

# Pacific Journal of Mathematics

**ON THE RADIAL LIMITS OF BLASCHKE PRODUCTS**

GERALD R. MAC LANE AND FRANK BEALL RYAN

# ON THE RADIAL LIMITS OF BLASCHKE PRODUCTS

G. R. MACLANE AND F. B. RYAN

**1. Introduction.** As is well known, a Blaschke product  $f(z)$  in  $\{|z| < 1\}$  has radial limits  $f(e^{i\theta})$  of modulus one almost everywhere on  $\{|z| = 1\}$ . The object of the present paper is to give a partial answer to the question: how many times does  $f(z)$  assume a given radial limit? We shall prove the following theorem.

**THEOREM A.** *Let  $E$  be a given closed set on  $\{|w| = 1\}$  and let  $E'$  be the complement of  $E$  relative to  $\{|w| = 1\}$ . Then there exists a Blaschke product  $f(z)$ , all of whose radial limits are of modulus one, and such that the set*

$$L(\beta) = \{\theta \mid f(e^{i\theta}) = e^{i\beta}\}$$

*has the power of the continuum for  $e^{i\beta} \in E$  and is countable for  $e^{i\beta} \in E'$ .*

Theorem A is a condensed statement of what we shall actually prove; Theorems 1, 2, and 3 contain somewhat more information on  $f(z)$ . The method of proof is to construct a suitable regularly-branched covering  $\mathscr{W}$  of  $\{|w| < 1\}$ , corresponding to an automorphic function  $w = f(z)$ , and then use the geometry of  $\mathscr{W}$  to obtain our results.

The question naturally arises as to whether one could prove Theorem A directly. That is: could one produce an  $f(z)$  with the desired properties by exhibiting its zeros instead of defining  $f(z)$  by means of a surface  $\mathscr{W}$ ? The answer to this question does not seem to be obvious.

**2. The surface  $\mathscr{W}$ .** Let  $E$  be a given *nonvoid* closed subset of  $\{|w| = 1\}$  and let  $\{a_n\}_1^\infty$  be an infinite sequence of points in  $\{|w| < 1\}$  whose derived set is  $E$ . Clearly, we may assume that  $a_n \neq 0$  and

$$(1) \quad \arg a_m \neq \arg a_n \quad (m \neq n).$$

Let  $\mathscr{W}$  be the simply-connected unbordered covering of  $\{|w| < 1\}$  which is regularly-branched over the points  $\{a_n\}$  with all branch points of multiplicity 2. It is well known [2, 3, 6] that such a covering, with any specified multiplicity or signature for each  $a_n$ , exists and is unique. Instead of appealing to the general theory of regularly-branched coverings, we shall construct the surface  $\mathscr{W}$  directly, since the details of the construction play a role in the proof of Theorem A.

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Let  $C_n$  be the radial segment  $\arg w = \arg a_n$ ,  $|a_n| \leq |w| < 1$ . The  $C_n$  are disjoint because of (1). We make cuts in  $\{|w| < 1\}$  along each  $C_n$  and so obtain a slit disc  $W$ , copies of which are joined together, according to the following specifications, to form the surface.

**0<sup>th</sup> level.** The surface  $\mathscr{W}_0$  consists of just one slit disc  $W$ . Note that  $\mathscr{W}_0$  is simply-connected.

**1<sup>st</sup> level.** The surface  $\mathscr{W}_1$  is obtained by adjoining an infinite sequence of *distinct* copies of  $W$ , namely  $W(1), W(2), \dots$ , to  $\mathscr{W}_0$ .  $W(n_1)$  is joined to  $\mathscr{W}_0$  along  $C_{n_1}$  so as to form a first-order branch-point over  $a_{n_1}$ . The surface  $\mathscr{W}_1 = \mathscr{W}_0 \cup \bigcup_n W(n)$  is simply-connected; for by adjoining the  $W(n)$  one at a time we obtain an increasing sequence of simply-connected surfaces which exhaust  $\mathscr{W}_1$ . We denote by  $\chi(n_1)$  the curve in  $\mathscr{W}_1$  along which  $W(n_1)$  and  $\mathscr{W}_0$  are identified.

**2<sup>nd</sup> level.** Along each free slit on the boundary of  $\mathscr{W}_1$  we adjoin a copy of  $W$ . More precisely, the sheet  $W(n_1, n_2)$  is adjoined to  $W(n_1)$  along the cut  $C_{n_2}$  in  $W(n_1)$ . The added sheets correspond one-to-one with all pairs  $(n_1, n_2)$  of positive integers such that  $n_1 \neq n_2$ . Again we see that the surface  $\mathscr{W}_2 = \mathscr{W}_1 \cup \bigcup W(n_1, n_2)$  is simply-connected. The curve over  $C_{n_2}$  along which  $W(n_1)$  and  $W(n_1, n_2)$  are joined is denoted by  $\chi(n_1, n_2)$ .

**$k^{\text{th}}$  level.** Continuing the construction, the surface  $\mathscr{W}_k$  consists of  $\mathscr{W}_{k-1}$  and copies of  $W$  denoted by  $W(n_1, n_2, \dots, n_k)$ ,  $n_i \neq n_{i+1}$ , which are joined to  $\mathscr{W}_{k-1}$ ;  $W(n_1, \dots, n_k)$  is adjoined to  $W(n_1, \dots, n_{k-1})$  along the cut  $C_{n_k}$  in  $W(n_1, \dots, n_{k-1})$ . Denote the curve along which those two sheets are joined by  $\chi(n_1, n_2, \dots, n_k)$ . Clearly  $\mathscr{W}_k$  is simply-connected.

We take the surface  $\mathscr{W}$  to be  $\lim \mathscr{W}_k$  as  $k \rightarrow \infty$ ; it is clear that  $\mathscr{W}$  is simply-connected as  $\mathscr{W}_k \uparrow \mathscr{W}$ . With the natural projection map onto  $\{|w| < 1\}$  it is clear that  $\mathscr{W}$  is a regularly-branched, unbordered, covering of  $\{|w| < 1\}$ . All points of  $\mathscr{W}$  over the  $a_n$  are branch-points of multiplicity 2, and  $\mathscr{W}$  has no other branch-points.

**3. The function  $f(z)$ .** Since  $\mathscr{W}$  is a covering of  $\{|w| < 1\}$  it is hyperbolic. Let  $w = f(z)$  be the holomorphic function which maps  $\{|z| < 1\}$  onto  $\mathscr{W}$ , with  $f(0) = 0 \in \mathscr{W}_0$  and  $f'(0) > 0$ . Clearly  $|f(z)| < 1$ . The radial limits of  $f(z)$  are *all* of modulus one, since if this were not the case a boundary point of  $\{|z| < 1\}$  would correspond to an interior point of  $\mathscr{W}$  which is unbordered. Thus  $f(z)$  is of class  $U$  [5, p. 32]. Applying Frostman's theorem [5, p. 33] we see that  $f(z)$  is a Blaschke product.

Also,  $f(z)$  is an automorphic function with respect to a Fuschian group  $F$ , since the decktransformations of  $\mathscr{W}$  correspond to linear

transformations preserving  $\{|z| < 1\}$ . It is easily shown that if  $E = \{|w| = 1\}$  then  $F$  is of the first kind: the limit points of  $F$  fill  $\{|z| = 1\}$ . If  $E \neq \{|w| = 1\}$  then  $F$  is of the second kind: the set of limit points of  $F$  is a perfect nowhere dense subset of  $\{|z| = 1\}$ .

The sheets  $W(n_1, n_2, \dots, n_k)$  of  $\mathscr{W}$  correspond to a set of fundamental regions  $R(n_1, \dots, n_k)$  of  $F$ . These are the fundamental regions which play a role in the proof; since these are defined via the function  $f$  it is not clear that they are the same as the fundamental regions obtained by any of the usual constructions in terms of  $F$ . Hence we must derive some properties of these regions.

**4. Properties of the fundamental regions.** For convenience we reduce the notations  $W(n_1, \dots, n_k)$ ,  $R(n_1, \dots, n_k)$ , and  $\chi(n_1, \dots, n_k)$  to  $W$ ,  $R$ , and  $\chi$  respectively. To each curve  $\chi$  in  $\mathscr{W}$  there corresponds a simple arc  $X$  in  $\{|z| < 1\}$ . It is evident that the fundamental regions  $R$  are bounded by the  $X$ 's and points of  $\{|z| = 1\}$ . We proceed with an investigation of the  $X$ 's.

First, *each  $X$  ends at two distinct points of  $\{|z| = 1\}$* . The two linear pieces of  $\chi$  correspond to two simple arcs  $X'$  and  $X''$ , and  $f(z)$  tends to a limit as  $|z| \rightarrow 1$  on  $X'$  and  $X''$ . Then by Koebe's lemma [1, p. 213] each of  $X'$  and  $X''$  must tend to a definite point of  $\{|z| = 1\}$ . The end points of  $X'$  and  $X''$  must be distinct. If not, let  $D$  be that part of  $\{|z| < 1\}$  bounded by  $X$  and a single point  $b$  on  $\{|z| = 1\}$ . Then the part of  $\mathscr{W}$  corresponding to  $D$  will contain an infinite number of sheets  $W$  joined along various  $\chi$ 's, which correspond to  $X$ 's, all ending at  $b$ . Thus  $f(z)$  would have infinitely many distinct asymptotic values, namely  $\exp(i \arg a_n)$ , at  $b$ ; but this would contradict the theorem of Lindelöf [4, p. 9] to the effect that a bounded holomorphic function can have at most one asymptotic value at a given point.

Thus each  $X$  is a crosscut of  $\{|z| < 1\}$ . A second property is that *no two  $X$ 's have a common endpoint*. To see this, suppose  $X_1$  and  $X_2$  are two distinct  $X$ 's with a common endpoint  $b$  on  $\{|z| = 1\}$ . Let the corresponding curves  $\chi_1$  and  $\chi_2$  in  $\mathscr{W}$  end at points  $\alpha_1$  and  $\alpha_2$ , respectively, over  $\{|w| = 1\}$ . If  $\alpha_1 \neq \alpha_2$  then we would again have a contradiction of Lindelöf's theorem. Now suppose  $\alpha_1 = \alpha_2$ . We may construct a sequence of arcs  $\Delta_n$  in  $\{|z| < 1\}$ , each joining a point of  $X_1$  to a point of  $X_2$ , such that  $\text{diam } \Delta_n \rightarrow 0$ . Since by Lindelöf's theorem  $f(z) \rightarrow \alpha_1$  uniformly between  $X_1$  and  $X_2$  we may also require  $\text{diam } \{f(\Delta_n)\} < 1/n$ . But from the structure of  $\mathscr{W}$  it is clear that there exists a curve  $\chi$  on  $\mathscr{W}$ , with endpoint  $\neq \alpha_1$ , such that *any* curve on  $\mathscr{W}$ , joining a point of  $\chi_1$  to a point of  $\chi_2$ , must intersect  $\chi$ . Since the projection of  $\chi$  into  $\{|w| < 1\}$  and the common projection of  $\chi_1$  and  $\chi_2$  are a positive distance  $\delta$  apart, we must have  $\text{diam } \{f(\Delta_n)\} \geq \delta$ , which is incompatible with  $\text{diam } \{f(\Delta_n)\} < 1/n$ .

Next, for any  $\varepsilon > 0$ , the set  $S = \{X \mid \text{diam } X > \varepsilon\}$  is finite. For, any disc  $\{|z| < 1 - \delta\}$  intersects only a finite number of the  $X$ 's. Hence if  $S$  were infinite there would exist an infinite sequence  $\{X_n\}_1^\infty$  of distinct crosscuts and a nondegenerate arc  $A$  on  $\{|z| = 1\}$  such that the radius joining  $z = 0$  to an arbitrary point of  $A$  crosses every  $X_n$ . Now any radial limit  $f(e^{i\theta}) = e^{i\alpha}$ ,  $e^{i\theta} \in A$ , forces the  $\chi_n$ , corresponding to  $X_n$  and ending at  $e^{i\alpha_n}$ , to satisfy  $\alpha_n \rightarrow \alpha$ . But then  $f(e^{i\theta}) = e^{i\alpha}$  for almost all  $e^{i\theta} \in A$ , which contradicts the theorem of F. and M. Riesz. The point of this paragraph is that if  $b$  is a limit point of  $F$ , then any neighborhood of  $b$  contains infinitely many complete fundamental regions  $R$ . There are at least some examples of Fuchsian groups possessing a set of fundamental regions (connected) whose diameters are bounded away from zero.

### 5. Properties of $f(z)$ on the boundary.

**THEOREM 1.** *Let  $b$  be a limit point of  $F$ ,  $U$  a neighborhood of  $b$ , and let  $e^{i\alpha} \in E$ . Then the set*

$$U \cap L(\alpha)$$

*has the power of the continuum.*

*Proof.* There exists a cross-cut  $X$ , corresponding to the curve  $\chi$  in  $\mathscr{W}$ , which separates  $\{|z| < 1\}$  into two domains, one of which,  $D$ , is contained in  $U$ . The corresponding part,  $\mathscr{D}$ , of  $\mathscr{W}$  contains infinitely many sheets. In  $\{|w| < 1\}$  we may select among the arcs  $C_n$  two sequences,  $\{C_n(0)\}_1^\infty$ , and  $\{C_n(1)\}_1^\infty$ , which satisfy either the following three conditions

(2) *the lengths of the  $C_n(0)$  and  $C_n(1)$  tend to zero,*

(3)  *$\arg C_n(0) \downarrow \alpha$ , and  $\arg C_n(1) \downarrow \alpha$ ,*

(4)  *$\arg C_{n+1}(0) < \arg C_n(1) < \arg C_n(0)$ ;*

or the same conditions with the arrows in (3) and the inequalities in (4) reversed. Such sequences  $\{C_n(\varepsilon)\}$ ,  $\varepsilon = 0, 1$  exist because of  $e^{i\alpha} \in E$ , the initial choice of  $\{a_n\}$ , and (1).

Now let  $\Gamma(\varepsilon) = \Gamma(\varepsilon_1, \varepsilon_2, \dots)$ ,  $\varepsilon_i = 0, 1$ , be an arc in  $\mathscr{D}$  with the properties:

(5)  *$\Gamma(\varepsilon)$  crosses, in order, curves  $\chi$  in  $\mathscr{D}$  over the arcs  $C_1(\varepsilon_1), C_2(\varepsilon_2), C_3(\varepsilon_3), \dots$ , and meets no other  $\chi$ 's.*

(6)  *$\Gamma(\varepsilon)$  tends to a point on the boundary of  $\mathscr{D}$  over  $e^{i\alpha}$ .*

This construction of  $\Gamma(\varepsilon)$  is possible by (2), (3), (4), and since all the curves  $\chi$  over  $\alpha < \arg w < \alpha + \delta$ ,  $\delta = \delta(\eta)$ , are of length  $< \eta$ .  $\Gamma(\varepsilon)$  corresponds to an arc  $A(\varepsilon)$  in  $\{|z| < 1\}$  which tends to a definite point  $b(\varepsilon) \in U \cap \{|z| = 1\}$ , since  $f(z) \rightarrow e^{i\alpha}$  on  $A(\varepsilon)$ . By a well-known theorem of Lindelöf [4, p. 10] then the radial limit of  $f(z)$  exists at  $b(\varepsilon)$  and has

the value of  $e^{i\alpha}$ .

By associating  $b(\varepsilon)$  with the dyadic expansion  $0.\varepsilon_1\varepsilon_2\varepsilon_3\cdots$ , we see that we have found a set of points  $b(\varepsilon)$  in  $U \cap \{|z| = 1\}$ , associated with the radial limit  $e^{i\alpha}$ , having the power of the continuum, provided that distinct sequences of  $\varepsilon$ 's correspond to distinct points  $b(\varepsilon)$ . To show that, let  $\{\varepsilon_i\}$  and  $\{\varepsilon'_i\}$  be two distinct sequences and let  $p$  be the smallest integer for which  $\varepsilon_p \neq \varepsilon'_p$ . Then  $C_p(\varepsilon_p)$  and  $C_p(\varepsilon'_p)$  are distinct and the corresponding crosscuts  $X_p(\varepsilon_p)$  and  $X_p(\varepsilon'_p)$  subtend two *disjoint* (recall the structure of  $\mathscr{W}$ ) closed arcs  $A_p$  and  $A'_p$  on  $U \cap \{|z| = 1\}$ . But  $b(\varepsilon) \in A_p$  and  $b(\varepsilon') \in A'_p$  and so  $b(\varepsilon) \neq b(\varepsilon')$ .

**THEOREM 2.** *Let  $b$  be a limit point of  $F$  and let  $U$  be a neighborhood of  $b$ . The set*

$$\{\theta | e^{i\theta} \in U, f(e^{i\theta}) \text{ does not exist}\}$$

*has the power of the continuum.*

*Proof.* Select three distinct arcs,  $C(0)$ ,  $C(1)$ ,  $C(2)$ , from among the arcs  $C_n$ . Suppose a curve  $\Gamma$  in  $\mathscr{W}$  meets, in succession, curves  $\chi$  over the arcs in the sequence

$$C(\varepsilon_1), C(\varepsilon_2), C(\varepsilon_3), \cdots \quad (\varepsilon_i = 0, 1, 2; \varepsilon_i \neq \varepsilon_{i+1})$$

and crosses no other  $\chi$ 's. To those curves  $\chi$  in  $\mathscr{W}$  which  $\Gamma$  meets there corresponds a sequence of crosscuts  $X_1, X_2, X_3, \cdots$ , which subtend arcs  $A_1, A_2, A_3, \cdots$  on  $\{|z| = 1\}$  satisfying the condition  $A_{n+1}^- \subset A_n^+$ . Also we choose  $\varepsilon_1 = 0$  and  $X_1$  fixed, in  $U$ , so that the image of  $\Gamma$  lies in  $U$ . The sequence  $\{\varepsilon_n\}$  then determines a unique point  $b(\varepsilon) = \bigcap A_n^o \in U$ . The radius to  $b(\varepsilon)$  intersects all  $X_n$ ; hence  $f(z)$  has no radial limit at  $b(\varepsilon)$ , for  $C(0)$ ,  $C(1)$ ,  $C(2)$  are all distinct and  $\varepsilon_i \neq \varepsilon_{i+1}$ . Now given the start of the sequence,  $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$ , there are two possible choices for  $\varepsilon_{p+1}$  and the two possible arcs  $A_{p+1}$  are disjoint. Thus distinct sequences  $\{\varepsilon_n\}$  yield distinct points  $b(\varepsilon)$ . The set of sequences  $\{\varepsilon_n\}$  has the power of the continuum.

**THEOREM 3.** *Let  $e^{i\alpha} \in E'$ . Then the set  $L(\alpha)$  is countable.*

*Proof.* Let  $U$  be a neighborhood of  $e^{i\alpha}$  containing none of the points  $a_n$ . Then  $\mathscr{W}$  contains a countable number of schlicht components  $\mathscr{U}_1, \mathscr{U}_2, \cdots$  over  $U \cap \{|w| < 1\}$ . Each  $\mathscr{U}_n$  maps onto  $V_n \subset \{|z| < 1\}$ , where  $V_n$  is bounded by an arc  $A_n$  of  $\{|z| = 1\}$  and a crosscut of  $\{|z| < 1\}$ . The function  $f(z)$  is holomorphic on  $A_n$  and there is just one radius, ending on  $A_n$ , associated with the radial limit  $e^{i\alpha}$ . Since  $\mathscr{W}$  contains only this countable collection of components over  $U$ , the result is clear.

We remark that if  $E$  is void, then the use of a two-point set  $\{a_1, a_2\}$  leads to a Blaschke product satisfying Theorem 3. With a three-point set we can satisfy both Theorem 2 and Theorem 3. Theorem 1 is of course vacuous.

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