Pacific Journal of Mathematics

ON THE ZEROS OF THE SOLUTIONS OF $w''(z) + p(z)w(z) = 0$

DAVID LONDON

Vol. 12, No. 3 March 1962

ON THE ZEROS OF THE SOLUTIONS OF $w''(z) + p(z)w(z) = 0$

DAVID LONDON

1. Introduction. Let $f(z)$ be a meromorphic function with, at most, simple poles in a simply-connected domain D, such that $f'(z) \neq 0$ for $z \in D$. Let

$$p(z) = \frac{1}{2}\{f(z), z\} , \tag{1.1}$$

where

$$\{f(z), z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

is the Schwarzian derivative of $f(z)$ with respect to z. In connection with the function $f(z)$, we consider the differential equation

$$w''(z) + p(z)w(z) = 0 . \tag{1.2}$$

The function $f(z)$ may be written in the form

$$f(z) = \frac{w_1(z)}{w_2(z)} ,$$

where $w_1(z)$ and $w_2(z)$ are linearly independent solutions of (1.2). The nontrivial solution

$$w(z) = Aw_1(z) + Bw_2(z)$$

vanishes at $z_1, z_2, \cdots, z_n$ if, and only if, $f(z)$ takes at these points the value $-BA^{-1}$. Hence, it follows that *$f(z)$ takes some value in D n times if, and only if, there exists a nontrivial solution of* (1.2) *having n zeros in D.*
This connection was pointed out by Nehari in [3].

DEFINITION 1. *The equation* (1.2) *is disconjugate in D if every solution of* (1.2) *vanishes in D not more than once.*
Hence, (1.2) is disconjugate in D if, and only if, $f(z)$ is univalent in D.

DEFINITION 2. *The equation* (1.2) *is nonoscillatory in D if every*

Received November 9, 1961. This paper represents part of a thesis submitted to the Senate of the Technion-Israel Institute of Technology in partial fulfillment of the requirements for the degree of Master of Science. The author wishes to thank Professor B. Schwarz for his guidance in the preparation of this paper. Acknowledgment is also due to Professor E. Netanyahu and Professor E. Jabotinsky for their help.

solution of (1.2) *vanishes in D at most a finite number of times.* Hence, (1.2) is nonoscillatory in D if, and only if, $f(z)$ is finitely-valent in D.

By imposing restrictions on $p(z)$, disconjugacy and nonoscillation theorems can be obtained. By the above connection, these theorems are equivalent to theorems about the distribution of the values of $f(z)$. In this paper these theorems will be formulated as disconjugacy and nonoscillation theorems only, and not as theorems regarding the values of $f(z)$.

2. A lemma.

The lemma to be proved in this paragraph yields an upper bound for $|p(z)|$ on $|z| = \rho$, $\rho < 1$. This bound is connected with the *area integral* $\iint |p(z)|\, dxdy$.

LEMMA 1. *Let $p(z)$ be analytic in $|z| < 1$. Then*

$$|p(z')| \leqq \frac{\iint_{|z|<1} |p(z)|\, dxdy}{\pi(1-|z'|^2)^2}\,, \qquad |z'| < 1\,, \qquad (z = x + iy)\,. \tag{2.1}$$

Proof. Let

$$p(z) = \sum_{k=0}^{\infty} a_k z^k\,.$$

Then

$$\int_0^{2\pi} p(\rho\, e^{i\theta}) d\theta = 2\pi a_0\,, \qquad 0 \leqq \rho < 1\,,$$

and therefore

$$2\pi |p(0)| \leqq \int_0^{2\pi} |p(\rho e^{i\theta})| d\theta\,.$$

Multiplying by $\rho d\rho$ and integrating, we obtain

$$|p(0)| \leqq \frac{\iint_{|z|<1} |p(z)| dxdy}{\pi}\,, \tag{2.2}$$

which proves (2.1) for $z' = 0$. The transformation

$$z = +\frac{\zeta + z'}{1 + \zeta\bar{z}'}\,, \qquad (\zeta = \xi + i\eta)$$

maps $|\zeta| < 1$ onto $|z| < 1$. Let

$$p_1(\zeta) = p[z(\zeta)]\left(\frac{dz}{d\zeta}\right)^2 = p(z)\left[\frac{1-|z'|^2}{(1+\zeta\bar{z}')^2}\right]^2\,. \tag{2.3}$$

We have

$$(2.4) \quad \iint_{|z|<1} |p(z)|dxdy = \iint_{|\zeta|<1} |p[z(\zeta)]| \left|\frac{dz}{d\zeta}\right|^2 d\xi d\eta = \iint_{|\zeta|<1} |p_1(\zeta)| d\xi d\eta ,$$

and

$$(2.5) \qquad p_1(0) = p(z')(1 - |z'|^2)^2 .$$

Applying (2.2) to the function $p_1(\zeta)$, and using (2.4) and (2.5) we obtain the required result (2.1).

REMARK 1. For the special case when the function $p(z)$ is a square of a function analytic in $|z| < 1$, Lemma 1 can be obtained using the Bergman kernel function of $|z| < 1$. (see [4], p. 261, ex. 4).

REMARK 2. (2.1) is a sharp inequality. It is easily proved that the sign of equality in (2.1) at a point $z' = z_0$, $|z_0| < 1$, occurs if (and only if) $p(z)$ is of the form

$$p(z) = c\left[\frac{1 - |z_0|^2}{(1 - z\bar{z}_0)^2}\right]^2 .$$

3. **A sufficient condition for disconjugacy (I).** In [3], Nehari proved the following theorem (Th. 1, [3], sufficient condition): *Let* $p(z)$ *be analytic in* $|z| < 1$. *A sufficient condition for* (1.2) *to be disconjugate in* $|z| < 1$ *is*:

$$(3.1) \qquad |p(z)| \leqq \frac{1}{(1 - |z|^2)^2} , \qquad |z| < 1 .$$

This theorem is sharp, as is shown by an example due to E. Hille [2].

From Lemma 1 and from Nehari's theorem, we obtain a disconjugacy theorem, in which the restriction on $p(z)$ is given by a condition on the area integral.

THEOREM 1. *Let* $p(z)$ *be analytic in* $|z| < 1$. *A sufficient condition for* (1.2) *to be disconjugate in* $|z| < 1$ *is*:

$$(3.2) \qquad \iint_{|z|<1} |p(z)|dxdy \leqq \pi .$$

Proof. From (2.1) and (3.2) it follows that

$$|p(z')| \leqq \frac{\iint_{|z|<1} |p(z)|dxdy}{\pi(1 - |z'|^2)^2} \leqq \frac{1}{(1 - |z'|^2)^2} , \qquad |z'| < 1 .$$

The assumption of Nehari's theorem is satisfied, and therefore (1.2) is disconjugate in $|z| < 1$.

REMARK 1. The question of the sharpness of Theorem 1 is still open. Although both, inequality (2.1) and Nehari's theorem, are sharp, it does not follow that Theorem 1, which is deduced from them, is sharp too.

REMARK 2. In [7] the following theorem (Th. 4, [7]) is proved: *Let* $p(z)$ *be analytic in* $|z| < 1$. *A sufficient condition for* (1.2) *to be disconjugate in* $|z| < 1$ *is*

$$\int_0^{2\pi} |p(e^{i\theta})|\, d\theta \leqq 4 \,. \tag{3.3}$$

The integral on the left-hand side of (3.3) is defined as the limit for $\rho \to 1$, of the nondecreasing function

$$A(\rho) = \int_0^{2\pi} |p(\rho e^{i\theta})|\, d\theta \,, \qquad \rho < 1 \,.$$

From our Theorem 1, it follows that the constant 4 in (3.3) can be improved to 2π. Indeed, if

$$\int_0^{\alpha\pi} |p(e^{i\theta})|\, d\theta \leqq 2\pi \,,$$

then

$$\int_0^{2\pi} |p(\rho e^{i\theta})|\, d\theta \leqq 2\pi \,, \qquad \rho < 1 \,.$$

This implies now the validity of (3.2), and therefore, by Theorem 1, (1.2) is disconjugate in $|z| < 1$.

The constant 2π is, however, not the best possible. In Theorem 6 it will be improved to 4π.

4. Invariance of the area integral.

We shall prove that the area integral is invariant under the transformation of (1.2), resulting from a linear mapping of the variable z.

Let

$$\zeta = \frac{az + b}{cz + d} \,, \qquad ad - bc \neq 0 \,, \qquad (\zeta = \xi + i\eta) \,, \tag{4.1}$$

be a linear transformation analytic in the simply-connected domain D. D is mapped by (4.1) onto D'. By this mapping (1.2) will be transformed into an equation of the form

$$W''(\zeta) + P(\zeta) W'(\zeta) + Q(\zeta) W(\zeta) = 0 \,, \tag{4.2}$$

where

$$W(\zeta)=w[z(\zeta)]\ .$$

By the further substitution

$$W(\zeta)=\frac{w_1(\zeta)}{a-\zeta c}\ , \tag{4.3}$$

equation (4.2) takes the form:

$$w_1''(\zeta)+p_1(\zeta)w_1(\zeta)=0\ . \tag{4.4}$$

The solutions $w(z)$ and $w_1(\zeta)$ vanish in D and D' respectively at points z and ζ connected by (4.1). By a simple calculation, or from the properties of the Schwarzian derivative, it follows that

$$p_1(\zeta)=p[z(\zeta)]\left(\frac{dz}{d\zeta}\right)^2\ , \tag{4.5}$$

and hence

$$\iint_D|p(z)|dxdy=\iint_{D'}|p[z(\zeta)]|\left|\frac{dz}{d\zeta}\right|^2 d\xi d\eta=\iint_{D'}|p_1(\zeta)|d\xi d\eta\ .$$

We have thus proved that

$$\iint_D|p(z)|dxdy=\iint_{D'}|p_1(\zeta)|d\xi d\eta\ . \tag{4.6}$$

The property expressed by (4.6) is the *invariance of the area integral.*

The invariance of the area integral yields the following generalization of Theorem 1:

THEOREM 1′. *Let $p(z)$ be analytic in D, where D is a circle or a half plane. A sufficient condition for* (1.2) *to be disconjugate in D is*

$$\iint_D|p(z)|dxdy\leqq\pi\ . \tag{3.2$'$}$$

Proof. By a suitable linear transformation, D will be mapped onto the unit circle. From the invariance of the area integral, from Theorem 1, and from the fact that the solutions of (1.2) and (4.4) vanish at corresponding points, the desired result follows.

5. A theorem about zeros on the boundary of a certain domain. Using the invariance of the area integral and a theorem of Grunsky, we obtain a result regarding the zeros of the solutions of (1.2) on the boundary of a domain bounded by two orthogonal circular arcs.

THEOREM 2. *Let $p(z)$ be analytic in $\bar{D}$, where D is a domain bounded by two orthogonal circular arcs. If*

$$\iint_D |p(z)|dxdy \leqq 1 , \tag{5.1}$$

then no nontrivial solution of (1.2) *vanishes twice on one of the arcs bounding D.*

Proof. Let Γ be one of the two orthogonal arcs bounding D. Assume that there exists a nontrivial solution $w(z)$ of (1.2) and $z_1, z_2 \in \Gamma$, such that:

$$w(z_1) = w(z_2) = 0 .$$

Let D_1 be the domain bounded by the arc z_1z_2 of Γ, and by the arc passing through z_1 and z_2, orthogonal to Γ and lying within D. Let D_2 be the upper half of the unit circle. A suitable linear transformation maps D_1 onto D_2 so that z_1 and z_2 are mapped on ± 1. As

$$\iint_{D_1} |p(z)|\, dxdy \leqq 1 ,$$

it follows from the invariance of the area integral that we may assume, without loss of generality, that D is the upper half of the unit circle, and z_1, z_2 are ± 1.

We shall make use of the following theorem of Grunsky [1]: *Let $g(z)$ be analytic in a convex domain D. Let $z_1, z_2 \in D$ be such that*

$$g(z_1) = g(z_2) = 0 .$$

Let Δ be the triangle with vertices z_1, z_2 and z', $z' \in D$. Let A be the area of Δ. Then

$$2Ag(z') = (z' - z_1)(z' - z_2)\iint_\Delta g''(z)dxdy , \qquad (z = x + iy) . \tag{5.2}$$

From (5.2) we obtain here

$$\begin{aligned} 2Aw(z') &= (z' - 1)(z' + 1)\iint_\Delta w''(z)dxdy \\ &= -(z' - 1)(z' + 1)\iint_\Delta p(z)w(z)dxdy , \end{aligned}$$

and therefore

$$2A|w(z')| < |z' - 1|\,|z' + 1|\iint_\Delta |p(z)|\,|w(z)|dxdy . \tag{5.3}$$

Let z^* be a point on the boundary of D, at which $|w(z)|$ takes its maximum value in D. There are two possibilities:

I. z^* belongs to the circular part of the boundary of D.

II. z^* belongs to the diameter of D.

In case I, we get from (5.3):

$$|w(z^*)| < \frac{|z^* - 1||z^* + 1|}{2A} |w(z^*)| \iint_\Delta |p(z)| dxdy .$$

Noting that

$$\frac{|z^* - 1||z^* + 1|}{2A} = 1 ,$$

we obtain:

$$\iint_\Delta |p(z)| dxdy > 1 . \tag{5.4}$$

Inequalities (5.1) and (5.4) are incompatible, proving our theorem in this case.

In case II, we use the linear transformation

$$\zeta = \frac{1}{z + i} , \qquad (\zeta = \xi + i\eta) .$$

Let D' be the lower half of the circle $|\zeta + i/2| < 1/2$. The above transformation maps D onto D', so that the circular part of the boundary of D is mapped onto the diameter of D', and the diameter of D is mapped onto the circular part of the boundary of D'. The point z^*, which according to our assumption belongs to the diameter of D, is mapped on the point ζ^*, which belongs to the circular part of the boundary of D'. The points $z = \pm 1$ are mapped on the points $a = 1/2 - i/2$, $b = -1/2 - i/2$, which are the two endpoints of the diameter of D'. Equation (1.2) is transformed into equation (4.4), for which we have

$$w_1(a) = w_1(b) = 0 .$$

From (4.3) we get

$$w_1(\zeta) = w[z(\zeta)](-\zeta) , \tag{5.5}$$

and therefore:

$$|w_1(\zeta)| < |\zeta| |w(z^*)| . \tag{5.6}$$

Let $\zeta' = \xi - i/2$ be any point on the diameter of D'. We have:

$$|\zeta'| < |\zeta^*| . \tag{5.7}$$

From (5.5), (5.6) and (5.7) we obtain:

$$|w_1(\zeta')| < |\zeta'| |w(z^*)| < |\zeta^*| |w(z^*)| = |w_1(\zeta^*)| .$$

As ζ' is any point on the diameter of D', and ζ^* is a point on the circular part of the boundary of D', we conclude, from the last inequality, that

$|w_1(\zeta)|$ takes its maximum value in D' on the circular part of the boundary. From the invariance of the area integral it follows that

$$\iint_{D'} |p_1(\zeta)|d\xi d\eta \leqq 1 .$$

We have now the same situation as in case I, for which the proof has already been completed.

6. A sufficient condition for nonoscillation. Inequality (3.2) was seen to be a sufficient condition for (1.2) to be disconjugate in $|z| < 1$. The following result shows that mere boundedness of the integral appearing in (3.2) is sufficient to assure nonoscillation in $|z| < 1$.

THEOREM 3. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for* (1.2) *to be nonoscillatory in $|z| < 1$ is*:

$$\iint_{|z|<1} |p(z)|dxdy < \infty . \tag{6.1}$$

Proof. Assume that there exists a nontrivial solution $w(z)$ of (1.2) with infinitely many zeros in $|z| < 1$. The set of these zeros has an accumulation point α on $|z| = 1$.

From (6.1) follows the existence of a number ρ, $0 \leqq \rho < 1$, such that

$$\iint_{\rho \leqq |z| < 1} |p(z)|dxdy \leqq 1 . \tag{6.2}$$

It is obvious that at least one of the two halves of a circle, having for diameter the segment connecting two internal points of a given circle, lies inside the given circle.

As α is an accumulation point of the set of zeros, we can choose two elements of that set, z_1 and z_2, so that the half circle D, for which the segment connecting z_1 and z_2 is a diameter, and which lies in $|z| < 1$, will also lie in the circular ring $\rho < |z| < 1$.

Inequality (6.2) implies inequality (5.1) for this half circle D. D is thus a half circle for which (5.1) is satisfied, and on its diameter there exist two zeros of (1.2). This last fact is a contradiction to Theorem 2, so that no such nontrivial solution $w(z)$ of (1.2) exists.

REMARK 1. In [5] the following theorem (Th. 3, [5]) is proved: *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for* (1.2) *to be nonoscillatory in $|z| < 1$ is*:

$$\int_0^{2\pi} |p(e^{i\theta})|d\theta < \infty . \tag{6.3}$$

(The integral in (6.3) is defined in paragraph 3).

This theorem can be deduced from our Theorem 3. Indeed, (6.3) implies the existence of a bound M such that

$$\int_0^{2\pi} |p(\rho e^{i\theta})|d\theta < M , \qquad 0 \leqq \rho < 1 ,$$

and, therefore, such that

$$\iint_{|z|<1} |p(z)|dxdy < \frac{M}{2} .$$

The assumption of Theorem 3 is satisfied, and therefore (1.2) is nonoscillatory in $|z| < 1$.

From the invariance of the area integral follows the validity of Theorem 3 for every circle and every half plane. In the following theorem, Theorem 3 will be extended to more general domains.

THEOREM 4. *Let* $p(z)$ *be analytic in a domain* D *bounded by an analytic Jordan curve. A sufficient condition for* (1.2) *to be nonoscillatory in* D *is*:

$$\iint_D |p(z)|dxdy < \infty . \tag{6.1$'$}$$

Proof. Let $\zeta = \psi(z)$ be a function mapping D onto $|\zeta| < 1$.

In paragraph 4 we described the transformation of (1.2) by a linear mapping. The transformation of (1.2) by a general mapping $\zeta = \psi(z)$ may be performed in a similar way. In the general case we have to change (4.3) into

$$W(\zeta) = w_1(\zeta)e^{-\frac{1}{2}\int P(\zeta)d\zeta} . \tag{4.3$'$}$$

Equation (1.2) is transformed into an equation of the form (4.4), but (4.5) becomes now

$$p_1(\zeta) = \frac{1}{[\psi'(z)]^2}\left[p(z) - \frac{1}{2}\{\psi(z), z\}\right] . \tag{4.5$'$}$$

As the corresponding solutions of (1.2) and (4.4) vanish at corresponding points, equation (1.2) is nonoscillatory in D if, and only if, equation (4.4) is nonoscillatory in $|\zeta| < 1$. In order to prove that (4.4) is nonoscillatory in $|\zeta| < 1$, it is sufficient, by Theorem 3, to show that

$$\iint_{|\zeta|<1} |p_1(\zeta)|d\xi d\eta < \infty . \tag{6.1$''$}$$

From (4.5)′ we have

$$\iint_{|\zeta|<1} |p_1(\zeta)| d\xi d\eta = \iint_D \left| p(z) - \frac{1}{2}\{\psi(z), z\} \right| dxdy \tag{6.4}$$
$$\leqq \iint_D |p(z)| dxdy + \iint_D |\{\psi(z), z\}| dxdy \ .$$

As D is bounded by an analytic Jordan curve, the function $\zeta = \psi(z)$ is analytic in $\bar{D}$, and for $z \in \bar{D}$ $\psi'(z) \neq 0$, so that:

$$\iint_D |\{\psi(z), z\}| dxdy < \infty \ . \tag{6.5}$$

Inequality (6.1)″ now follows from (6.1)′, (6.4) and (6.5).

7. A theorem of Pokornyi. In [8] Pokornyi obtained a sufficient condition for (1.2) to be disconjugate in $|z| < 1$. This sufficient condition follows also from a more general theorem of Nehari (Th. 1, [6]). We give here an additional proof of Pokornyi's theorem.

THEOREM 5. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for (1.2) to be disconjugate in $|z| < 1$ is:*

$$|p(z)| \leqq \frac{2}{1 - |z|^2}\ , \qquad |z| < 1 \ . \tag{7.1}$$

Proof. Assume that there exists a nontrivial solution $w(z)$ of (1.2), and z_1, z_2, $|z_1|, |z_2| < 1$, $z_1 \neq z_2$, such that

$$w(z_1) = w(z_2) = 0 \ .$$

The points z_1 and z_2 determine uniquely a circle passing through them and orthogonal to $|z| = 1$. We denote by C the part of this circle within $|z| = 1$. We may assume, without loss of generality, that C is in the upper half plane and symmetric with respect to the imaginary axis (see [6]).

Let $i\rho$ be the point of C on the imaginary axis. The linear transformation

$$\zeta = \frac{z - i\rho}{1 + i\rho z}$$

maps the unit circle onto itself, and maps z_1 and z_2 on ρ_1 and ρ_2, $-1 < \rho_1 < \rho_2 < +1$. Equation (1.2) is transformed into equation (4.4), for which

$$w_1(\rho_1) = w_1(\rho_2) = 0 \ .$$

By (4.5) we have:

$$|p_1(\zeta)| = |p(z)|\left|\frac{dz}{d\zeta}\right|^2 = |p(z)|\left(\frac{1-|z|^2}{1-|\zeta|^2}\right)^2 \leqq \frac{2}{1-|z|^2}\left(\frac{1-|z|^2}{1-|\zeta|^2}\right)^2$$
$$= \frac{2}{1-|\zeta|^2}\,\frac{1-|z|^2}{1-|\zeta|^2}\,.$$

For $-1 < \zeta < +1$, we have

$$|\zeta| \leqq |z|\,,$$

and therefore

$$|p_1(\zeta)| \leqq \frac{2}{1-|\zeta|^2}\,, \qquad -1 < \zeta < +1\,. \tag{7.2}$$

Let $0 < R < 1$ be such that $-R < \rho_1 < \rho_2 < R$. From (7.2) it follows that:

$$|p_1(\zeta)| < \frac{2}{R^2-|\zeta|^2}\,, \qquad \rho_1 \leqq \zeta \leqq \rho_2\,. \tag{7.3}$$

The strict inequality in (7.3) assures the existence of an $\varepsilon > 0$ and of a neighbourhood D of the segment $[\rho_1, \rho_2]$, such that

$$|p_1(\zeta)| \leqq \frac{2}{R^2-|\zeta|^2} - \varepsilon\,, \qquad \xi \in D\,. \tag{7.4}$$

From Grunsky's theorem, quoted above, we obtain

$$2Aw_1(\zeta') = -(\zeta'-\rho_1)(\zeta'-\rho_2)\iint_\Delta p_1(\zeta)w_1(\zeta)d\xi d\eta,\ |\zeta'| < 1\,. \tag{7.5}$$

The domain of integration Δ is the triangle with vertices at ρ_1, ρ_2, ζ'. A is the area of Δ. Let D_b be an ellipse having $[\rho_1, \rho_2]$ as its major axis, and let the magnitude of its minor axis be $2b$, $b > 0$. For a small enough b, we have $D_b \subset D$. Let ζ_b be a point on the boundary of D_b, at which $|p_1(\zeta)w_1(\zeta)|$ takes its maximum value in D_b. From (7.5) it follows that

$$2A|w_1(\zeta_b)| < |\zeta_b-\rho_1|\,|\zeta_b-\rho_2|\,|p_1(\zeta_b)|\,|w_1(\zeta_b)|A\,.$$

Hence,

$$|p_1(\zeta_b)| > \frac{2}{|\zeta_b-\rho_1|\,|\zeta_b-\rho_2|}\,,$$

and therefore

$$|p_1(\zeta_b)| > \frac{2}{|R^2-\zeta_b^2|}\,. \tag{7.6}$$

We define the number δ_b by the equation

$$|R^2 - \zeta_b^2| = R^2 - |\zeta_b|^2 + \delta_b \,. \tag{7.7}$$

It is obvious that $\delta_b > 0$, and that $\delta_b \to 0$, for $b \to 0$. For a small enough b we have

$$\frac{2}{R^2 - |\zeta_b|^2 + \delta_b} > \frac{2}{R^2 - |\zeta_b|^2} - \varepsilon \,. \tag{7.8}$$

From (7.6), (7.7) and (7.8) it follows that

$$|p_1(\zeta_b)| > \frac{2}{R^2 - |\zeta_b|^2} - \varepsilon \,. \tag{7.9}$$

As $\zeta_b \in D$, (7.4) and (7.9) are incompatible, so that no such nontrivial solution $w(z)$ of (1.2) exists.

8. A sufficient condition for disconjugacy (II). In [4], p. 127, ex. 8, the following theorem is mentioned: *Let $p(z)$ be analytic in $|z| \leqq 1$. Then*

$$|p(z)| \leqq \frac{\int_0^{2\pi} |p(e^{i\theta})|d\theta}{2\pi(1 - |z|^2)} \,, \qquad |z| < 1 \,. \tag{8.1}$$

In paragraph 3 the integral $\int_0^{2\pi} |p(e^{i\theta})|d\theta$ was defined for functions analytic in the open unit circle. It is easily seen that if we use the above definition for the integral in the right-hand side of (8.1), then (8.1) is also valid for functions analytic in the open unit circle.

From (8.1) and from Theorem 5, we obtain now the following disconjugacy theorem:

THEOREM 6. *Let $p(z)$ be analytic in $|z| < 1$. A sufficient condition for* (1.2) *to be disconjugate in $|z| < 1$ is*:

$$\int_0^{2\pi} |(p(e^{i\theta})|d\theta \leqq 4\pi \,. \tag{8.2}$$

(The integral in (8.2) was defined in paragraph 3).

Proof. By (8.1), for functions analytic in $|z| < 1$, and by (8.2), we have:

$$|p(z)| \leqq \frac{\int_0^{2\pi} |p(e^{i\theta})|d\theta}{2\pi(1 - |z|^2)} \leqq \frac{2}{1 - |z|^2} \,, \qquad |z| < 1 \,.$$

The validity of (7.1) is thus proved, and therefore, by Theorem 5, (1.2) is disconjugate in $|z| < 1$.

REMARK 1. Theorem 6 improves Theorem 4 in [7]. (see Remark 2 in paragraph 3).

REMARK 2. The question whether the constant 4π in (8.2) is the best possible is left open. That it cannot be improved too much is shown by the example $p(z) \equiv \pi^2/4$. The corresponding equation (1.2) is disconjugate in $|z| < 1$, and

$$\int_0^{2\pi} |p(e^{i\theta})| d\theta = \frac{\pi^3}{2} \approx 4.9\pi .$$

REFERENCES

1. H. Grunsky, *Eine Funktionentheoretische Integralformel*, Math. Zeitschr., **63** (1955), 320-323.
2. E. Hille, *Remarks on a paper by Zeev Nehari*, Bull. Amer. Math. Soc., **55** (1949), 552-553.
3. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc., **55** (1949), 545-551.
4. ———, *Conformal Mapping*, 1st ed., Mc Graw-Hill Book Company, INC., 1952.
5. ———, *On the zeros of solutions of second order linear differential equations*, Amer. J. Math., **76** (1954), 689-697.
6. ———, *Some criteria of univalence*, Proc. Amer. Math. Soc., **5** (1954), 700-704.
7. ———, *Univalent functions and linear differential equations.* Lectures on Functions of a Complex Variable, Ann. Arbor. (1955), 49-60.
8. V. V. Pokornyi, *On some sufficient conditions for univalence*, Doklady Akademii Nauk SSSR (N.S.) **79** (1951), 743-746.

TECHNION-ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Pacific Journal of Mathematics
Vol. 12, No. 3 March, 1962