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## **THE SPECTRA OF MINIMAL SELF-ADJOINT EXTENSIONS OF A SYMMETRIC OPERATOR**

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**1. Introduction.** Let  $T$  be a closed symmetric operator with domain  $D_T$  dense in a Hilbert space  $\mathcal{H}$ . A (*generalized*) *spectral resolution* of  $T$  is a family of bounded self-adjoint operators  $E_\mu$  defined for  $-\infty < \mu < \infty$  and such that:

(a)  $E_\mu$  is nondecreasing, continuous from the right, and  $E_{-\infty} = 0$ ,  $E_\infty = 1$ .

(b) For  $u \in D_T$  and  $v \in \mathcal{H}$ ,

$$(Tu, v) = \int_{-\infty}^{\infty} \mu d(E_\mu u, v), \quad \|Tu\|^2 = \int_{-\infty}^{\infty} \mu^2 d(E_\mu u, u).$$

When in particular  $T$  is self-adjoint, it possesses only one generalized spectral resolution, namely the *orthogonal* spectral resolution where  $E_\mu$  is for each  $\mu$  an orthogonal projection. For an account of the theory of generalized resolutions see [1], Appendix I.

M. A. Naimark has shown that for each generalized resolution  $E_\mu$  there is at least one self-adjoint extension  $T^+$  of  $T$  in a Hilbert space  $\mathcal{H}^+ \supset \mathcal{H}$  with the following property: If  $E_\mu^+$  is the orthogonal resolution of  $T^+$  and  $P$  is the projection onto the subspace  $\mathcal{H}$  of  $\mathcal{H}^+$ , then  $E_\mu = PE_\mu^+$ . We shall usually require that  $T^+$  be a *minimal* self-adjoint extension of  $T$ , i.e. that  $\mathcal{H}^+$  be the closed linear hull of the set of vectors  $E_\mu^+ \mathcal{H}$ , ( $-\infty < \mu < \infty$ ); (see § 3). The minimal extension  $T^+$  corresponding to a given  $E_\mu$  is determined by  $E_\mu$  uniquely, up to unitary equivalence ([8], § 4). We shall denote it by  $T^+ = \psi(E_\mu)$ .

In this paper we investigate certain questions regarding the *spectrum*  $\Sigma$  of  $T^+ = \psi(E_\mu)$ . In view of the above mentioned unitary equivalence, the point set  $\Sigma$  depends only upon  $E_\mu$ ; it may in fact be characterized directly as the set of points of increase of  $E_\mu$  (see § 3). Parts of the spectrum—e.g. eigenvalues and essential spectrum—may likewise be characterized directly in terms of  $E_\mu$ . It will be convenient to refer to the spectrum of  $T^+$  as the *spectrum of  $E_\mu$* .

We are interested in comparing the spectra of various resolutions of a given  $T$ . In order to describe the situation precisely, one refers to A. V. Štraus' extension theory of symmetric operators [10]. For any complex  $\lambda$ , let  $A_T(\lambda)$  denote the range of  $T - \lambda$ . By definition, the *defect subspace*  $M(\lambda)$  is the orthogonal complement in  $\mathcal{H}$  of  $A_T(\lambda)$ .

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Štraus has associated with each generalized resolution  $E_\mu$  of  $T$  a family of contraction operators  $F_\lambda$ , mapping  $M(i)$  into  $M(-i)$ , and such that  $F_\lambda$  is analytic on  $\mathcal{S}_\lambda > 0$  with  $\|F_\lambda\| \leq 1$  there. Conversely each such family of contractions is associated with some  $E_\mu$ . A constant unitary  $F$  corresponds by this association to an orthogonal resolution  $E_\mu$ , and for these Štraus' extension theory reduces to that of J. von Neumann. (For a complete description, see § 2)

We characterize the spectral resolutions  $E_\mu$  of  $T$  by the behavior near the real axis of the corresponding  $F_\lambda$ . Specifically we single out two extreme cases, where  $F_\lambda$  satisfies, respectively, conditions  $\alpha$  and  $\beta$  or condition  $\gamma$  as defined in § 4. These are local conditions, defined for an open real interval  $\Delta$ . When  $E_\mu$  is an orthogonal resolution, conditions  $\alpha$  and  $\beta$  hold on the entire real axis.

In §§ 4–6 we consider a symmetric operator  $T$  with equal finite-defect numbers  $(n, n)$ . In § 4 we extend to generalized resolutions of  $T$ , satisfying conditions  $\alpha$  and  $\beta$  on an interval  $\Delta$ , the theorem of H. Weyl [13] on the invariance of essential spectrum. In § 6 we obtain a parallel theorem on the invariance of absolutely continuous spectrum, proved for  $T$  a singular second order ordinary differential operator. This extends a theorem of N. Aronszajn [2]. (The theorems of Weyl and Aronszajn both concern self-adjoint extensions of  $T$  in  $\mathcal{H}$ , hence orthogonal resolutions.)

When  $F_\lambda$  is such that  $\alpha$  and  $\beta$  fail everywhere on an interval  $\Delta$  an altogether different pattern emerges, for in this case  $\Delta$  lies entirely within the spectrum of  $E_\mu$ . In § 5 we adopt the more stringent assumption that condition  $\gamma$  holds on  $\Delta$ . In particular, suppose  $F_\lambda$  is a family of strict contractions, i.e. satisfies condition  $\gamma$  on the entire real axis. Suppose that  $T - \mu$  has a bounded inverse for each real  $\mu$ . Then  $T^+ = \psi(E_\mu)$  is unitarily equivalent to the  $n$ -fold direct sum of  $iD$  with itself, where  $D$  is the differentiation operator in  $L_2(-\infty, \infty)$ . This generalizes a theorem proved by Coddington and Gilbert ([4], Theorem 14) for  $T$  a regular ordinary differential operator of order  $n$ . As is indicated in § 6, the situation is more complicated when  $T$  is a singular differential operator.

The study of the spectrum of  $E_\mu$  requires an analysis of the behavior of the resolvent  $R_\lambda$  of  $E_\mu$  near the real axis. The generalized resolvent  $R_\lambda$  of a spectral resolution  $E_\mu$  is defined for  $\mathcal{S}_\lambda \neq 0$  by

$$(1.1) \quad R_\lambda = \int_{-\infty}^{\infty} (\mu - \lambda)^{-1} dE_\mu .$$

Thus  $R_\lambda$  is a bounded operator with domain  $\mathcal{H}$ , analytic on each half plane  $\mathcal{S}_\lambda > 0$ ,  $\mathcal{S}_\lambda < 0$ . Inversely,  $E_\mu$  is determined by  $R_\lambda$  through the formula

$$(1.2) \quad (E(\Delta)u, u) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\pi} \int_{\Delta} \mathcal{S}(R_{\mu+i\varepsilon}u, u) d\mu$$

where  $\Delta$  is an interval  $(\mu_1, \mu_2]$ ,  $\mu_1$  and  $\mu_2$  are continuity points of  $E_\mu$ , and  $E(\Delta) = E_{\mu_2} - E_{\mu_1}$ . When  $T$  is self-adjoint, the (generalized) resolvent  $R_\lambda$  of its orthogonal resolution  $E_\mu$  coincides with the *resolvent* of  $T$ , i.e.  $R_\lambda = (T - \lambda)^{-1}$  for  $\mathcal{S}\lambda \neq 0$ .

Let  $T^+ = \check{\nu}(E_\mu)$ . The resolvents  $R_\lambda^+$  and  $R_\lambda$  of  $E_\mu^+$  and  $E_\mu$  are related, when  $\mathcal{S}\lambda \neq 0$ , by (see [1])

$$(1.3) \quad R_\lambda = PR_\lambda^+.$$

A. V. Štraus [10] has given another characterization of  $R_\lambda$ , when  $\mathcal{S}\lambda \neq 0$ , as the resolvent of a certain *quasi-self-adjoint extension*  $T_\lambda$  in  $\mathcal{H}$  of  $T$ . (For precise definition, see § 2). Thus

$$(1.4) \quad R_\lambda = (T_\lambda - \lambda)^{-1}.$$

In § 2 we investigate limit values, as  $\lambda$  tends to the real axis, of  $R_\lambda$ . It is found that in general the interpretation (1.3) fails for limit values while (1.4) retains its meaning. The interpretation (1.3) remains valid on a real interval  $\Delta$  precisely when  $R_\lambda$  can be continued analytically through  $\Delta$ , and this is possible precisely when  $\Delta$  lies in the complement of the spectrum of  $E_\mu$  (theorem 3.1).

It is a pleasure to express here my indebtedness to E. A. Coddington, who first drew my attention to generalized resolutions and in particular suggested that the theorem of Coddington and Gilbert, referred to above, might be valid in a broader setting. During the course of the work I have had access to his library and frequent benefit of his counsel.

**2. Limit values of the resolvent.** We shall designate an arbitrary one of the half planes  $\mathcal{S}\lambda > 0$ ,  $\mathcal{S}\lambda < 0$  by  $\pi^+$  and the other by  $\pi^-$ . Choose any  $\lambda_0 \in \pi^+$  and any contraction operator  $F$  (i.e.  $\|F\| \leq 1$ ) with domain  $M(\lambda_0)$  and values in  $M(\bar{\lambda}_0)$ . The operator  $\hat{T}$ , defined by

$$(2.1) \quad \begin{aligned} T &\subset \hat{T} \subset T^*, \\ D_{\hat{T}} &= \{u: u = u_0 + \phi - F\phi, u_0 \in D_T, \phi \in M(\lambda_0)\} \end{aligned}$$

has been called by A. V. Štraus a *quasi-self-adjoint extension* of  $T$ . The class  $C^+$  of operators  $\hat{T}$  obtained by holding  $\lambda_0$  fixed and varying  $F$  is, in fact, independent of the choice of  $\lambda_0 \in \pi^+$ . (See Štraus [10], Lemma 9 and the discussion preceding it). A second, and in general different, class  $C^-$  of quasi-self-adjoint extensions of  $T$  is obtained by taking  $\lambda_0 \in \pi^-$ .

Let  $R_\lambda$  be the resolvent of  $E_\mu$  of  $T$ . Štraus has proved that, to

each  $\lambda \in \pi^+$  corresponds a quasi-self-adjoint extension  $T_\lambda \in C^+$  such that

$$(2.2) \quad (T_\lambda - \lambda)^{-1} = R_\lambda, \quad \lambda \in \pi^+.$$

For a fixed choice of  $\lambda_0 \in \pi^+$ , the corresponding contraction  $F_\lambda = F_{\lambda}(\lambda_0)$  is analytic in  $\lambda$  on  $\pi^+$ . Conversely, any analytic contraction  $F_\lambda$  carrying  $M(\lambda_0)$  into  $M(\bar{\lambda}_0)$  gives rise, through (2.1) and (2.2) to a resolvent  $R_\lambda$  of  $T$ . The relation  $R_{\bar{\lambda}} = R_\lambda^*$  (which follows from (1.1) or (1.3)) has as its correspondent the relation

$$(2.3) \quad F_{\bar{\lambda}}(\bar{\lambda}_0) = [F_\lambda(\lambda_0)]^*$$

defining a contraction taking  $M(\bar{\lambda}_0)$  into  $M(\lambda_0)$ .

The following theorem shows that these statements remain valid in a limiting sense on the real axis.

**THEOREM 2.1.** *Let  $\lambda_1, \lambda_2, \dots$  in  $\pi^+$  tend to  $\hat{\lambda}$  on the real axis.*

(A) *Suppose that for a certain  $\lambda_0 \in \pi^+$  the sequence of contractions  $F_{\lambda_k}(\lambda_0)$  converges in norm as  $k \rightarrow \infty$ . Then the same is true for every  $\lambda'_0 \in \pi^+$ . The limit, also a contraction taking  $M(\lambda_0)$  into  $M(\bar{\lambda}_0)$ , will be denoted by  $F_{\hat{\lambda}+} = F_{\hat{\lambda}+}(\lambda_0)$ . It defines a quasi-self-adjoint extension in  $C^+$  of  $T$ , and the extension  $T_{\hat{\lambda}+}$  so obtained does not depend upon the particular  $\lambda_0 \in \pi^+$  figuring in its construction.*

(B) *Necessary and sufficient for the convergence in norm of  $R_{\lambda_k}$  to a limit, denoted by  $R_{\hat{\lambda}+}$ , is:*

- (i) *Convergence in norm of  $F_{\lambda_k}$ , and*
- (ii) *Existence of  $(T_{\hat{\lambda}+} - \hat{\lambda})^{-1}$  as a bounded operator with domain  $\mathcal{H}$ .*

*In this case,*

$$(2.4) \quad R_{\hat{\lambda}+} = (T_{\hat{\lambda}+} - \hat{\lambda})^{-1}.$$

(C) *In any subset of  $[\pi^+$  plus the real axis] in which both  $R_\lambda$  and  $F_\lambda$  are defined (by extension), the single-valuedness and continuity of either implies that of the other.*

(D) *When, as above,  $R_{\lambda_k}$  and  $F_{\lambda_k}(\lambda_0)$  tend to limits in norm, then the same is true of  $R_{\bar{\lambda}_k}$  and  $F_{\bar{\lambda}_k}(\bar{\lambda}_0)$ , and*

$$(2.5) \quad R_{\hat{\lambda}-} = [R_{\hat{\lambda}+}]^*, \quad F_{\hat{\lambda}-}(\bar{\lambda}_0) = [F_{\hat{\lambda}+}(\lambda_0)]^*.$$

*Proof.* (A) Let  $W(\lambda_0)$  denote the Cayley transform of a quasi-self-adjoint extension  $\hat{T}$  of  $T$ . Thus  $W = U \oplus F$ , where

$$U(\lambda_0) = (T - \bar{\lambda}_0)(T - \lambda_0)^{-1}$$

is the Cayley transform of  $T$ . One easily shows ([10], equation (5.22)), that for  $\lambda_0$  and  $\lambda'_0$  in  $\pi^+$ ,

$$(2.6) \quad W(\lambda'_0) = [(\bar{\lambda}'_0 - \bar{\lambda}_0) - (\bar{\lambda}'_0 - \lambda_0)W(\lambda_0)][(\lambda'_0 - \bar{\lambda}_0) - (\lambda'_0 - \lambda_0)W(\lambda_0)]^{-1}$$

where the inverse shown is a bounded operator with domain  $\mathcal{H}$ . Since this equation holds between  $W_{\lambda_k}(\lambda_0) = U(\lambda_0) \oplus F_{\lambda_k}(\lambda_0)$  and  $W_{\lambda_k}(\lambda'_0)$ , therefore by continuity  $W_{\hat{\lambda}+}(\lambda'_0) = \lim W_{\lambda_k}(\lambda'_0)$  exists. Furthermore  $W_{\hat{\lambda}+}(\lambda_0)$  and  $W_{\hat{\lambda}+}(\lambda'_0)$  are related by (2.6) and hence are Cayley transforms relative to  $\lambda_0$  and  $\lambda'_0$ , of the same  $\hat{T} = T_{\hat{\lambda}+}$ . Since  $W_{\lambda_k}(\lambda'_0) = U(\lambda'_0) \oplus F_{\lambda_k}(\lambda'_0)$  therefore  $F_{\hat{\lambda}+}(\lambda'_0) = \lim F_{\lambda_k}(\lambda'_0)$  exists, and  $W_{\hat{\lambda}+}(\lambda'_0) = U(\lambda'_0) \oplus F_{\hat{\lambda}+}(\lambda'_0)$ . Thus  $F_{\hat{\lambda}+}(\lambda_0)$  and  $F_{\hat{\lambda}+}(\lambda'_0)$  define the same extension of  $T$ .

(B)<sub>1</sub> Here we establish the necessity of the condition. Let  $\lambda_0 \in \pi^+$ . It follows from (2.2) that, for  $\lambda \in \pi^+$ ,

$$T_\lambda - \lambda_0 = (T_\lambda - \lambda) + (\lambda - \lambda_0) = [1 + (\lambda - \lambda_0)R_\lambda](T_\lambda - \lambda)$$

and therefore that

$$(T_\lambda - \lambda_0)^{-1} = R_\lambda[1 + (\lambda - \lambda_0)R_\lambda]^{-1}, \quad \lambda \in \pi^+.$$

Here  $[1 + (\lambda - \lambda_0)R_\lambda]^{-1}$  is bounded with domain  $\mathcal{H}$ . ([10], equation (5.30), footnote.). By assumption,  $\lambda_k \rightarrow \hat{\lambda}$  on the real axis, and  $R_{\lambda_k} \rightarrow R_{\hat{\lambda}+}$  in norm. By choosing a *special*  $\lambda_0$  for which  $|\hat{\lambda} - \lambda_0| \cdot \|R_{\hat{\lambda}+}\| < 1$ , we guarantee that  $[1 + (\hat{\lambda} - \lambda_0)R_{\hat{\lambda}+}]^{-1}$  exists, is bounded, and has domain  $\mathcal{H}$ . Consequently the operator

$$G_\lambda = R_\lambda[1 + (\lambda - \lambda_0)R_\lambda]^{-1}$$

is well defined for  $\lambda = \hat{\lambda} +$  as well as  $\lambda \in \pi^+$ , and  $G_{\lambda_k} \rightarrow G_{\hat{\lambda}+}$  in norm.

The Cayley transform  $W_\lambda(\lambda_0)$  of  $T_\lambda$  for  $\lambda \in \pi^+$ , is given by

$$W_\lambda = (T_\lambda - \bar{\lambda}_0)(T_\lambda - \lambda_0)^{-1} = 1 + (\lambda_0 - \bar{\lambda}_0)(T_\lambda - \lambda_0)^{-1}.$$

Hence

$$(2.7) \quad W_\lambda = 1 + (\lambda_0 - \bar{\lambda}_0)G_\lambda, \quad \text{for } \lambda \in \pi^+.$$

We define the transformation  $W_{\hat{\lambda}+}$  also by this formula, and show that  $W_{\hat{\lambda}+}$  is a quasi-unitary extension, with  $\|W_{\lambda+}\| \leq 1$  of the Cayley transform  $U(\lambda_0)$  of  $T$ . In fact, the statements

$$\|W_\lambda\| \leq 1; \quad W_\lambda f = U(\lambda_0)f \quad \text{for } f \in \mathcal{D}_T(\lambda_0)$$

are valid for  $\lambda \in \pi^+$  and, since by (2.7)  $W_{\lambda_k} \rightarrow W_{\hat{\lambda}+}$ , are valid for  $\lambda = \hat{\lambda} +$  as well. But by [10], Lemma 8, these statements imply that  $W_{\hat{\lambda}+}$  is a quasi-unitary extension of  $U(\lambda_0)$ .

Consequently  $W_{\hat{\lambda}+}$  is the Cayley transform of a quasi-self-adjoint extension (of class  $C^+$ ) of  $T$ . From the relation

$$W_\lambda = U \oplus F_\lambda \quad \text{for } \lambda \in \pi^+$$

it follows, since  $W_{\lambda_k} \rightarrow W_{\hat{\lambda}+}$ , that:  $F_{\hat{\lambda}+} = \lim F_{\lambda_k}$  exists and

$$W_{\hat{\lambda}+} = U \oplus F_{\hat{\lambda}+}.$$

Thus  $W_{\hat{\lambda}_+}$  is the Cayley transform of the extension which we have denoted (in  $A$ ) by  $T_{\hat{\lambda}_+}$ .

From the relation between any quasi-self-adjoint extension and its Cayley transform we have

$$W_{\hat{\lambda}_+} = (T_{\hat{\lambda}} - \bar{\lambda}_0)(T_{\hat{\lambda}_+} - \lambda_0)^{-1} = 1 + (\lambda_0 - \bar{\lambda}_0)(T_{\hat{\lambda}_+} - \lambda_0)^{-1}.$$

Comparing this relation with (2.7), we conclude that

$$(T_{\hat{\lambda}_+} - \lambda_0)^{-1} = G_{\hat{\lambda}_+} = R_{\hat{\lambda}_+}[1 + (\hat{\lambda} - \lambda_0)R_{\hat{\lambda}_+}]^{-1}.$$

From this it immediately follows that  $R_{\hat{\lambda}_+}^{-1}$  exists and that

$$T_{\hat{\lambda}_+} - \lambda_0 = [1 + (\hat{\lambda} - \lambda_0)R_{\hat{\lambda}_+}]R_{\hat{\lambda}_+}^{-1} = R_{\hat{\lambda}_+}^{-1} + (\hat{\lambda} - \lambda_0).$$

Hence

$$T_{\hat{\lambda}_+} - \hat{\lambda} = R_{\hat{\lambda}_+}^{-1},$$

or

$$R_{\hat{\lambda}_+} = (T_{\hat{\lambda}_+} - \hat{\lambda})^{-1}.$$

This shows the necessity of conditions (i) and (ii) for the special choice of  $\lambda_0$  made in the course of the argument. But from part (A), already proved, it follows that the conditions hold as well for any other  $\lambda_0 \in \pi^+$ .

(B)<sub>2</sub> In order to prove the sufficiency of the conditions (i) and (ii), we make use of the inverse relation between  $T_\lambda$  and its Cayley transform, namely

$$T_\lambda = (\lambda_0 W_\lambda - \bar{\lambda}_0)(W_\lambda - 1)^{-1} \quad \text{for } \lambda \in \pi^+ \text{ or } \lambda = \hat{\lambda} + .$$

(For notation, see the proof of part A). Hence

$$T_\lambda - \lambda = [(\lambda_0 - \lambda)W_\lambda + (\lambda - \bar{\lambda}_0)](W_\lambda - 1)^{-1}.$$

and, since  $(T_\lambda - \lambda)^{-1}$  exists (condition ii), therefore

$$(2.8) \quad (T_\lambda - \lambda)^{-1} = (W_\lambda - 1)[(\lambda - \bar{\lambda}_0) - (\lambda - \lambda_0)W_\lambda]^{-1} \quad \text{for } \lambda \in \pi^+ \text{ or } \lambda = \hat{\lambda} + .$$

Furthermore, since the inverse appearing on the left side of this equation is bounded with domain  $\mathcal{H}$ , the same is true for the inverse appearing on the right side. This fact, together with  $W_{\lambda_k} \rightarrow W_{\hat{\lambda}_+}$  in norm, shows that

$$R_{\hat{\lambda}_k} = (T_{\lambda_k} - \lambda_k)^{-1} \rightarrow (T_{\hat{\lambda}_+} - \hat{\lambda})^{-1},$$

which proves the proposition.

(C) This is a direct consequence of the reciprocal relations (2.7) and (2.8), namely

$$R_\lambda = (W_\lambda - 1)[(\lambda_0 - \lambda) - (\lambda - \bar{\lambda}_0)W_\lambda]^{-1}$$

$$W_\lambda = 1 + (\lambda - \lambda_0)R_\lambda[1 + (\lambda - \lambda_0)R_\lambda]^{-1}.$$

These are valid in  $\pi^+$  and under the assumptions of (C), are valid, in the limiting sense, on the entire set considered. Since the inverses displayed are bounded operators with domain  $\mathcal{H}$ , the assertion regarding continuity is evident.

REMARK 2.1. When  $R_{\lambda^+}$  exists (as a limit in norm) it is, by Theorem 2.1, an extension of  $(T - \hat{\lambda})^{-1}$ . This implies that  $\hat{\lambda}$  is a *point of regular type* of  $T$ , i.e. that  $(T - \hat{\lambda})^{-1}$  exists and is bounded. In particular (see [1], Chap. 7), the defect numbers of  $T$  are equal.

REMARK 2.2. Necessary and sufficient for the continuity of  $R_\lambda$  across an open interval  $\Delta$  of the real axis is:

- (i) Continuity of  $R_\lambda$  down to  $\Delta$  in  $\pi^+$  and
- (ii) Self-adjointness of  $R_{\lambda^+}$  on  $\Delta$ , i.e.  $R_{\lambda^+} = R_{\lambda^-}$ .

In the presence of (i), condition (ii) is equivalent to

- (ii)' Unitarity of  $F_{\lambda^+}$  on  $\Delta$ , i.e.  $(F_{\lambda^+})^{-1} = F_{\lambda^-}$ .

Under these conditions  $R_\lambda$  is in fact analytic across  $\Delta$ .

(One has only to consider  $(R_\lambda f, f)$ , which is analytic in  $\pi^+$  and  $\pi^-$  and continuous across  $\Delta$ )

**3. Resolvent set and spectrum.** By the *resolvent set* of a spectral resolution will be meant the points of  $\pi^+ \cup \pi^-$  plus any real point  $\lambda_0$  contained in an open real interval  $\Delta$  across which  $R_\lambda$  may be continued analytically. The *resolvent*  $R_\lambda$  at  $\lambda = \lambda_0$  is the common value of the limits  $R_{\lambda_0^+}$  and  $R_{\lambda_0^-}$  there.

In this paragraph we characterize the resolvent set, showing that it is the complementary point set of the *spectrum* of  $E_\mu$ , described in the introduction.

According to M. A. Naimark, the spectral family  $E_\lambda$  in  $\mathcal{H}$  may be regarded as the projection on  $\mathcal{H}$  of an orthogonal family  $E_\lambda^+$  in an enclosing space  $\mathcal{H}^+ \supset \mathcal{H}$ . Thus  $E_\lambda = PE_\lambda^+$ , where  $P$  is the orthogonal projection onto  $\mathcal{H}$ :  $P\mathcal{H}^+ = \mathcal{H}$ . The family  $E_\lambda^+$  is the spectral resolution of a self-adjoint operator  $T^+$  in  $\mathcal{H}^+$ . In the following we shall assume that  $T^+$  is a *minimal* self-adjoint extension of  $T$ , thus we assume that the set of vectors

$$\{E^+(\Delta)h: \Delta \text{ is any interval, } h \in \mathcal{H}\}$$

is fundamental in  $\mathcal{H}^+$ . In other words,  $\mathcal{H}^+$  is the closed linear hull of this set. (See Naimark [8], § 4).

LEMMA 3.1. *Let  $\Delta$  be a (possibly degenerate) interval of the real*



axis. Then

(A). The set of vectors

$$Z(\Delta) = \{E^+(\Delta')h: \Delta' \subset \Delta, h \in \mathcal{H}\}$$

is fundamental in  $E^+(\Delta)\mathcal{H}^+$ .

(B)  $E^+(\Delta) = 0$  if and only if  $E(\Delta) = 0$ .

*Proof.* (A) Given  $f \in E^+(\Delta)\mathcal{H}^+$ . For any  $\varepsilon > 0$  there exists  $g = \sum_{k=1}^n E^+(\Delta_k)g_k$ , for certain intervals  $\Delta_k$  and certain  $g_k \in \mathcal{H}$ , such that  $\|f - g\| < \varepsilon$ . We can write  $E^+(\Delta_k)g_k = E^+(\Delta_k \cap \Delta)g_k + E^+(\Delta - \Delta_k)g_k$ , and thus  $g = g^{(1)} + g^{(2)}$  with  $g^{(1)} = \sum_{j=1}^{n_1} E^+(\Delta'_j)g_j^{(1)}$  and  $g^{(2)} = \sum_{j=1}^{n_2} E^+(\Delta''_j)g_j^{(2)}$ , where  $\Delta'_j \subset \Delta$  and  $\Delta''_j \cap \Delta = 0$ . Thus  $g^{(1)} \in E^+(\Delta)\mathcal{H}^+$  while  $g^{(2)} \perp E^+(\Delta)\mathcal{H}^+$ , and  $\|f - g\|^2 = \|f - g^{(1)}\|^2 + \|g^{(2)}\|^2$ . It follows that  $\|f - g^{(1)}\| < \varepsilon$ , proving the proposition.

(B) (i) Suppose  $E(\Delta) > 0$ . Then there exists  $h \in \mathcal{H}$  such that  $0 < (E(\Delta)h, h) = (PE^+(\Delta)h, h) = (E^+(\Delta)h, h) = \|E^+(\Delta)h\|^2$ . Thus  $E^+(\Delta) > 0$ .

(ii) Suppose  $E(\Delta) = 0$ . Then for  $\Delta' \subset \Delta$ ,  $E(\Delta') = 0$  also. Hence for  $h \in \mathcal{H}$ ,  $0 = (E(\Delta')h, h) = (E^+(\Delta')h, h)$ , i.e.  $E^+(\Delta')h = 0$ . By part (A) this implies that  $E^+(\Delta) = 0$ .

**THEOREM 3.1.** A real point  $\hat{\lambda}$  of the resolvent set of the spectral family  $E_\lambda$  of  $T$  may be characterized in these equivalent ways:

(A)  $R_\lambda$  may be continued analytically across some open real interval  $\Delta$  containing  $\hat{\lambda}$ .

(B)  $E(\Delta) = 0$ , for some real interval  $\Delta$  containing  $\hat{\lambda}$ .

(C)  $\hat{\lambda}$  is in the resolvent set of a minimal self-adjoint extension  $T^+ = \psi(E_\lambda)$  of  $T$ .

In this case,  $R_{\hat{\lambda}} = PR_{\hat{\lambda}}^+$ , where  $R_{\hat{\lambda}}^+$  is the resolvent of  $T^+$ .

*Proof.* (A  $\rightarrow$  B) This is a consequence of the formula (1.2).

(B  $\rightarrow$  C) By the lemma,  $E^+(\Delta) = 0$ . Since  $E_\lambda^+$  is an orthogonal resolution of the identity, this implies that the points of  $\Delta$  are in the resolvent set of  $T^+$ .

(C  $\rightarrow$  A) If  $\Delta$  is in the resolvent set of  $T^+$  then  $R_{\hat{\lambda}}^+$  exists for  $\hat{\lambda} \in \Delta$ , and  $PR_{\hat{\lambda}}^+$  is well defined for points in  $\Delta$  as well as for nonreal points. Since  $R_{\hat{\lambda}}^+$  is analytic across  $\Delta$ , the same is true of  $PR_{\hat{\lambda}}^+$ . But for nonreal  $\lambda$ ,  $R_\lambda = PR_{\hat{\lambda}}^+$ . Hence  $R_\lambda$  can be continued analytically through  $\Delta$ , and will then equal  $PR_{\hat{\lambda}}^+$  there.

**REMARK 3.1.** The representation  $R_\lambda = PR_{\hat{\lambda}}^+$  throughout the resolvent set allows the establishment of a number of formulas already known for nonreal points:

$$(i) (R_\lambda f, g) = \int_{-\infty}^{\infty} \frac{d(E_\mu f, g)}{\mu - \lambda}$$

- (ii) For  $f \in \mathcal{A}_T(\lambda)$ ,  $(R_\mu - R_\lambda)f = (\mu - \lambda)R_\mu R_\lambda f$
- (iii)  $\mathcal{A}_T(\mu) = [1 + (\lambda - \mu)R_\lambda]\mathcal{A}_T(\lambda)$ .

We next obtain a result concerning the point spectrum of a minimal self-adjoint extension  $T^+$  of  $T$ . In the following theorem,  $\dim \mathcal{E}$  denotes the dimension ( $\leq \infty$ ) of the manifold  $\mathcal{E}(\hat{\lambda})$  of solutions of  $T^*u = \hat{\lambda}u$ . Also  $E[\lambda] = E_{\lambda^+} - E_{\lambda^-}$ .

**THEOREM 3.2.** *Let  $M^+(\hat{\lambda})$  be the characteristic manifold in  $\mathcal{H}^+$  corresponding to an eigenvalue  $\hat{\lambda}$  of  $T^+$ , a minimal self-adjoint extension of  $T$ . Then  $\dim M^+(\hat{\lambda}) = \dim E[\hat{\lambda}]\mathcal{H} \leq \dim \mathcal{E}(\hat{\lambda})$ .*

*Proof.* (i)  $E[\hat{\lambda}]\mathcal{H} \subset \mathcal{E}(\hat{\lambda})$ ; proving the inequality in the theorem. To verify this let  $h \in \mathcal{H}$  and choose  $f \in D_T$ . Then  $Tf = T^+f$ , and  $(E[\hat{\lambda}]h, Tf) = (E^+[\hat{\lambda}]h, Tf) = (T^+E^+[\hat{\lambda}]h, f) = (\hat{\lambda}E^+[\hat{\lambda}]h, f) = \hat{\lambda}(E[\hat{\lambda}]h, f)$ . Thus  $E[\hat{\lambda}]h \in D_{T^*}$  and  $T^*E[\hat{\lambda}]h = \hat{\lambda}E[\hat{\lambda}]h$ .

(ii) By Lemma 3.1,  $E^+[\hat{\lambda}]\mathcal{H}$  is dense on  $M^+(\hat{\lambda})$ . Thus  $\dim M^+(\hat{\lambda}) = \dim E^+[\hat{\lambda}]\mathcal{H}$ . The theorem will be proved by showing  $\dim E^+[\hat{\lambda}]\mathcal{H} = \dim E[\hat{\lambda}]\mathcal{H}$ .

Suppose  $f_1, \dots, f_m$  are vectors in  $\mathcal{H}$  such that  $E^+[\hat{\lambda}]f_1, \dots, E^+[\hat{\lambda}]f_m$  are linearly independent. Then  $E[\hat{\lambda}]f_1, \dots, E[\hat{\lambda}]f_m$  are also linearly independent. For otherwise there would be constants  $c_1, \dots, c_m$ , not all zero, such that

$$P \sum c_k E^+[\hat{\lambda}]f_k = \sum c_k E[\hat{\lambda}]f_k = 0 .$$

This would then imply that  $f = \sum c_k E^+[\hat{\lambda}]f_k$  was a characteristic vector of  $T^+$  such that  $f \in \mathcal{H}^+ \ominus \mathcal{H}$ . But that cannot be, since no reducing manifold of a *minimal* extension can lie in  $\mathcal{H}^+ \ominus \mathcal{H}$  (see Naimark [8], § 4.)

On the other hand  $E^+[\hat{\lambda}]f_1, \dots, E^+[\hat{\lambda}]f_m$  are obviously independent when their projections  $E[\hat{\lambda}]f_1, \dots, E[\hat{\lambda}]f_m$  are. Thus  $\dim E^+[\hat{\lambda}]\mathcal{H} = \dim E[\hat{\lambda}]\mathcal{H}$ , proving the theorem.

**REMARK 3.2.** Because of the unitary equivalence of all minimal self-adjoint extensions  $T^+$  associated with a given spectral resolution  $E_\mu$  of  $T$ , it is natural to associate with  $E_\mu$  the various aspects of the spectrum of  $T^+$ . Thus by the *spectrum, point spectrum, essential spectrum, etc.* of  $E_\mu$  will be meant the corresponding point sets in the spectrum of  $T^+$ . An *eigenvalue* of  $E_\mu$  will mean an eigenvalue of  $T^+$ , with its *multiplicity* the dimension of the corresponding manifold in  $\mathcal{H}^+$ .

From the theorems of this paragraph it follows that certain aspects of spectrum may be simply characterized *directly in terms of  $E_\mu$* . We mention especially:

- (i) *Spectrum:* The points of increase of  $E_\mu$

- (ii) *Eigenvalues*: Points of jump of  $E_\mu$ . The multiplicity of an eigenvalue  $\hat{\lambda}$  is  $\dim E[\hat{\lambda}]_{\mathcal{H}}$ .
- (iii) *Point Spectrum*: Closure of the set of eigenvalues.
- (iv) *Essential Spectrum*: Cluster points of the spectrum, plus eigenvalues of infinite multiplicity.

**4. Essential spectrum.** Let  $E_\mu$  be a generalized resolution of the identity associated with a symmetric operator  $T$ . From Remark 2.2, a *necessary* condition for an open real interval  $\Delta$  to belong to the resolvent set of  $E_\mu$  is that the associated family of contractions  $F_\lambda$  from  $M(\lambda_0)$  to  $M(\bar{\lambda}_0)$  have the properties:

- ( $\alpha$ )  $F_\lambda$  is continuous from  $\pi^+$  down to  $\Delta$ , and
- ( $\beta$ )  $F_{\lambda^+}$  is unitary on  $\Delta$ .

These properties obviously cannot hold for any  $\Delta$  unless the defect spaces  $M(\lambda_0)$  and  $M(\bar{\lambda}_0)$  have the same dimension. Hence, *when  $T$  has unequal defect numbers, the spectrum of any resolution  $E_\mu$  consists of the entire real axis.*

On the other hand, when  $T$  has equal defect numbers the properties ( $\alpha$ ) and ( $\beta$ ) may well hold; in particular, when  $F_\lambda$  is a constant unitary operator, thus when  $E_\mu$  is an *orthogonal* resolution, the properties are valid for every interval  $\Delta$ .

In the remainder of the paper we shall consider a symmetric operator  $A$  with equal *finite* defect numbers. We recall that the essential spectrum  $\Sigma_e$  is the same point set for all orthogonal resolutions of  $A$ , that is, for all self-adjoint extensions in  $\mathcal{H}$  of  $A$ . This is the classical theorem of H. Weyl, ([13] p. 251), proved originally for ordinary differential operators, and later extended to abstract operators by E. Heinz [6]. The principal theorem of this paragraph extends Weyl's result to generalized resolutions which satisfy ( $\alpha$ ) and ( $\beta$ ).

**THEOREM 4.1.** *Let the symmetric operator  $A$  have defect numbers  $(n, n)$  with  $n < \infty$ , and let  $\Sigma_e$  denote the points of the essential spectrum of any (hence every) orthogonal resolution of  $A$ . If  $E_\mu$  be an arbitrarily chosen (generalized) resolution of  $A$  with essential spectrum  $\Sigma'_e$ , then:*

- (i)  $\Sigma'_e \supset \Sigma_e$ .
- (ii) *When ( $\alpha$ ) and ( $\beta$ ) hold on  $\Delta$  for the family of contractions associated with  $E_\mu$ , then  $\Sigma'_e$  and  $\Sigma_e$  coincide on  $\Delta$ .*
- (iii) *If ( $\alpha$ ) and ( $\beta$ ) fail on every subinterval of  $\Delta$ , then  $\Delta \subset \Sigma'_e$ .*

We remark that the hypothesis of (iii) holds in particular under the condition:

- ( $\gamma$ )  $F_\lambda$  is continuous from  $\pi^+$  down to the open real interval  $\Delta$  and  $\|F_{\lambda^+}\| < 1$  on  $\Delta$ .

The proof will be based upon two lemmas of independent interest.

For any complex  $\lambda$ , let  $\mathcal{E}(\lambda)$  denote the eigenspace of solutions of  $T^*u = \lambda u$ . Thus  $\mathcal{E}(\lambda) = M(\bar{\lambda})$ .

LEMMA 4.1. *Let  $\hat{T}$  be a quasi-self-adjoint extension of  $T$  defined by  $F: M(\lambda_0) \rightarrow M(\bar{\lambda}_0)$ . For  $f, g \in D_{T^*}$  introduce the form  $\langle f, g \rangle = (T^*f, g) - (f, T^*g)$ . Then the domains of  $\hat{T}$  and  $\hat{T}^*$  have the following characterization:*

$$D_{\hat{T}} = \{u: u \in D_{T^*} \text{ and } \langle u, \phi - F^*\phi \rangle = 0 \text{ for all } \phi \in \mathcal{E}(\lambda_0)\}$$

$$D_{\hat{T}^*} = \{u: u \in D_{T^*} \text{ and } \langle u, \psi - F\psi \rangle = 0 \text{ for all } \psi \in \mathcal{E}(\bar{\lambda}_0)\}.$$

*Proof.* The proof of Theorem 1 in Coddington [3] is directly adaptable.

LEMMA 4.2. *Consider a symmetric  $T$  with equal finite defect numbers  $(n, n)$ , and suppose that  $\lambda$  is a real point of regular type of  $T$ , i.e. that  $T - \lambda$  has a bounded inverse. For any quasi-s.a. extension  $\hat{T}$ , if  $(\hat{T} - \lambda)^{-1}$  exists, it is a bounded operator with domain  $\mathcal{H}$ .*

*Proof.*  $(\hat{T} - \lambda)^{-1}$  is defined on  $\Delta_{\hat{T}}(\lambda) = \Delta_T(\lambda) \oplus [\Delta_{\hat{T}}(\lambda) \ominus \Delta_T(\lambda)]$ . It is bounded on the first since  $(T - \lambda)^{-1}$  is bounded at a point of regular type, and bounded on the second since the enclosing subspace  $M(\lambda)$  has dimension  $n$ . Hence  $(\hat{T} - \lambda)^{-1}$  is bounded on the sum of these orthogonal manifolds.

It remains to show that  $\Delta_{\hat{T}}(\lambda) = \mathcal{H}$ . Since  $\Delta_T(\lambda)$  is closed, the problem reduces to showing that  $\Delta_{\hat{T}}(\lambda) \ominus \Delta_T(\lambda)$  is  $n$ -dimensional. By (2.1), which gives the domain of  $\hat{T}$ , and by the existence of  $(\hat{T} - \lambda)^{-1}$ , it follows that  $\Delta_{\hat{T}}(\lambda)$  contains  $n$  vectors which are linearly independent mod  $\Delta_T(\lambda)$ . Their projections onto  $\Delta_{\hat{T}}(\lambda) \ominus \Delta_T(\lambda)$  are therefore linearly independent. Q.E.D.

*Proof of Theorem 4.1.* The statement that in general  $\Sigma'_e \supset \Sigma_e$  follows from a result of Hartman, ([5], § 3, proof of proposition (iii)): He has shown that, when  $\hat{\lambda} \in \Sigma_e$  (and  $n$  is finite), there exists a sequence  $f_n \in D_A$  such that  $\|f_n\| = 1, f_n \rightarrow 0$  weakly (in  $\mathcal{H}$ ) and  $(A - \hat{\lambda})f_n \rightarrow 0$  strongly. Consequently for any extension  $A^+$  in  $\mathcal{H}^+, f_n \in D_{A^+}, f_n \rightarrow 0$  weakly (in  $\mathcal{H}^+$ ), and  $(A^+ - \hat{\lambda})f_n \rightarrow 0$  strongly. Thus by Weyl's criterion ([9], § 133),  $\hat{\lambda}$  is in the essential spectrum of  $A^+$ , and (by Remark 3.2) in the essential spectrum of the corresponding  $E_{\lambda}$ .

Next we show that, under the conditions (ii) on  $F_{\lambda}$ , when  $\hat{\lambda} \notin \Sigma_e$  it cannot belong to  $\Sigma'_e$ . Since  $\hat{\lambda} \notin \Sigma_e$ , therefore the eigenspace of  $A$  at  $\hat{\lambda}$  is finite dimensional at most. We can depress  $\mathcal{H}$  and every  $\mathcal{H}^+$  to the orthogonal complement of this manifold without changing any essential spectrum. Hence it may be assumed from the beginning that

$\hat{\lambda}$  is not an eigenvalue of  $A$ . Hence, by [5], § 3, property (ii), there exists a self-adjoint extension  $\mathring{A}$  in  $\mathcal{H}$  of  $A$  for which  $\hat{\lambda}$  is not an eigenvalue. Since  $\hat{\lambda} \notin \Sigma_e$ , it cannot be a cluster point of the spectrum of  $\mathring{A}$ ; consequently  $\hat{\lambda}$  is in the resolvent set of  $\mathring{A}$ . Let  $\Delta$  about  $\hat{\lambda}$  be an open real interval in which  $\mathring{R}_\lambda = (\mathring{A} - \lambda)^{-1}$  is analytic. We shall show that  $R_\lambda$  (corresponding to the given  $E_\lambda$ ) is analytic in  $\Delta$  except at isolated points. Since  $\hat{\lambda}$  has at most finite multiplicity (by Theorem 3.2) as an eigenvalue of  $E_\lambda$ , it follows that  $\hat{\lambda} \notin \Sigma'_e$ .

It will be enough to show that  $(A_\lambda - \lambda)\varphi = 0$  has a nonzero solution at only isolated points  $\lambda$  in  $\Delta$ . For, by Lemma 4.2,  $R_\lambda$  will then exist except at these isolated points and, by the conditions of the theorem, and Remark 2.2, will be analytic.

Following M. G. Krein (see [1], § 84), we introduce an analytical basis  $\phi_1(\lambda), \dots, \phi_n(\lambda)$  for  $\mathcal{E}(\lambda), \lambda \in \pi^+ \cup \pi^- \cup \Delta$ , by

$$\phi_k(\lambda) = [1 + (\lambda - \lambda_0)R_\lambda]\phi_k(\lambda_0), \quad k = 1, 2, \dots, n.$$

Here  $\phi_1(\lambda_0), \dots, \phi_n(\lambda_0)$  form a basis (for convenience assumed *orthonormal*) for  $\mathcal{E}(\lambda_0)$ , with  $\lambda_0 \in \pi^+$ .

The solution space of  $(A_\lambda - \lambda)\varphi = 0$  is  $\mathcal{E}(\lambda) \cap D_{A_\lambda}$ . According to Lemma 4.1, this subspace contains a nonzero vector at just those points  $\lambda \in \Delta$  which are zeroes in  $(\Delta-)$  of

$$(4.1) \quad \det \langle \phi_j(\lambda), \phi_k(\bar{\lambda}_0) - F_\lambda^* \phi_k(\bar{\lambda}_0) \rangle, \quad \lambda \in \pi^- \cup (\Delta-).$$

As noted, the expression is meaningful also in  $\pi^-$ , indeed is analytic there and continuous in  $\pi^- \cup (\Delta-)$ . Thus the theorem can be proved by showing that (4.1), (which is nonvanishing in  $\pi^-$ ), can be continued analytically across  $\Delta$ .

For  $\lambda \in \pi^+ \cup (\Delta+)$  we have

$$F_\lambda \phi_k(\bar{\lambda}_0) = \sum_{l=1}^n F_{kl}(\lambda) \phi_l(\lambda_0), \text{ where } F_{kl}(\lambda) = (F_\lambda \phi_k(\bar{\lambda}_0), \phi_l(\lambda_0)).$$

The coefficient determinant,  $\det (F_{kl}(\lambda))$  is analytic on  $\pi^+$  and continuous on  $\pi^+ \cup (\Delta+)$ . It is non-vanishing wherever  $F_\lambda^{-1}$  exists, hence in particular on  $\Delta+$ .

We shall show that the expression, defined for  $\lambda \in \pi^+ \cup (\Delta+)$ ,

$$(4.2) \quad (-1)^n \det (\phi_l(\lambda_0), F_\lambda \phi_k(\bar{\lambda}_0)) \cdot \det \langle \phi_j(\lambda), \phi_k(\lambda_0) - F_\lambda^* \phi_k(\lambda_0) \rangle$$

coincides on  $\Delta$  with (4.1). Since this expression (4.2) is analytic on  $\pi^+$  and continuous on  $\pi^+ \cup \Delta+$ , it furnishes the desired continuation of (4.1) across  $\Delta$ .

Since  $F_{\lambda^+}^{-1} = F_{\lambda^+}^*$  on  $\Delta$ , therefore

$$\begin{aligned} \phi_k(\bar{\lambda}_0) - F_{\lambda_+} \phi_k(\bar{\lambda}_0) &= F_{\lambda_+}^* F_{\lambda_+} \phi_k(\bar{\lambda}_0) - F_{\lambda_+} \phi_k(\bar{\lambda}_0) \\ &= \sum_l F_{ki}(\lambda_+) [F_{\lambda_+}^* \phi_l(\lambda_0) - \phi_l(\lambda_0)] . \end{aligned}$$

Noting that  $F_{\lambda_-}^* = F_{\lambda_+}$ , this permits writing the limit value of (4.1) in the form

$$(4.3) \quad (-1)^n \det(\overline{F_{ki}(\lambda_+)}) \det \langle \phi_j(\lambda), \phi_i(\lambda_0) - F_{\lambda_+}^* \phi_l(\lambda_0) \rangle .$$

But

$$\begin{aligned} \overline{F_{ki}(\lambda_+)} &= \overline{(F_{\lambda_+} \phi_k(\bar{\lambda}_0), \phi_i(\lambda_0))} \\ &= (\phi_i(\lambda_0), F_{\lambda_+} \phi_k(\bar{\lambda}_0)) \end{aligned}$$

so that (4.3) is identical with the limit value on  $\Delta$  of (4.2).

The theorem is proved.

**5. Strict contractions.** In this paragraph we shall examine spectral resolutions of a symmetric operator  $A$  satisfying the conditions

(I)  $A$  has equal finite defect numbers  $(n, n)$ .

(II) Every point  $\hat{\lambda}$  on a real interval  $\Delta$  is of regular type for  $A$ , i.e.  $(A - \hat{\lambda})^{-1}$  exists and is bounded.

Condition II can be stated in the following equivalent form:

(II') Any self-adjoint extension in  $\mathcal{H}$  of  $A$  has in  $\Delta$  only isolated points of its spectrum. No point of  $\Delta$  is common to the spectra of all such extensions.

The equivalence of II and II' follows from Hartman ([5], prop. (ii))

Let  $E_\mu$  be a spectral resolution of  $A$ , and  $F_\lambda$  be the associated family of contractions of  $M(\lambda_0)$  into  $M(\bar{\lambda}_0)$ . It follows from theorem 4.1 that, on any sub-interval of  $\Delta$  where  $(\alpha)$  and  $(\beta)$  hold, the spectrum of  $E_\mu$  will contain only isolated points.

Our interest here, however, will be in resolutions for which condition  $(\gamma)$  of § 4 holds on  $\Delta$ . In this case, by Theorem 4.1 (iii), the spectrum of  $E_\mu$  includes  $\Delta$ . We first state a result valid when  $\Delta$  is the entire real axis  $\mathcal{R}$ . When  $(\gamma)$  holds on  $\mathcal{R}$  we shall describe  $F_\lambda$  as a family of strict contractions.

**THEOREM 5.1.** *Suppose that  $A$  satisfies (I), and (II) on  $\mathcal{R}$ . Let  $E_\mu$  be a resolution of  $A$  for which the associated family of contractions  $F_\lambda$  is strict. Then:*

*The associated minimal self-adjoint extension of  $A$  is unitarily equivalent to the  $n$ -fold direct sum of  $iD$  with itself,  $D$  being the differential operator  $d/dx$  on  $\mathcal{L}_2(-\infty, \infty)$ .*

**REMARK 5.1.** This theorem generalizes results of Coddington and Gilbert [4] for ordinary differential operators on a closed bounded inter-

val. Their method of proof appears to be adaptable to handle certain other ordinary differential operators satisfying I and II, in particular, singular operators in Weyl's limit circle case.

REMARK 5.2. Condition II on  $\mathcal{R}$  of course implies that  $A$  has no eigenvalues. However it is easy to analyze the more general situation in which eigenvalues do occur, provided II on  $\mathcal{R}$  holds for the restriction of  $A$  to the manifold orthogonal to the eigenvectors. In that case the minimal self-adjoint extension is equivalent to the direct sum of the discrete part of  $A$  with the operator described in Theorem 5.1.

We shall prove Theorem 5.1 as a special case of a more general theorem. We now suppose that I, II, and  $(\gamma)$  hold on  $\mathcal{A}$ . By assumption,  $F_{\lambda+}$ , and hence  $A_{\lambda+}$ , exists for every  $\lambda \in \mathcal{A}$ . The assumptions that  $\|F_{\lambda}\| < 1$  and that  $A$  has no eigenvalues on  $\mathcal{A}$  imply that  $D_{A_{\lambda+}} \cap \mathcal{E}(\lambda) = \{0\}$  for  $\lambda \in \mathcal{A}$ , and hence that  $(A_{\lambda+} - \lambda)^{-1}$  exists. This statement follows from the fact, noted by Hartman [5], that when  $f \in \mathcal{E}(\lambda)$ , for  $\lambda \in \mathcal{R}$ , is written in the form

$$f = f_0 + f^+ + f^-, \quad \text{where } f_0 \in D_{\mathcal{A}}, f^+ \in \mathcal{E}(\lambda_0), f^- \in \mathcal{E}(\bar{\lambda}_0)$$

for  $\lambda_0 \in \pi^+$ , then  $\|f^+\| = \|f^-\|$ . Then, by assumption I and Lemma 4.2,  $R_{\lambda}$  exists, and is continuous in  $\lambda$ , on  $\pi^+ \cup (\mathcal{A}+)$ .

One may define a basis for  $\mathcal{E}(\lambda)$ ,  $\lambda \in \pi^+ \cup \mathcal{A}$ , by

$$(5.1) \quad \phi_k(\lambda) = [1 + (\lambda - \lambda_0)R_{\lambda}]\phi_k(\lambda_0), \quad k = 1, 2, \dots, n.$$

Here  $\lambda_0 \in \pi^+$ , and  $\phi_1(\lambda_0), \dots, \phi_n(\lambda_0)$  form a basis for  $\mathcal{E}(\lambda_0)$ . That  $\phi_k(\lambda)$  is in  $\mathcal{E}(\lambda)$  follows from  $(A^* - \lambda)R_{\lambda} = 1$ . That  $\phi_1(\lambda), \dots, \phi_n(\lambda)$  are independent follows from the fact that

$$1 + (\lambda - \lambda_0)R_{\lambda} = (A_{\lambda} - \lambda_0)(A_{\lambda} - \lambda)^{-1}$$

has an inverse.<sup>1</sup>

We shall henceforth identify  $\pi^+$  with the half-plane  $\mathcal{S}(\lambda) > 0$ . The basis (5.1) allows a simple representation for  $\mathcal{S}R_{\lambda+} = 1/2i [R_{\lambda+} - R_{\lambda-}]$ :

LEMMA 5.1. *Assume that  $A$  satisfies I, II, and  $F_{\lambda}$  satisfies  $(\gamma)$ . Then for every  $\lambda \in \mathcal{A}$  and every  $f \in \mathcal{H}$ ,*

$$(5.2) \quad \mathcal{S}R_{\lambda+}f = \sum_{j,k=1}^n \Phi_{jk}(\lambda)(f, \phi_j(\lambda))\phi_k(\lambda).$$

*The matrix  $\Phi(\lambda)$  is positive definite and continuous in  $\lambda$ . Here  $\pi^+$  has been identified with the half plane  $\mathcal{S}(\lambda) > 0$ .*

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<sup>1</sup> In what follows only the *existence* of a continuous basis is needed, not its relation (5.1) to  $R_{\lambda}$ .

*Proof.* Since for every  $f \in \mathcal{H}$ ,  $(A^* - \lambda)R_{\lambda+}f = f$ , therefore  $\mathcal{S}R_{\lambda+}f \in \mathcal{E}(\lambda)$ . In terms of an orthonormal basis  $\tilde{\phi}_1, \dots, \tilde{\phi}_n$  for  $\mathcal{E}(\lambda)$ ,  $\mathcal{S}R_{\lambda+}f = \Sigma C_k \tilde{\phi}_k$ , where

$$C_k = (\mathcal{S}R_{\lambda+}f, \tilde{\phi}_k) = (f, \mathcal{S}R_{\lambda+}\tilde{\phi}_k) = (f, \psi_k)$$

for some  $\psi_k \in \mathcal{E}(\lambda)$ . Writing  $\tilde{\phi}_1, \dots, \tilde{\phi}_n, \psi_1, \dots, \psi_n$  as linear combinations of  $\phi_1(\lambda), \dots, \phi_n(\lambda)$  establishes the form of (5.2).

From the known relation

$$(\mathcal{S}R_{\lambda}f, f) \geq 0 \quad \text{for} \quad \mathcal{S}\lambda > 0$$

follows

$$(5.3) \quad (\mathcal{S}R_{\lambda+}f, f) \geq 0 \quad \text{for} \quad \mathcal{S}\lambda = 0 .$$

Recalling that  $\mathcal{E}(\lambda)$  is invariant under  $\mathcal{S}R_{\lambda+}$ , let  $\{\mathcal{S}R_{\lambda+}\}$  denote the restriction of  $\mathcal{S}R_{\lambda+}$  to  $\mathcal{E}(\lambda)$ . We assert that

$$(5.4) \quad (\{\mathcal{S}R_{\lambda+}\}\phi, \phi) > 0 \quad \text{when} \quad \|\phi\| > 0, \quad \phi \in \mathcal{E}(\lambda) .$$

In view of (5.3) it is sufficient to show that

$$(5.5) \quad \{\mathcal{S}R_{\lambda}\}\psi = 0 \quad \text{implies} \quad \psi = 0 .$$

Suppose that  $\{\mathcal{S}R_{\lambda}\}\psi = 0$ . Then  $g = R_{\lambda+}\psi = R_{\lambda-}\psi$  belongs to  $D(A_{\lambda+}) \cap D(A_{\lambda-})$ . Writing  $g$  in the form

$$g = g_0 + g^+ + g^-, \quad g_0 \in D(A), \quad g^+ \in \mathcal{E}(\lambda_0), \quad g^- \in \mathcal{E}(\bar{\lambda}_0)$$

then, by the definition of  $D(A_{\lambda\pm})$ ,

$$-g^+ = F_{\lambda+}g^-, \quad -g^- = F_{\lambda-}g^+ .$$

Since  $\|F_{\lambda+}\|, \|F_{\lambda-}\| < 1$ , this implies that  $g^+ = g^- = 0$ , i.e.  $g \in D(A)$ . Since  $R_{\lambda+}\psi \in D(A)$  therefore  $\psi \in \mathcal{A}_A(\lambda)$ , the orthogonal complement of  $\mathcal{E}(\lambda)$ . Thus  $\psi = 0$ . This proves (5.4).

Now let  $\phi$  be an arbitrary element of  $\mathcal{E}(\lambda)$  and put

$$\xi_k = (\phi, \phi_k(\lambda)), \quad k = 1, 2, \dots, n .$$

In view of the independence of  $\phi_1(\lambda), \dots, \phi_n(\lambda)$ , this relation is a one-to-one linear mapping of  $\mathcal{E}(\lambda)$  onto the  $n$ -dimensional space of vectors  $\xi = (\xi_1, \dots, \xi_n)$ . Thus relation (5.4) is equivalent, because of the form of (5.2), to

$$\Sigma \phi_{jk}(\lambda) \xi_j \xi_k > 0 \quad \text{when} \quad \|\xi\| \neq 0 .$$

That is, the matrix  $\phi(\lambda)$  is positive-definite.

It remains only to prove the continuity of  $\phi(\lambda)$ . This follows directly from the relation



$$(\mathcal{R}R_{\lambda+\phi_{\mu}(\lambda)}, \phi_{\nu}(\lambda)) = \sum_{j,k} \Phi_{jk}(\lambda)(\phi_{\mu}(\lambda), \phi_j(\lambda))\overline{(\varphi_j(\lambda), \varphi_k(\lambda))} ,$$

since  $\det (\phi_{\mu}(\lambda), \phi_j(\lambda)) \neq 0$ .

**THEOREM 5.2.** *Suppose that on an interval  $\Delta$  the operator  $A$  satisfies I, II and that  $E_{\mu}$  is a spectral resolution of  $A$  for which the corresponding mapping  $F_{\lambda}$  satisfies  $(\gamma)$ . Let  $A^+$  in  $\mathcal{H}^+$  be a minimal s.a. extension of  $A$  with orthogonal resolution  $E_{\mu}^+$  satisfying  $E_{\mu} = PE_{\mu}^+$ . For  $\mu \in \Delta$  define  $\rho(\mu) = 1/\pi \int \Phi(\mu)d\mu$ . Then the part of  $A^+$  on  $E^+(\Delta)\mathcal{H}^+$  is unitarily equivalent to the multiplication operator on  $\mathcal{L}_2(\rho(\mu))$ ,  $\mu \in \Delta$ .*

*Proof of the Theorems.* It is pointed out by Coddington and Gilbert [4] that the multiplication operator in  $\mathcal{L}_2(\rho)$  (where  $\rho$  is strictly increasing and continuous in  $\lambda$  on  $\mathcal{S}$ ) is unitarily equivalent to the  $n$ -fold direct product of  $iD$  with itself,  $D$  being the differential operator  $d/dx$  on  $\mathcal{L}_2(-\infty, \infty)$ . Thus Theorem 5.1 is a corollary of Theorem 5.2.

For every  $f \in \mathcal{H}$  and every bounded real interval  $\Delta' \subset \Delta$ ,

$$\begin{aligned} (E(\Delta')f, f) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\Delta'} (\mathcal{R}R_{\lambda+i\varepsilon}f, f)d\lambda \\ &= \frac{1}{\pi} \int_{\Delta'} (\mathcal{R}R_{\lambda+}f, f)d\lambda . \end{aligned}$$

Here we have used the continuity of  $R_{\lambda}$  on  $\pi^+ \cup (\Delta+)$  and of  $E_{\lambda}$  on  $\Delta$ . Let  $g(\lambda) = \{g_k(\lambda)\}_{k=1}^n$  be defined by  $g_k(\lambda) = (f, \phi_k(\lambda))$ . Hence

$$\begin{aligned} (E(\Delta')f, f) &= \frac{1}{\pi} \int_{\Delta'} \sum \Phi_{jk}(\lambda)g_j(\lambda)\bar{g}_k(\lambda)d\lambda ; \\ (5.6) \quad \|E^+(\Delta')f\|^2 &= \int_{\Delta'} \sum g_j(\lambda)\bar{g}_k(\lambda)d\rho_{jk}(\lambda) . \end{aligned}$$

Now suppose  $f \in \mathcal{H}$  is in  $E^+(\Delta)\mathcal{H}^+$ . Thus  $E(\Delta)f = f$ . Consider  $V: E^+(\Delta')f \rightarrow \chi_{\Delta'}(\lambda)g(\lambda)$ , where  $\chi_{\Delta'}(\lambda)$  is the characteristic function of the interval  $\Delta' \subset \Delta$ . From (5.6)  $V$  is an isometric mapping of  $Z(\Delta)$  (see Lemma 3.1) into  $\mathcal{L}_2(\rho(\lambda))$ , ( $\lambda \in \Delta$ ), which carries  $E^+(\Delta')$  into the operation of multiplication by  $\chi_{\Delta'}(\lambda)$ . Since  $Z(\Delta)$  is fundamental on  $E^+(\Delta)\mathcal{H}^+$ , Theorem 5.2 follows.

**6. Differential operators.** Let  $Lu = -(pu)'+qu$  be an ordinary differential expression on the positive axis  $0 \leq x \leq \infty$ , with  $p$  and  $q$  real measurable functions such that  $p(x) > 0$ ,

$$\int_0^b p(x)^{-1}dx < \infty , \quad \int_0^b |q(x)| dx < \infty$$

for any  $b > 0$ . With a suitably prescribed<sup>2</sup> minimal domain in  $\mathcal{L}_2(0, \infty)$ ,  $L$  defines a symmetric quasi-differential operator  $L_0$  with defect numbers (1, 1) or (2, 2). It is easily seen that  $L_0$  has no eigenvalues. When the defect numbers are (2, 2), the Conditions I and II of § 5 are automatically satisfied on  $\mathcal{R}$ , so that the results of that section hold.

We shall assume that  $L_0$  has defect numbers (1, 1), i.e. is in the limit point case, and shall study the absolutely continuous spectrum of a minimal self-adjoint extension  $L_0^+$  of  $L_0$ . As before (§ 3),  $L_0^+$  operates in a space  $\mathcal{H}^+$  containing  $\mathcal{H}$  and has a spectral family of projections denoted by  $E_\mu^+$ .

Let  $M_a$  and  $M_s$  be the absolutely continuous and singular subspaces of  $\mathcal{H}^+$  with respect to  $L_0^+$  (see [7] for definitions). Thus  $M_a$  and  $M_s$  reduce  $L_0^+$ , are orthogonal, and  $\mathcal{H}^+ = M_a \oplus M_s$ . For any  $u \in M_a$  [ $u \in M_s$ ], the function  $(E_\mu^+ u, u)$  is absolutely continuous [singular] with respect to Lebesgue measure on  $-\infty < \mu < \infty$ . Let  $E_\mu$  be a generalized resolution of  $L_0$  for which  $E_\mu = PE_\mu^+$ . By the *absolutely continuous spectrum* of  $L_0^+$  (or of  $E_\mu$ ) will be meant the spectrum of the part of  $L_0^+$  in  $M_a$ . The *singular spectrum* is defined similarly.

It has been proved by N. Aronszajn [2] that the absolutely continuous spectrum is the same point set for all orthogonal resolutions of the differential operator  $L_0$ . The following theorem extends Aronszajn's result to generalized resolutions in a way parallel to Theorem 4.1 for essential spectrum. Clause (iii) contains a partial extension, for differential operators, of Theorem 5.1.

**THEOREM 6.1.** *Let  $L_0$  be a quasi-differential operator as described above and let  $\Sigma_a$  denote the points of the absolutely continuous spectrum of any (hence every) orthogonal resolution of  $L_0$ . If  $E_\mu$  is an arbitrarily chosen (generalized) resolution of  $L_0$  with absolutely continuous spectrum  $\Sigma'_a$  and singular spectrum  $\Sigma'_s$ , then:*

(i)  $\Sigma'_a \supset \Sigma_a$

(ii) *When  $(\alpha)$  and  $(\beta)$  hold on  $\Delta$  for the family of contractions associated with  $E_\mu$ , then  $\Sigma'_a$  and  $\Sigma_a$  coincide on  $\Delta$ .*

(iii) *When  $(\gamma)$  holds on  $\Delta$ , then  $\Delta \subset \Sigma'_a$ , while  $\Delta \cap \Sigma'_s = 0$ .*

The proof depends upon

**LEMMA 6.1.** *Let  $\rho(\mu) = [\rho_{jk}(\mu)]_{j,k=1}^2$  be a nondecreasing Hermitian matrix ( $-\infty < \mu < \infty$ ) and  $\Delta$  the multiplication operator with maximal domain in  $\mathcal{L}_2(\rho)$ . Let*

$$\rho(\mu) = \rho_a(\mu) + \rho_s(\mu)$$

*be the Lebesgue decomposition of  $\rho$  into its absolutely continuous and*

<sup>2</sup> A precise specification may be found in [1], Appendix II.

singular parts, defined by the corresponding decomposition of the components of  $\rho$ .

Then  $\rho_a$  and  $\rho_s$  are nondecreasing Hermitian matrices.  $\mathcal{L}_2(\rho_a)$  and  $\mathcal{L}_2(\rho_s)$  are, respectively, the absolutely continuous and singular subspaces of  $\mathcal{L}_2(\rho)$  with respect to  $\Delta$ . Thus the absolutely continuous spectrum of  $\Delta$  consists of the points of increase of  $\rho_a$ , or equivalently of its trace  $\text{tr } \rho_a$ . A similar statement holds for the singular spectrum.

We shall omit the proof of Lemma 6.1.

*Proof of Theorem 6.1.* For  $\mathcal{I}\lambda \neq 0$ , let  $\psi_\lambda = \psi(x, \lambda)$  denote the  $\mathcal{L}_2$  solution of  $L\psi = \lambda\psi$  which is determined by

$$[p(x)\psi'(x, \lambda)]_{x=0} = -1.$$

Put  $\psi(0, \lambda) = m(\lambda)$ . Each generalized resolution  $E_\mu$  of  $L_0$  is now specified by a family of contractions  $F_\lambda: M(i) \rightarrow M(-i)$  of the form

$$F_\lambda \psi_{-i} = W(\lambda)\psi_i$$

where  $W(\lambda)$  is analytic and  $|W(\lambda)| \leq 1$  for  $\mathcal{I}\lambda > 0$ . Define

$$\theta(\lambda) = \frac{W(\lambda)m(i) - m(-i)}{1 - W(\lambda)}, \quad \mathcal{I}\lambda > 0.$$

Since  $\mathcal{I}m(i) > 0$  and  $m(-i) = \bar{m}(i)$ , therefore  $\mathcal{I}\theta(\lambda) \geq 0$  (with  $\theta = \infty$  when  $W = 1$ ).

A. V. Štraus [11] has associated with each  $E_\mu$  a spectral matrix  $\rho(\mu) = [\rho_{jk}(\mu)]_{j,k=1,2}$ ,  $-\infty < \mu < \infty$ , which is Hermitian nondecreasing, and such that

$$\text{tr } \rho(\mu) = \frac{1}{\pi} \lim_{\varepsilon \rightarrow +0} \int_0^\mu \mathcal{I} \phi(\gamma + i\varepsilon) d\gamma$$

where

$$(6.1) \quad \phi(\lambda) = \frac{m(\lambda)\theta(\lambda) - 1}{\theta(\lambda) + m(\lambda)}, \quad \mathcal{I}\lambda > 0.$$

In particular, when  $W(\lambda) \equiv 1$ ,  $\phi(\lambda)$  reduces to  $m(\lambda)$ .

Let  $\Delta$  be the multiplication operator with maximum domain in  $\mathcal{L}_2(\rho)$  and let  $L_0^+$  be a minimal self adjoint extension of  $L_0$ , with  $E_\mu = PE_\mu^+$ . By the reasoning in [4], § 4,  $\Delta$  is unitarily equivalent to  $L_0^+$ .

Therefore, by Lemma 6.1, the problem is reduced to a consideration of the absolutely continuous and singular parts of  $\text{tr } \rho$ . But such consideration is possible along the lines of [2].

A set  $G$  is a support of a real measure  $\nu$  when  $\nu(\mathcal{R} - G) = 0$ . It is a minimal support when for every support  $G_1 \subset G$ , the Lebesgue

measure  $|G - G_1| = 0$ . It is easy to prove that when  $\nu$  and  $\nu'$  are absolutely continuous measures with minimal supports  $G \subset G'$  then  $\nu < \nu'$  (i.e.  $\nu'(s) = 0$  implies  $\nu(s) = 0$ ).

The following disjoint sets  $G_a$  and  $G_s$  are minimal supports for, respectively, the absolutely continuous and singular parts of  $\text{tr } \rho$  (compare [2]):

$$G_a = \{ \mu \in \mathcal{R} : \lim_{\lambda \rightarrow \mu} \phi(\lambda) \text{ exists finitely and } \lim_{\lambda \rightarrow \mu} \mathcal{I} \phi(\lambda) > 0 \}$$

$$G_s = \{ \mu \in \mathcal{R} : \mathcal{I} \phi(\lambda) \rightarrow \infty \text{ when } \lambda \rightarrow \mu \} .$$

(Here it is understood that  $\lambda \rightarrow \mu$  with the constraint that  $\varepsilon < \text{Arg}(\lambda - \mu) < \pi - \varepsilon$  for some fixed  $\varepsilon > 0$ .)

We shall compare the sets  $G_a, G_s$  corresponding to an arbitrarily chosen resolution  $E_\mu$  with the special sets  $G_a^0, G_s^0$  corresponding to the orthogonal resolution for which  $W \equiv 1$ . Thus in the definitions of  $G_a^0, G_s^0, \phi(\lambda)$  is replaced by  $m(\lambda)$ .

We note first that  $\mathcal{I} \theta, \mathcal{I} \phi$  and  $\mathcal{I} m$  are all  $\geq 0$  when  $\mathcal{I} \lambda > 0$ .

Since  $\lim_{\lambda \rightarrow \mu} \theta(\lambda)$  exists finitely except for  $\lambda$  on a certain set  $S_0$  of Lebesgue measure zero, inspection of (6.1) and the formula

$$\mathcal{I} \phi = \frac{(1 + |\theta|^2) \mathcal{I} m + (1 + |m|^2) \mathcal{I} \theta}{|m + \theta|^2}$$

reveals that  $G_a \supset G_a^0 - S_0$ . Since these are minimal support it follows that  $\text{tr } \rho_a > \text{tr } \rho_a^0$ , implying the statement (i) of the theorem.

Next, assume that  $(\alpha)$  and  $(\beta)$  hold on  $\Delta$ . Therefore  $\theta(\lambda)$  may be continued down to  $\Delta$  with  $\mathcal{I} \theta(\mu+) = 0$  on  $\Delta$ . Inverting (6.1) one obtains the formulas

$$m = \frac{\phi \theta + 1}{\theta - \phi} , \quad \mathcal{I} m = \frac{(1 + |\theta|^2) \mathcal{I} \phi - (1 + |\phi|^2) \mathcal{I} \theta}{|\theta - \phi|^2} ,$$

which show on inspection that  $G_a \cap \Delta \subset G_a^0$ . Together with the earlier obtained inclusion, this implies that  $G_a$  and  $G_a^0$  coincide on  $\Delta$ . Since these are minimal supports, (ii) follows.

Finally, assume  $(\gamma)$  holds on  $\Delta$ . In this case  $\theta(\lambda)$  may be continued down to  $\Delta$  with  $\mathcal{I} \theta(\mu+) > 0$  for  $\mu \in \Delta$ . Equation (6.1) shows that  $\phi(\lambda)$  remains bounded as  $\lambda \rightarrow \mu$  on  $\Delta$  and hence that  $G_s \cap \Delta = 0$ . Thus  $\Sigma'_s \cap \Delta = 0$ . At the same time, by Theorem 4.1 (iii),  $\Delta$  does belong to the spectrum of  $E_\mu$ , and hence must belong to the absolutely continuous spectrum  $\Sigma'_a$ . Q.E.D.

REMARK 6.1. A. V. Štraus [12] has shown that when  $\theta(\lambda)$  may be continued to real limit values on the entire real axis—equivalent to the assertion that  $(\alpha)$  and  $(\beta)$  hold on  $\mathcal{R}$ —then  $E_\mu^+$  has simple spectrum,

This, together with Theorem 6.1 (ii), implies the *unitary equivalence of the absolutely continuous parts* of minimal self-adjoint extensions corresponding to resolutions  $E_\mu$  satisfying  $(\alpha)$  and  $(\beta)$  on  $\mathcal{R}$ .

**REMARK 6.2.** Assume  $(\gamma)$  holds on  $\mathcal{R}$ . If conditions I and II of §5 hold for  $L_0$  then, by Theorem 5.1, the multiplicity of spectrum of  $L_0^+$  will be 1, and the operator equivalent to  $iD$ . Simple examples show that in general (i.e. without Conditions I and II) the multiplicity of spectrum may well be 2 (the maximum consistent with  $\rho$  being a  $2 \times 2$  matrix) and that  $L_0^+$  may even be equivalent to  $iD \oplus iD$ .

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