Pacific Journal of Mathematics

OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH SPACE

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Vol. 12, No. 3 March 1962

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1. Let X be a reflexive Banach space and $F(X)$ the Banach algebra of all uniform limits of operators of finite rank, in X. Bonsall [1] has characterized $F(X)$ as a simple, $B^{\#}$-annihilator algebra: $F(X)$ contains no proper closed two-sided ideals, every proper, closed right (left) ideal of $F(X)$ has a nonzero left (right) annihilator, and, given any $T \in F(X)$, there exists $T^{\#} \in F(X)$ such that

$$||T|| \, ||T^{\#}|| = ||(TT^{\#})^n||^{1/n}, \qquad n = 1, 2, 3, \cdots .$$

In this note, we obtain a new characterization for $F(X)$ (Theorem 3.2): a Banach algebra A is the algebra $F(X)$ of all uniform limits of operators of finite rank in a reflexive Banach space X if and only if A is a simple, weakly compact, $B^{\#}$-algebra with minimal ideals (A is weakly compact if left- and right-multiplications by every $a \in A$ are weakly compact operators). In the process of proving this result, we obtain a characterization of reflexive Banach spaces which seems to be of some independent interest (Theorem 2.2): a Banach space X is reflexive if and only if every operator in X of rank 1 is a weakly compact element of $B(X)$.

2. Let X be a Banach space and $B = B(X)$ the Banach algebra of all bounded operators in X with the uniform topology. For $T \in B$, let R_T denote the operator in B obtained by multiplying elements of B on the right by T: $R_T(A) = AT$ for $A \in B$.

Suppose that T is a fixed operator of rank 1 in X with $H = [x \in X: Tx = 0]$. Then H is a closed hyperplane in X and if x_0 is an element of X such that $Tx_0 \neq 0$, then $X = H \oplus (x_0)$ and we may assume that $||x_0|| = 1$. Write $B' = [S \in B: S(H) = (0)]$. For each $S \in B'$, we define an element x_S of X by setting $x_S = S(x_0)$. The mapping $S \to x_S$ is clearly linear.

LEMMA 2.1. *The linear mapping $S \to x_S$ is a homeomorphism of B' onto X.*

Proof. It is clear that the mapping is one-to-one and, since $||S(x_0)|| \leqq ||S||$, it is continuous. It is also onto; in fact, let $\varphi \in X^*$ be such that $\varphi(H) = (0)$, $\varphi(x_0) = 1$. Then for given $x \in X$, the operator S_x defined by setting $S_x(y) = \varphi(y)x$, $y \in X$ belongs to B' and is mapped into x by the mapping $S \to S(x_0)$. Hence, by the closed graph theorem, the

Received November 9, 1961.

mapping is bicontinuous and the proof is complete.

Let B_1 denote the unit ball in B, so that $R_T(B_1) = [PT \in B: ||P|| \leqq 1]$.

LEMMA 2.2. $R_T(B_1) = [A \in B': ||Ax_0|| \leqq ||Tx_0||]$.

Proof. It is clear that $R_T(B_1) \subset [A \in B': ||Ax_0|| \leqq ||Tx_0||]$. Now let $A \in B'$ with $||Ax_0|| \leqq ||Tx_0||$; we find $P \in B_1$ such that $A = PT$. There exists $\psi \in X^*$ such that $||\psi|| = 1$ and $\psi(Tx_0) = ||Tx_0||$. We define P by setting $Px = \psi(x)Ax_0/||Tx_0||$. Then $PTx = 0$ if $x \in H$ and $PTx_0 = Ax_0$. Thus PT and A coincide in the subspace (x_0) and must therefore coincide everywhere in X. Finally $||P|| = \sup_{||x|| \leq 1} ||\psi(x)Ax_0||/||Tx_0|| \leqq 1$; hence $P \in B_1$ and $R_T(B_1) = [A \in B': ||Ax_0|| \leqq ||Tx_0||]$.

LEMMA 2.3. *Let F be any subset of B'. If $F^{B'}$ denotes the closure of F with respect to the weak topology of B' and F^B the closure of F with respect to the weak topology of B, then $F^{B'} = F^B$.*

Proof. Let $P_0 \in F^{B'}$ and let

$$\begin{aligned} N &= N(P_0; \Phi_1, \Phi_2, \cdots, \Phi_n; \varepsilon) \\ &= [P \in B: |\Phi_k(P - P_0)| < \varepsilon; k = 1,2, \cdots, n; \Phi_k \in B^*] \end{aligned}$$

be an arbitrary neighborhood of P_0 in B. Then the neighborhood $N' = N(P_0; \Phi'_1, \Phi'_2, \cdots, \Phi'_n; \varepsilon)$ of P_0 obtained by taking the restriction of Φ_k to B' for each k, contains a point P of F. Since P must therefore belong to N, it follows that $F^{B'} \subseteqq F^B$.

Now suppose that $P_0 \in F^B$. Then $P_0 \in B'$ since B' is closed with respect to the weak topology of $B(X)$ (being linear and strongly closed). Let $N' = [P \in B': |\varphi_k(P - P_0)| < \varepsilon, k = 1, 2, \cdots, n; \varphi_k \in (B')^*]$ be an arbitrary neighborhood of P_0 in B'. Then again, by considering the neighborhood $N = [P \in B: |\Phi_k(P - P_0)| < \varepsilon, k = 1, 2, \cdots, n, \Phi_k \in B^*]$ obtained by extending φ_k to Φ_k, for each k, on the whole of B, we can find $P \in F$ such that $P \in N'$. Hence $F^B \subseteqq F^{B'}$. This completes the proof.

THEOREM 2.1. *A Banach space X is reflexive if and only if every operator in X of rank 1 is a right weakly compact element of $B(X)$.*

Proof. If X is reflexive and T is of rank 1, then by Lemma 2.1, B' is homeomorphic with X under the correspondence $S \leftrightarrow S(x_0)$. Now the image of B_1 under R_T is a bounded subset of B' which is therefore contained in a set U which is compact with respect to the weak topology of B' and by Lemma 2.3, with respect to the weak topology of $B(X)$. Thus R_T is a weakly compact operator in $B(X)$ and T is a right weakly compact element of $B(X)$.

Now suppose that R_T is weakly compact in $B(X)$. Then $R_T(B_1)$ is contained in a set $V \subset B'$ which is compact with respect to the weak topology of $B(X)$ and hence also with respect to the weak topology of B'. Now the ball $Q = [A \in B' \colon \|A\| \leqq \|Tx_0\|/\|x_0\|]$ is contained in $R_T(B_1) \subset V$ and is weakly closed. Hence Q is compact with respect to the weak topology of B' and therefore B' is reflexive. Since B' is homeomorphic with X, it follows that X is reflexive and the proof is complete.

COROLLARY 2.1. *If X is a reflexive Banach space, then the algebra $F(X)$ of all uniform limits of operators of finite rank in X is a weakly compact algebra.*

COROLLARY 2.2. (Ogasawara [2] Theorem 4.) *Let H be a Hilbert space and $B(H)$ the Banach algebra of all bounded operators in H. If T is a compact operator in H, then T is a weakly compact element of $B(H)$.*

3. This section is devoted to the study of simple, weakly compact, $B^{\#}$-algebras with minimal ideals.

LEMMA 3.1. *Let A be a simple Banach algebra with minimal ideals. Then every maximal regular left ideal M of A has a nonzero right annihilator.*

Proof. Since A is a simple Banach algebra, there exists an idempotent $e \in A$ such that $M \cap Ae = (0)$ and $M \oplus Ae = A$. Since M is regular, there is $j \in A$ such that $xj - x \in M$ for every $x \in A$. For some $a_0 \in A$ and $m_0 \in M$, $j = m_0 + a_0 e$, $a_0 e \neq 0$. Suppose now that m is an arbitrary element in M. We have $mj - m \in M$ and $mj - ma_0 e = mm_0 \in M$, from which it follows that $m - ma_0 e \in M$. Now, $m \in M$ and hence $ma_0 e \in M$. However, $ma_0 e \in Ae$ since Ae is a left ideal, thus $ma_0 e \in M \cap Ae = (0)$ and since m is arbitrary in M, the lemma is proved.

LEMMA 3.2. *Let A be a simple Banach algebra with minimal right ideals. If $j \in A$ and j has no left reverse, then there exists $a \neq 0$ such that $ja = a$.*

Proof. Let $J = [yj - y \colon y \in A]$. Then J is a regular left ideal of A which is proper since $j \notin J$. Hence by Lemma 3.1, there exists $a \in A$, $a \neq 0$ such that $Ja = (0)$, i.e. such that $yja - ya = 0$ for all $y \in A$ or $A(ja - a) = (0)$. Since $(A)_r = (0)$, this implies that $ja = a$.

LEMMA 3.3. *Let A be a simple $B^{\#}$-algebra with minimal right* *ideals. If* $|\ \ |$ *is any other norm in* A *with* $|a| \leqq ||a||$ *for each* $a \in A$, *then* $|\ \ | = ||\ \ ||$.

Proof. Lemma 3.2 implies that if $|\ \ |$ is any other norm in A, then $\lim_{n\to\infty} |a^n|^{1/n} = \lim_{n\to\infty} ||a^n||^{1/n}$ for every $a \in$ A (Cf [4], Lemma 3.1). Then since A is a $B^\#$-algebra, we have

$$|a^\#|\,|a| \geqq |a^\# a| \geqq \lim_{n\to\infty} |(a^\# a)^n|^{1/n}$$
$$= \lim_{n\to\infty} ||(a^\# a)^n||^{1/n} = ||a||\,||a|| ,$$

and since $|a^\#| \leqq ||a^\#||$ and $|a| \leqq ||a||$, the result follows.

THEOREM 3.1. *A Banach algebra* A *is the algebra* $F(X)$ *of all uniform limits of operators of finite rank in a reflexive Banach space* X *if and only if* A *is a simple, weakly compact,* $B^\#$*-algebra with minimal right ideals.*

Proof. Let A be a simple, weakly compact, $B^\#$-algebra with eA a minimal right ideal, e a primitive idempotent. We represent A as an algebra of operators $\mathscr{A}$ in eA, the latter regarded as a Banach space. Corresponding to each $a \in A$, we define an operator $\bar{a} \in \mathscr{A}$ by $\bar{a}: x \longrightarrow xa$ for $x \in eA$. The correspondence $a \longrightarrow \bar{a}$ is obviously an isomorphism and if we take $||\bar{a}|| = \sup_{||x||\leqq 1} ||xa||$, $x \in eA$, the correspondence is an isometry in view of Lemma 3.3. Thus A is isomorphic and isometric to $\mathscr{A}$ and A is the uniform closure of $\mathscr{A}$.

Next we show that eA is a reflexive Banach space. Now e has no left reverse in A; hence by Lemma 3.2, there exists $a \in A$, $a \neq 0$ such that $ea = a$. The set $P = [a \in A: ea = a]$ is a right ideal of A and since $P \subseteqq eA$, we must have $P = eA$ since eA is minimal. If e is now regarded as a left weakly compact operator on A, then it is clear that the set $P = eA$ is a reflexive Banach space.

Our next step is to show that in the representation described above, $\mathscr{A}$ contains all operators of finite rank in eA. Corresponding to each $a \in Ae$, there exists a continuous linear functional φ_a on eA satisfying $\varphi_a(x)e = xa$, $x \in eA$. Let $G = [\varphi_a \in (eA)^*: a \in A]$; then G is a linear subspace of $(eA)^*$. We show that G is closed with respect to the usual norm in $(eA)^*$ defined by $||\varphi|| = \sup_{||x||\leqq 1} |\varphi(x)|$ $x \in eA$. For $a \in Ae$, we have $xa = \varphi_a(x)e$, $x \in eA$, and since $||a|| = ||\bar{a}||$ for each $a \in A$, we have

$$
\begin{aligned}
||a|| = ||\bar{a}|| &= \sup_{||x||\leqq 1} ||xa|| \qquad\qquad a \in Ae \\
&= \sup_{||x||\leqq 1} ||\varphi_a(x)e|| \\
&= \sup_{||x||\leqq 1} |\varphi_a(x)|\,||e||
\end{aligned}
$$

$$= \|\varphi_a\| \cdot \|e\| .$$

Thus G is topologically equivalent to Ae and hence closed. Having proved that G is a closed linear subspace of $(eA)^*$, we now show that G is in fact the whole of $(eA)^*$. Suppose that there exists $\varphi' \in (eA)^*$ such that $\varphi' \notin G$. Since G is closed, there exists $\Phi \in (eA)^{**}$ such that $\Phi(\varphi_a) = 0$ for all $\varphi_a \in G$ and $\Phi(\varphi') = 1$. However, eA is a reflexive Banach space: hence there exists $u_0 \in eA$, $u_0 \neq 0$ such that $\Phi(\varphi) = \varphi(u_0)$ for all $\varphi \in (eA)^*$. In particular, for $\varphi_a \in G$, this implies that $0 = \varphi_a(u_0)e = u_0 a$ for all $a \in Ae$, which in turn implies that $u_0 \in (Ae)_l = (0)$ which is absurd. Hence $G = (eA)^*$. From this it follows that $\mathscr{A}$ contains all operators of rank 1 and hence all operators of finite rank in eA, since if T is an operator of rank 1 in eA, then there exists $\varphi \in (eA)^*$ and $u_0 \in eA$ such that $xT = \varphi(x)u_0$, $x \in eA$. Since $\varphi \in G$, there exist $a \in Ae$ and $\varphi_a \in (eA)^*$ such that $\varphi = \varphi_a$ and $xa = \varphi_a(x)e$. Let $u_0 = ea_0$ for some $a_0 \in A$; we have $xT = \varphi_a(x)u_0 = \varphi_a(x)ea_0 = xaa_0$, and since $aa_0 \in A$, the operator $aa_0 = T$ belongs to $\mathscr{A}$.

Finally, the uniform closure of the set of all operators of finite rank in eA is a closed two-sided ideal of $\mathscr{A}$ which must coincide with $\mathscr{A}$ since $\mathscr{A}$ is simple. Thus the "if" part of the theorem is proved.

That $F(X)$ is a simple, weakly compact $B^\sharp$-algebra with minimal ideals follows form corollary 1 and a result due to Bonsall and Goldie [1], Theorem 2. This completes the proof of the theorem.

REMARKS. 1. The problems discussed here were suggested by reading Ogasawara and Yoshinaga [2,3] and Bonsall [1].

2. Work on this paper was started at University College, Ibadan, and completed at Yale University. The author wishes to express his gratitude to the Carnegie Corporation of New York and to Yale University for financial support.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics
Vol. 12, No. 3 March, 1962