable linear transformation maps D_1 onto D_2 so that z_1 and z_2 are mapped on ± 1 . As

$$\iint_{D_1} |p(z)| dx dy \leq 1 ,$$

it follows from the invariance of the area integral that we may assume, without loss of generality, that D is the upper half of the unit circle, and z_1, z_2 are ± 1 .

We shall make use of the following theorem of Grunsky [1]: *Let $g(z)$ be analytic in a convex domain D . Let $z_1, z_2 \in D$ be such that*

$$g(z_1) = g(z_2) = 0 .$$

Let Δ be the triangle with vertices z_1, z_2 and $z', z' \in D$. Let A be the area of Δ . Then

$$(5.2) \quad 2Ag(z') = (z' - z_1)(z' - z_2) \iint_{\Delta} g''(z) dx dy , \quad (z = x + iy) .$$

From (5.2) we obtain here

$$\begin{aligned} 2Aw(z') &= (z' - 1)(z' + 1) \iint_{\Delta} w''(z) dx dy \\ &= -(z' - 1)(z' + 1) \iint_{\Delta} p(z)w(z) dx dy , \end{aligned}$$

and therefore

$$(5.3) \quad 2A|w(z')| < |z' - 1| |z' + 1| \iint_{\Delta} |p(z)| |w(z)| dx dy .$$

Let z^* be a point on the boundary of D , at which $|w(z)|$ takes its maximum value in D . There are two possibilities:

- I. z^* belongs to the circular part of the boundary of D .
- II. z^* belongs to the diameter of D .

In case I, we get from (5.3):

$$|w(z^*)| < \frac{|z^* - 1||z^* + 1|}{2A} |w(z^*)| \iint_D |p(z)| dx dy .$$

Noting that

$$\frac{|z^* - 1||z^* + 1|}{2A} = 1 ,$$

we obtain:

$$(5.4) \quad \iint_D |p(z)| dx dy > 1 .$$

Inequalities (5.1) and (5.4) are incompatible, proving our theorem in this case.

In case II, we use the linear transformation

$$\zeta = \frac{1}{z + i} , \quad (\zeta = \xi + i\eta) .$$

Let D' be the lower half of the circle $|\zeta + i/2| < 1/2$. The above transformation maps D onto D' , so that the circular part of the boundary of D is mapped onto the diameter of D' , and the diameter of D is mapped onto the circular part of the boundary of D' . The point z^* , which according to our assumption belongs to the diameter of D , is mapped on the point ζ^* , which belongs to the circular part of the boundary of D' . The points $z = \pm 1$ are mapped on the points $a = 1/2 - i/2$, $b = -1/2 - i/2$, which are the two endpoints of the diameter of D' . Equation (1.2) is transformed into equation (4.4), for which we have

$$w_1(a) = w_1(b) = 0 .$$

From (4.3) we get

$$(5.5) \quad w_1(\zeta) = w[z(\zeta)](-\zeta) ,$$

and therefore:

$$(5.6) \quad |w_1(\zeta)| < |\zeta| |w(z^*)| .$$

Let $\zeta' = \xi - i/2$ be any point on the diameter of D' . We have:

$$(5.7) \quad |\zeta'| < |\zeta^*| .$$

From (5.5), (5.6) and (5.7) we obtain:

$$|w_1(\zeta')| < |\zeta'| |w(z^*)| < |\zeta^*| |w(z^*)| = |w_1(\zeta^*)| .$$

As ζ' is any point on the diameter of D' , and ζ^* is a point on the circular part of the boundary of D' , we conclude, from the last inequality, that

$|w_1(\xi)|$ takes its maximum value in D' on the circular part of the boundary. From the invariance of the area integral it follows that

$$\iint_{D'} |p_1(\xi)| d\xi d\eta \leq 1 .$$

We have now the same situation as in case I, for which the proof has already been completed.

6. A sufficient condition for nonoscillation. Inequality (3.2) was seen to be a sufficient condition for (1.2) to be disconjugate in $|z| < 1$. The following result shows that mere boundedness of the integral appearing in (3.2) is sufficient to assure nonoscillation in $|z| < 1$.

THEOREM 3. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be nonoscillatory in $|z| < 1$ is:*

$$(6.1) \quad \iint_{|z| < 1} |p(z)| dx dy < \infty .$$

Proof. Assume that there exists a nontrivial solution $w(z)$ of (1.2) with infinitely many zeros in $|z| < 1$. The set of these zeros has an accumulation point α on $|z| = 1$.

From (6.1) follows the existence of a number $\rho, 0 \leq \rho < 1$, such that

$$(6.2) \quad \iint_{\rho \leq |z| < 1} |p(z)| dx dy \leq 1 .$$

It is obvious that at least one of the two halves of a circle, having for diameter the segment connecting two internal points of a given circle, lies inside the given circle.

As α is an accumulation point of the set of zeros, we can choose two elements of that set, z_1 and z_2 , so that the half circle D , for which the segment connecting z_1 and z_2 is a diameter, and which lies in $|z| < 1$, will also lie in the circular ring $\rho < |z| < 1$.

Inequality (6.2) implies inequality (5.1) for this half circle D . D is thus a half circle for which (5.1) is satisfied, and on its diameter there exist two zeros of (1.2). This last fact is a contradiction to Theorem 2, so that no such nontrivial solution $w(z)$ of (1.2) exists.

REMARK 1. In [5] the following theorem (Th. 3, [5]) is proved: *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be nonoscillatory in $|z| < 1$ is:*

$$(6.3) \quad \int_0^{2\pi} |p(e^{i\theta})| d\theta < \infty .$$

(The integral in (6.3) is defined in paragraph 3).

This theorem can be deduced from our Theorem 3. Indeed, (6.3) implies the existence of a bound M such that

$$\int_0^{2\pi} |p(\rho e^{i\theta})| d\theta < M, \quad 0 \leq \rho < 1,$$

and, therefore, such that

$$\iint_{|z| < 1} |p(z)| dx dy < \frac{M}{2}.$$

The assumption of Theorem 3 is satisfied, and therefore (1.2) is non-oscillatory in $|z| < 1$.

From the invariance of the area integral follows the validity of Theorem 3 for every circle and every half plane. In the following theorem, Theorem 3 will be extended to more general domains.

THEOREM 4. *Let $p(z)$ be analytic in a domain D bounded by an analytic Jordan curve. A sufficient condition for (1.2) to be nonoscillatory in D is:*

$$(6.1)' \quad \iint_D |p(z)| dx dy < \infty.$$

Proof. Let $\zeta = \psi(z)$ be a function mapping D onto $|\zeta| < 1$.

In paragraph 4 we described the transformation of (1.2) by a linear mapping. The transformation of (1.2) by a general mapping $\zeta = \psi(z)$ may be performed in a similar way. In the general case we have to change (4.3) into

$$(4.3)' \quad W(\zeta) = w_1(\zeta) e^{-\frac{1}{2} \int P(\zeta) d\zeta}.$$

Equation (1.2) is transformed into an equation of the form (4.4), but (4.5) becomes now

$$(4.5)' \quad p_1(\zeta) = \frac{1}{[\psi'(z)]^2} \left[p(z) - \frac{1}{2} \{\psi'(z), z\} \right].$$

As the corresponding solutions of (1.2) and (4.4) vanish at corresponding points, equation (1.2) is nonoscillatory in D if, and only if, equation (4.4) is nonoscillatory in $|\zeta| < 1$. In order to prove that (4.4) is nonoscillatory in $|\zeta| < 1$, it is sufficient, by Theorem 3, to show that

$$(6.1)'' \quad \iint_{|\zeta| < 1} |p_1(\zeta)| d\xi d\eta < \infty.$$

From (4.5)' we have

$$(6.4) \quad \iint_{|\xi| < 1} |p_1(\xi)| d\xi d\eta = \iint_D \left| p(z) - \frac{1}{2} \{\psi(z), z\} \right| dx dy \\ \leq \iint_D |p(z)| dx dy + \iint_D |\{\psi(z), z\}| dx dy .$$

As D is bounded by an analytic Jordan curve, the function $\zeta = \psi(z)$ is analytic in \bar{D} , and for $z \in \bar{D}$ $\psi'(z) \neq 0$, so that:

$$(6.5) \quad \iint_D |\{\psi(z), z\}| dx dy < \infty .$$

Inequality (6.1)'' now follows from (6.1)', (6.4) and (6.5).

7. A theorem of Pokornyi. In [8] Pokornyi obtained a sufficient condition for (1.2) to be disconjugate in $|z| < 1$. This sufficient condition follows also from a more general theorem of Nehari (Th. 1, [6]). We give here an additional proof of Pokornyi's theorem.

THEOREM 5. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be disconjugate in $|z| < 1$ is:*

$$(7.1) \quad |p(z)| \leq \frac{2}{1 - |z|^2}, \quad |z| < 1 .$$

Proof. Assume that there exists a nontrivial solution $w(z)$ of (1.2), and $z_1, z_2, |z_1|, |z_2| < 1, z_1 \neq z_2$, such that

$$w(z_1) = w(z_2) = 0 .$$

The points z_1 and z_2 determine uniquely a circle passing through them and orthogonal to $|z| = 1$. We denote by C the part of this circle within $|z| = 1$. We may assume, without loss of generality, that C is in the upper half plane and symmetric with respect to the imaginary axis (see [6]).

Let $i\rho$ be the point of C on the imaginary axis. The linear transformation

$$\zeta = \frac{z - i\rho}{1 + i\rho z}$$

maps the unit circle onto itself, and maps z_1 and z_2 on ρ_1 and ρ_2 , $-1 < \rho_1 < \rho_2 < +1$. Equation (1.2) is transformed into equation (4.4)_r for which

$$w_1(\rho_1) = w_1(\rho_2) = 0 .$$

By (4.5) we have:

$$\begin{aligned} |p_1(\zeta)| &= |p(z)| \left| \frac{dz}{d\zeta} \right|^2 = |p(z)| \left(\frac{1 - |z|^2}{1 - |\zeta|^2} \right)^2 \leq \frac{2}{1 - |z|^2} \left(\frac{1 - |z|^2}{1 - |\zeta|^2} \right)^2 \\ &= \frac{2}{1 - |\zeta|^2} \frac{1 - |z|^2}{1 - |\zeta|^2}. \end{aligned}$$

For $-1 < \zeta < +1$, we have

$$|\zeta| \leq |z|,$$

and therefore

$$(7.2) \quad |p_1(\zeta)| \leq \frac{2}{1 - |\zeta|^2}, \quad -1 < \zeta < +1.$$

Let $0 < R < 1$ be such that $-R < \rho_1 < \rho_2 < R$. From (7.2) it follows that:

$$(7.3) \quad |p_1(\zeta)| < \frac{2}{R^2 - |\zeta|^2}, \quad \rho_1 \leq \zeta \leq \rho_2.$$

The strict inequality in (7.3) assures the existence of an $\varepsilon > 0$ and of a neighbourhood D of the segment $[\rho_1, \rho_2]$, such that

$$(7.4) \quad |p_1(\zeta)| \leq \frac{2}{R^2 - |\zeta|^2} - \varepsilon, \quad \xi \in D.$$

From Grunsky's theorem, quoted above, we obtain

$$(7.5) \quad 2Aw_1(\zeta') = -(\zeta' - \rho_1)(\zeta' - \rho_2) \iint_{\Delta} p_1(\zeta)w_1(\zeta)d\xi d\eta, \quad |\zeta'| < 1.$$

The domain of integration Δ is the triangle with vertices at ρ_1, ρ_2, ζ' . A is the area of Δ . Let D_b be an ellipse having $[\rho_1, \rho_2]$ as its major axis, and let the magnitude of its minor axis be $2b$, $b > 0$. For a small enough b , we have $D_b \subset D$. Let ζ_b be a point on the boundary of D_b , at which $|p_1(\zeta)w_1(\zeta)|$ takes its maximum value in D_b . From (7.5) it follows that

$$2A|w_1(\zeta_b)| < |\zeta_b - \rho_1| |\zeta_b - \rho_2| |p_1(\zeta_b)| |w_1(\zeta_b)| A.$$

Hence,

$$|p_1(\zeta_b)| > \frac{2}{|\zeta_b - \rho_1| |\zeta_b - \rho_2|},$$

and therefore

$$(7.6) \quad |p_1(\zeta_b)| > \frac{2}{|R^2 - \zeta_b^2|}.$$

We define the number δ_b by the equation

$$(7.7) \quad |R^2 - \zeta_b^2| = R^2 - |\zeta_b|^2 + \delta_b .$$

It is obvious that $\delta_b > 0$, and that $\delta_b \rightarrow 0$, for $b \rightarrow 0$. For a small enough b we have

$$(7.8) \quad \frac{2}{R^2 - |\zeta_b|^2 + \delta_b} > \frac{2}{R^2 - |\zeta_b|^2} - \varepsilon .$$

From (7.6), (7.7) and (7.8) it follows that

$$(7.9) \quad |p_1(\zeta_b)| > \frac{2}{R^2 - |\zeta_b|^2} - \varepsilon .$$

As $\zeta_b \in D$, (7.4) and (7.9) are incompatible, so that no such nontrivial solution $w(z)$ of (1.2) exists.

8. A sufficient condition for disconjugacy (II). In [4], p. 127, ex. 8, the following theorem is mentioned: *Let $p(z)$ be analytic in $|z| \leq 1$. Then*

$$(8.1) \quad |p(z)| \leq \frac{\int_0^{2\pi} |p(e^{i\theta})| d\theta}{2\pi(1 - |z|^2)}, \quad |z| < 1 .$$

In paragraph 3 the integral $\int_0^{2\pi} |p(e^{i\theta})| d\theta$ was defined for functions analytic in the open unit circle. It is easily seen that if we use the above definition for the integral in the right-hand side of (8.1), then (8.1) is also valid for functions analytic in the open unit circle.

From (8.1) and from Theorem 5, we obtain now the following disconjugacy theorem:

THEOREM 6. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be disconjugate in $|z| < 1$ is:*

$$(8.2) \quad \int_0^{2\pi} |p(e^{i\theta})| d\theta \leq 4\pi .$$

(The integral in (8.2) was defined in paragraph 3).

Proof. By (8.1), for functions analytic in $|z| < 1$, and by (8.2), we have:

$$|p(z)| \leq \frac{\int_0^{2\pi} |p(e^{i\theta})| d\theta}{2\pi(1 - |z|^2)} \leq \frac{2}{1 - |z|^2}, \quad |z| < 1 .$$

The validity of (7.1) is thus proved, and therefore, by Theorem 5, (1.2)

is disconjugate in $|z| < 1$.

REMARK 1. Theorem 6 improves Theorem 4 in [7]. (see Remark 2 in paragraph 3).

REMARK 2. The question whether the constant 4π in (8.2) is the best possible is left open. That it cannot be improved too much is shown by the example $p(z) \equiv \pi^2/4$. The corresponding equation (1.2) is disconjugate in $|z| < 1$, and

$$\int_0^{2\pi} |p(e^{i\theta})| d\theta = \frac{\pi^3}{2} \approx 4.9\pi .$$

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50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 12, No. 3

March, 1962

Alfred Aeppli, <i>Some exact sequences in cohomology theory for Kähler manifolds</i>	791
Paul Richard Beesack, <i>On the Green's function of an N-point boundary value problem</i>	801
James Robert Boen, <i>On p-automorphic p-groups</i>	813
James Robert Boen, Oscar S. Rothaus and John Griggs Thompson, <i>Further results on p-automorphic p-groups</i>	817
James Henry Bramble and Lawrence Edward Payne, <i>Bounds in the Neumann problem for second order uniformly elliptic operators</i>	823
Chen Chung Chang and H. Jerome (Howard) Keisler, <i>Applications of ultraproducts of pairs of cardinals to the theory of models</i>	835
Stephen Urban Chase, <i>On direct sums and products of modules</i>	847
Paul Civin, <i>Annihilators in the second conjugate algebra of a group algebra</i>	855
J. H. Curtiss, <i>Polynomial interpolation in points equidistributed on the unit circle</i>	863
Marion K. Fort, Jr., <i>Homogeneity of infinite products of manifolds with boundary</i>	879
James G. Glimm, <i>Families of induced representations</i>	885
Daniel E. Gorenstein, Reuben Sandler and William H. Mills, <i>On almost-commuting permutations</i>	913
Vincent C. Harris and M. V. Subba Rao, <i>Congruence properties of $\sigma_r(N)$</i>	925
Harry Hochstadt, <i>Fourier series with linearly dependent coefficients</i>	929
Kenneth Myron Hoffman and John Wermer, <i>A characterization of $C(X)$</i>	941
Robert Weldon Hunt, <i>The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order</i>	945
Edward Takashi Kobayashi, <i>A remark on the Nijenhuis tensor</i>	963
David London, <i>On the zeros of the solutions of $w''(z) + p(z)w(z) = 0$</i>	979
Gerald R. Mac Lane and Frank Beall Ryan, <i>On the radial limits of Blaschke products</i>	993
T. M. MacRobert, <i>Evaluation of an E-function when three of its upper parameters differ by integral values</i>	999
Robert W. McKelvey, <i>The spectra of minimal self-adjoint extensions of a symmetric operator</i>	1003
Adegoke Olubummo, <i>Operators of finite rank in a reflexive Banach space</i>	1023
David Alexander Pope, <i>On the approximation of function spaces in the calculus of variations</i>	1029
Bernard W. Roos and Ward C. Sangren, <i>Three spectral theorems for a pair of singular first-order differential equations</i>	1047
Arthur Argyle Sagle, <i>Simple Malcev algebras over fields of characteristic zero</i>	1057
Leo Sario, <i>Meromorphic functions and conformal metrics on Riemann surfaces</i>	1079
Richard Gordon Swan, <i>Factorization of polynomials over finite fields</i>	1099
S. C. Tang, <i>Some theorems on the ratio of empirical distribution to the theoretical distribution</i>	1107
Robert Charles Thompson, <i>Normal matrices and the normal basis in abelian number fields</i>	1115
Howard Gregory Tucker, <i>Absolute continuity of infinitely divisible distributions</i>	1125
Elliot Carl Weinberg, <i>Completely distributed lattice-ordered groups</i>	1131
James Howard Wells, <i>A note on the primes in a Banach algebra of measures</i>	1139
Horace C. Wisner, <i>Decomposition and homogeneity of continua on a 2-manifold</i>	1145