

# Pacific Journal of Mathematics

**OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH  
SPACE**

ADEGOKE OLUBUMMO

# OPERATORS OF FINITE RANK IN A REFLEXIVE BANACH SPACE

A. OLUBUMMO

1. Let  $X$  be a reflexive Banach space and  $F(X)$  the Banach algebra of all uniform limits of operators of finite rank, in  $X$ . Bonsall [1] has characterized  $F(X)$  as a simple,  $B^*$ -annihilator algebra:  $F(X)$  contains no proper closed two-sided ideals, every proper, closed right (left) ideal of  $F(X)$  has a nonzero left (right) annihilator, and, given any  $T \in F(X)$ , there exists  $T^\# \in F(X)$  such that

$$\|T\| \|T^\#\| = \|(TT^\#)^n\|^{1/n}, \quad n = 1, 2, 3, \dots$$

In this note, we obtain a new characterization for  $F(X)$  (Theorem 3.2): a Banach algebra  $A$  is the algebra  $F(X)$  of all uniform limits of operators of finite rank in a reflexive Banach space  $X$  if and only if  $A$  is a simple, weakly compact,  $B^*$ -algebra with minimal ideals ( $A$  is weakly compact if left- and right-multiplications by every  $a \in A$  are weakly compact operators). In the process of proving this result, we obtain a characterization of reflexive Banach spaces which seems to be of some independent interest (Theorem 2.2): a Banach space  $X$  is reflexive if and only if every operator in  $X$  of rank 1 is a weakly compact element of  $B(X)$ .

2. Let  $X$  be a Banach space and  $B = B(X)$  the Banach algebra of all bounded operators in  $X$  with the uniform topology. For  $T \in B$ , let  $R_T$  denote the operator in  $B$  obtained by multiplying elements of  $B$  on the right by  $T$ :  $R_T(A) = AT$  for  $A \in B$ .

Suppose that  $T$  is a fixed operator of rank 1 in  $X$  with  $H = [x \in X: Tx = 0]$ . Then  $H$  is a closed hyperplane in  $X$  and if  $x_0$  is an element of  $X$  such that  $Tx_0 \neq 0$ , then  $X = H \oplus (x_0)$  and we may assume that  $\|x_0\| = 1$ . Write  $B' = [S \in B: S(H) = (0)]$ . For each  $S \in B'$ , we define an element  $x_s$  of  $X$  by setting  $x_s = S(x_0)$ . The mapping  $S \rightarrow x_s$  is clearly linear.

**LEMMA 2.1.** *The linear mapping  $S \rightarrow x_s$  is a homeomorphism of  $B'$  onto  $X$ .*

*Proof.* It is clear that the mapping is one-to-one and, since  $\|S(x_0)\| \leq \|S\|$ , it is continuous. It is also onto; in fact, let  $\varphi \in X^*$  be such that  $\varphi(H) = (0)$ ,  $\varphi(x_0) = 1$ . Then for given  $x \in X$ , the operator  $S_x$  defined by setting  $S_x(y) = \varphi(y)x$ ,  $y \in X$  belongs to  $B'$  and is mapped into  $x$  by the mapping  $S \rightarrow S(x_0)$ . Hence, by the closed graph theorem, the

mapping is bicontinuous and the proof is complete.

Let  $B_1$  denote the unit ball in  $B$ , so that  $R_T(B_1) = [PT \in B: \|P\| \leq 1]$ .

LEMMA 2.2.  $R_T(B_1) = [A \in B': \|Ax_0\| \leq \|Tx_0\|]$ .

*Proof.* It is clear that  $R_T(B_1) \subset [A \in B': \|Ax_0\| \leq \|Tx_0\|]$ . Now let  $A \in B'$  with  $\|Ax_0\| \leq \|Tx_0\|$ ; we find  $P \in B_1$  such that  $A = PT$ . There exists  $\psi \in X^*$  such that  $\|\psi\| = 1$  and  $\psi(Tx_0) = \|Tx_0\|$ . We define  $P$  by setting  $Px = \psi(x)Ax_0/\|Tx_0\|$ . Then  $PTx = 0$  if  $x \in H$  and  $PTx_0 = Ax_0$ . Thus  $PT$  and  $A$  coincide in the subspace  $(x_0)$  and must therefore coincide everywhere in  $X$ . Finally  $\|P\| = \sup_{\|x\| \leq 1} \|\psi(x)Ax_0\|/\|Tx_0\| \leq 1$ ; hence  $P \in B_1$  and  $R_T(B_1) = [A \in B': \|Ax_0\| \leq \|Tx_0\|]$ .

LEMMA 2.3. *Let  $F$  be any subset of  $B'$ . If  $F^{B'}$  denotes the closure of  $F$  with respect to the weak topology of  $B'$  and  $F^B$  the closure of  $F$  with respect to the weak topology of  $B$ , then  $F^{B'} = F^B$ .*

*Proof.* Let  $P_0 \in F^{B'}$  and let

$$\begin{aligned} N &= N(P_0; \Phi_1, \Phi_2, \dots, \Phi_n; \varepsilon) \\ &= [P \in B: |\Phi_k(P - P_0)| < \varepsilon; k = 1, 2, \dots, n; \Phi_k \in B^*] \end{aligned}$$

be an arbitrary neighborhood of  $P_0$  in  $B$ . Then the neighborhood  $N' = N(P_0; \Phi'_1, \Phi'_2, \dots, \Phi'_n; \varepsilon)$  of  $P_0$  obtained by taking the restriction of  $\Phi_k$  to  $B'$  for each  $k$ , contains a point  $P$  of  $F$ . Since  $P$  must therefore belong to  $N$ , it follows that  $F^{B'} \subseteq F^B$ .

Now suppose that  $P_0 \in F^B$ . Then  $P_0 \in B'$  since  $B'$  is closed with respect to the weak topology of  $B(X)$  (being linear and strongly closed). Let  $N' = [P \in B': |\varphi_k(P - P_0)| < \varepsilon, k = 1, 2, \dots, n; \varphi_k \in (B')^*]$  be an arbitrary neighborhood of  $P_0$  in  $B'$ . Then again, by considering the neighborhood  $N = [P \in B: |\Phi_k(P - P_0)| < \varepsilon, k = 1, 2, \dots, n, \Phi_k \in B^*]$  obtained by extending  $\varphi_k$  to  $\Phi_k$ , for each  $k$ , on the whole of  $B$ , we can find  $P \in F$  such that  $P \in N'$ . Hence  $F^B \subseteq F^{B'}$ . This completes the proof.

THEOREM 2.1. *A Banach space  $X$  is reflexive if and only if every operator in  $X$  of rank 1 is a right weakly compact element of  $B(X)$ .*

*Proof.* If  $X$  is reflexive and  $T$  is of rank 1, then by Lemma 2.1,  $B'$  is homeomorphic with  $X$  under the correspondence  $S \mapsto S(x_0)$ . Now the image of  $B_1$  under  $R_T$  is a bounded subset of  $B'$  which is therefore contained in a set  $U$  which is compact with respect to the weak topology of  $B'$  and by Lemma 2.3, with respect to the weak topology of  $B(X)$ . Thus  $R_T$  is a weakly compact operator in  $B(X)$  and  $T$  is a right weakly compact element of  $B(X)$ .

Now suppose that  $R_T$  is weakly compact in  $B(X)$ . Then  $R_T(B_1)$  is contained in a set  $V \subset B'$  which is compact with respect to the weak topology of  $B(X)$  and hence also with respect to the weak topology of  $B'$ . Now the ball  $Q = [A \in B': \|A\| \leq \|Tx_0\|/\|x_0\|]$  is contained in  $R_T(B_1) \subset V$  and is weakly closed. Hence  $Q$  is compact with respect to the weak topology of  $B'$  and therefore  $B'$  is reflexive. Since  $B'$  is homeomorphic with  $X$ , it follows that  $X$  is reflexive and the proof is complete.

**COROLLARY 2.1.** *If  $X$  is a reflexive Banach space, then the algebra  $F(X)$  of all uniform limits of operators of finite rank in  $X$  is a weakly compact algebra.*

**COROLLARY 2.2.** (Ogasawara [2] Theorem 4.) *Let  $H$  be a Hilbert space and  $B(H)$  the Banach algebra of all bounded operators in  $H$ . If  $T$  is a compact operator in  $H$ , then  $T$  is a weakly compact element of  $B(H)$ .*

3. This section is devoted to the study of simple, weakly compact,  $B^*$ -algebras with minimal ideals.

**LEMMA 3.1.** *Let  $A$  be a simple Banach algebra with minimal ideals. Then every maximal regular left ideal  $M$  of  $A$  has a nonzero right annihilator.*

*Proof.* Since  $A$  is a simple Banach algebra, there exists an idempotent  $e \in A$  such that  $M \cap Ae = (0)$  and  $M \oplus Ae = A$ . Since  $M$  is regular, there is  $j \in A$  such that  $xj - x \in M$  for every  $x \in A$ . For some  $a_0 \in A$  and  $m_0 \in M$ ,  $j = m_0 + a_0e$ ,  $a_0e \neq 0$ . Suppose now that  $m$  is an arbitrary element in  $M$ . We have  $mj - m \in M$  and  $mj - ma_0e = mm_0 \in M$ , from which it follows that  $m - ma_0e \in M$ . Now,  $m \in M$  and hence  $ma_0e \in M$ . However,  $ma_0e \in Ae$  since  $Ae$  is a left ideal, thus  $ma_0e \in M \cap Ae = (0)$  and since  $m$  is arbitrary in  $M$ , the lemma is proved.

**LEMMA 3.2.** *Let  $A$  be a simple Banach algebra with minimal right ideals. If  $j \in A$  and  $j$  has no left reverse, then there exists  $a \neq 0$  such that  $ja = a$ .*

*Proof.* Let  $J = [yj - y: y \in A]$ . Then  $J$  is a regular left ideal of  $A$  which is proper since  $j \notin J$ . Hence by Lemma 3.1, there exists  $a \in A$ ,  $a \neq 0$  such that  $Ja = (0)$ , i.e. such that  $yja - ya = 0$  for all  $y \in A$  or  $A(ja - a) = (0)$ . Since  $(A)_r = (0)$ , this implies that  $ja = a$ .

**LEMMA 3.3.** *Let  $A$  be a simple  $B^*$ -algebra with minimal right*

ideals. If  $\|\cdot\|$  is any other norm in  $A$  with  $\|a\| \leq \|a\|$  for each  $a \in A$ , then  $\|\cdot\| = \|\cdot\|$ .

*Proof.* Lemma 3.2 implies that if  $\|\cdot\|$  is any other norm in  $A$ , then  $\lim_{n \rightarrow \infty} \|a^n\|^{1/n} = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$  for every  $a \in A$  (Cf [4], Lemma 3.1). Then since  $A$  is a  $B^*$ -algebra, we have

$$\begin{aligned} \|a^*\| \|a\| &\geq \|a^*a\| \geq \lim_{n \rightarrow \infty} \|(a^*a)^n\|^{1/n} \\ &= \lim_{n \rightarrow \infty} \|(a^*a)^n\|^{1/n} = \|a\| \|a\|, \end{aligned}$$

and since  $\|a^*\| \leq \|a^*\|$  and  $\|a\| \leq \|a\|$ , the result follows.

**THEOREM 3.1.** *A Banach algebra  $A$  is the algebra  $F(X)$  of all uniform limits of operators of finite rank in a reflexive Banach space  $X$  if and only if  $A$  is a simple, weakly compact,  $B^*$ -algebra with minimal right ideals.*

*Proof.* Let  $A$  be a simple, weakly compact,  $B^*$ -algebra with  $eA$  a minimal right ideal,  $e$  a primitive idempotent. We represent  $A$  as an algebra of operators  $\mathcal{A}$  in  $eA$ , the latter regarded as a Banach space. Corresponding to each  $a \in A$ , we define an operator  $\bar{a} \in \mathcal{A}$  by  $\bar{a}: x \rightarrow xa$  for  $x \in eA$ . The correspondence  $a \rightarrow \bar{a}$  is obviously an isomorphism and if we take  $\|\bar{a}\| = \sup_{\|x\| \leq 1} \|xa\|$ ,  $x \in eA$ , the correspondence is an isometry in view of Lemma 3.3. Thus  $A$  is isomorphic and isometric to  $\mathcal{A}$  and  $A$  is the uniform closure of  $\mathcal{A}$ .

Next we show that  $eA$  is a reflexive Banach space. Now  $e$  has no left reverse in  $A$ ; hence by Lemma 3.2, there exists  $a \in A$ ,  $a \neq 0$  such that  $ea = a$ . The set  $P = [a \in A: ea = a]$  is a right ideal of  $A$  and since  $P \subseteq eA$ , we must have  $P = eA$  since  $eA$  is minimal. If  $e$  is now regarded as a left weakly compact operator on  $A$ , then it is clear that the set  $P = eA$  is a reflexive Banach space.

Our next step is to show that in the representation described above,  $\mathcal{A}$  contains all operators of finite rank in  $eA$ . Corresponding to each  $a \in Ae$ , there exists a continuous linear functional  $\varphi_a$  on  $eA$  satisfying  $\varphi_a(x)e = xa$ ,  $x \in eA$ . Let  $G = [\varphi_a \in (eA)^*: a \in A]$ ; then  $G$  is a linear subspace of  $(eA)^*$ . We show that  $G$  is closed with respect to the usual norm in  $(eA)^*$  defined by  $\|\varphi\| = \sup_{\|x\| \leq 1} |\varphi(x)|$ ,  $x \in eA$ . For  $a \in Ae$ , we have  $xa = \varphi_a(x)e$ ,  $x \in eA$ , and since  $\|a\| = \|\bar{a}\|$  for each  $a \in A$ , we have

$$\begin{aligned} \|a\| &= \|\bar{a}\| = \sup_{\|x\| \leq 1} \|xa\| && a \in Ae \\ &= \sup_{\|x\| \leq 1} \|\varphi_a(x)e\| \\ &= \sup_{\|x\| \leq 1} |\varphi_a(x)| \|e\| \end{aligned}$$

$$= \|\varphi_a\| \cdot \|e\|.$$

Thus  $G$  is topologically equivalent to  $Ae$  and hence closed. Having proved that  $G$  is a closed linear subspace of  $(eA)^*$ , we now show that  $G$  is in fact the whole of  $(eA)^*$ . Suppose that there exists  $\varphi' \in (eA)^*$  such that  $\varphi' \notin G$ . Since  $G$  is closed, there exists  $\Phi \in (eA)^{**}$  such that  $\Phi(\varphi_a) = 0$  for all  $\varphi_a \in G$  and  $\Phi(\varphi') = 1$ . However,  $eA$  is a reflexive Banach space: hence there exists  $u_0 \in eA$ ,  $u_0 \neq 0$  such that  $\Phi(\varphi) = \varphi(u_0)$  for all  $\varphi \in (eA)^*$ . In particular, for  $\varphi_a \in G$ , this implies that  $0 = \varphi_a(u_0)e = u_0a$  for all  $a \in Ae$ , which in turn implies that  $u_0 \in (Ae)_i = (0)$  which is absurd. Hence  $G = (eA)^*$ . From this it follows that  $\mathcal{A}$  contains all operators of rank 1 and hence all operators of finite rank in  $eA$ , since if  $T$  is an operator of rank 1 in  $eA$ , then there exists  $\varphi \in (eA)^*$  and  $u_0 \in eA$  such that  $xT = \varphi(x)u_0$ ,  $x \in eA$ . Since  $\varphi \in G$ , there exist  $a \in Ae$  and  $\varphi_a \in (eA)^*$  such that  $\varphi = \varphi_a$  and  $xa = \varphi_a(x)e$ . Let  $u_0 = ea_0$  for some  $a_0 \in A$ ; we have  $xT = \varphi_a(x)u_0 = \varphi_a(x)ea_0 = xaa_0$ , and since  $aa_0 \in A$ , the operator  $aa_0 = T$  belongs to  $\mathcal{A}$ .

Finally, the uniform closure of the set of all operators of finite rank in  $eA$  is a closed two-sided ideal of  $\mathcal{A}$  which must coincide with  $\mathcal{A}$  since  $\mathcal{A}$  is simple. Thus the "if" part of the theorem is proved.

That  $F(X)$  is a simple, weakly compact  $B^*$ -algebra with minimal ideals follows from corollary 1 and a result due to Bonsall and Goldie [1], Theorem 2. This completes the proof of the theorem.

REMARKS. 1. The problems discussed here were suggested by reading Ogasawara and Yoshinaga [2,3] and Bonsall [1].

2. Work on this paper was started at University College, Ibadan, and completed at Yale University. The author wishes to express his gratitude to the Carnegie Corporation of New York and to Yale University for financial support.

## REFERENCES

1. F. F. Bonsall, *A minimal property of the norm in some Banach algebras*, Journal London Math. Soc., **29** (1954), 156-164.
2. T. Ogasawara, *Finite dimensionality of certain Banach algebras*, Journal of Science, Hiroshima University Series A **17** (1953), 359-364.
3. T. Ogasawara and K. Yoshinaga, *Weakly completely continuous Banach  $*$ -Algebras*, Journal of Science, Hiroshima University Series A **18** (1954), 15-36.
4. A. Olubummo, *Left completely continuous  $B^*$ -algebras*, Journal London Math. Soc., **32** (1957), 270-276.

UNIVERSITY COLLEGE, IBADAN, NIGERIA  
AND YALE UNIVERSITY



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

RALPH S. PHILLIPS

Stanford University  
Stanford, California

M. G. ARSOVE

University of Washington  
Seattle 5, Washington

A. L. WHITEMAN

University of Southern California  
Los Angeles 7, California

LOWELL J. PAIGE

University of California  
Los Angeles 24, California

## ASSOCIATE EDITORS

E. F. BECKENBACH

T. M. CHERRY

D. DERRY

M. OHTSUKA

H. L. ROYDEN

E. SPANIER

E. G. STRAUS

F. WOLF

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CALIFORNIA RESEARCH CORPORATION  
SPACE TECHNOLOGY LABORATORIES  
NAVAL ORDNANCE TEST STATION

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.



Alfred Aeppli, <i>Some exact sequences in cohomology theory for Kähler manifolds</i> .....	791
Paul Richard Beesack, <i>On the Green's function of an <math>N</math>-point boundary value problem</i> .....	801
James Robert Boen, <i>On <math>p</math>-automorphic <math>p</math>-groups</i> .....	813
James Robert Boen, Oscar S. Rothaus and John Griggs Thompson, <i>Further results on <math>p</math>-automorphic <math>p</math>-groups</i> .....	817
James Henry Bramble and Lawrence Edward Payne, <i>Bounds in the Neumann problem for second order uniformly elliptic operators</i> .....	823
Chen Chung Chang and H. Jerome (Howard) Keisler, <i>Applications of ultraproducts of pairs of cardinals to the theory of models</i> .....	835
Stephen Urban Chase, <i>On direct sums and products of modules</i> .....	847
Paul Civin, <i>Annihilators in the second conjugate algebra of a group algebra</i> .....	855
J. H. Curtiss, <i>Polynomial interpolation in points equidistributed on the unit circle</i> .....	863
Marion K. Fort, Jr., <i>Homogeneity of infinite products of manifolds with boundary</i> .....	879
James G. Glimm, <i>Families of induced representations</i> .....	885
Daniel E. Gorenstein, Reuben Sandler and William H. Mills, <i>On almost-commuting permutations</i> .....	913
Vincent C. Harris and M. V. Subba Rao, <i>Congruence properties of <math>\sigma_r(N)</math></i> .....	925
Harry Hochstadt, <i>Fourier series with linearly dependent coefficients</i> .....	929
Kenneth Myron Hoffman and John Wermer, <i>A characterization of <math>C(X)</math></i> .....	941
Robert Weldon Hunt, <i>The behavior of solutions of ordinary, self-adjoint differential equations of arbitrary even order</i> .....	945
Edward Takashi Kobayashi, <i>A remark on the Nijenhuis tensor</i> .....	963
David London, <i>On the zeros of the solutions of <math>w''(z) + p(z)w(z) = 0</math></i> .....	979
Gerald R. Mac Lane and Frank Beall Ryan, <i>On the radial limits of Blaschke products</i> .....	993
T. M. MacRobert, <i>Evaluation of an <math>E</math>-function when three of its upper parameters differ by integral values</i> .....	999
Robert W. McKelvey, <i>The spectra of minimal self-adjoint extensions of a symmetric operator</i> .....	1003
Adegoke Olubummo, <i>Operators of finite rank in a reflexive Banach space</i> .....	1023
David Alexander Pope, <i>On the approximation of function spaces in the calculus of variations</i> .....	1029
Bernard W. Roos and Ward C. Sangren, <i>Three spectral theorems for a pair of singular first-order differential equations</i> .....	1047
Arthur Argyle Sagle, <i>Simple Malcev algebras over fields of characteristic zero</i> .....	1057
Leo Sario, <i>Meromorphic functions and conformal metrics on Riemann surfaces</i> .....	1079
Richard Gordon Swan, <i>Factorization of polynomials over finite fields</i> .....	1099
S. C. Tang, <i>Some theorems on the ratio of empirical distribution to the theoretical distribution</i> .....	1107
Robert Charles Thompson, <i>Normal matrices and the normal basis in abelian number fields</i> .....	1115
Howard Gregory Tucker, <i>Absolute continuity of infinitely divisible distributions</i> .....	1125
Elliot Carl Weinberg, <i>Completely distributed lattice-ordered groups</i> .....	1131
James Howard Wells, <i>A note on the primes in a Banach algebra of measures</i> .....	1139
Horace C. Wiser, <i>Decomposition and homogeneity of continua on a 2-manifold</i> .....	1145