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1. Introduction. Malcev algebras are a natural generalization of Lie algebras suggested by introducing the commutator of two elements as a new multiplicative operation in an alternative algebra [3]. The defining identities obtained in this way for a Malcev algebra A are

$$xy = -yx \tag{1.1}$$

$$xy \cdot xz = (xy \cdot z)x + (yz \cdot x)x + (zx \cdot x)y \tag{1.2}$$

for all $x, y, z \in A$. Since Albert [1] has shown that every simple alternative ring which contains an idempotent not its unity quantity is either associative or the split Cayley-Dickson algebra C, it is natural to see if a simple Malcev algebra can be obtained from C. In [3] a seven dimensional simple non-Lie Malcev algebra A^* is obtained from C and is discussed in detail. In this paper we shall prove the following

THEOREM. *Let A be a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field of characteristic zero. Furthermore assume A contains an element u such that the right multiplication by u, R_u, is not a nilpotent linear transformation. Then A is isomorphic to A^*.*

The necessary identities and notation from [3] for any algebra A are repeated here for convenience:

$$\text{Commutator,} \quad (x, y) = [x, y] = xy - yx \tag{1.3}$$

$$\text{Associator,} \quad (x, y, z) = xy \cdot z - x \cdot yz \tag{1.4}$$

$$\text{Jacobian,} \quad J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y \tag{1.5}$$

for $x, y, z \in A$. If $h(x_1, \cdots, x_n)$ is a function of n indeterminates such that for any n subsets B_i of A and $b_i \in B_i$, the elements $h(b_1, \cdots, b_n)$ are in A, then $h(B_1, \cdots, B_n)$ will denote the linear subspace of A spanned by all of the elements $h(b_1, \cdots, b_n)$.

For a Malcev algebra A of characteristic not 2 or 3, we shall use the following identities and theorems from [3]:

$$J(x, y, xz) = J(x, y, z)x \tag{1.6}$$

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$$J(x, y, wz) + J(w, y, xz) = J(x, y, z)w + J(w, y, z)x \tag{1.7}$$

$$2wJ(x, y, z) = J(w, x, yz) + J(w, y, zx) + J(w, z, xy) \tag{1.8}$$

$$J(wx, y, z) = wJ(x, y, z) + J(w, y, z)x - 2J(yz, w, x) \tag{1.9}$$

$$xy \cdot zw = x(wy \cdot z) + w(yz \cdot x) + y(zx \cdot w) + z(xw \cdot y) \tag{1.10}$$

for all $w, x, y, z \in A$. If $N = \{x \in A\colon J(x, A, A) = 0\}$, then it is shown in [3] that N is an ideal of A which is a Lie subalgebra and furthermore for $a, b \in A$

$$J(a, b, A) = 0 \quad \text{implies } ab \in N\,. \tag{1.11}$$

It is also shown in [3] that $J(A, A, A)$ is an ideal of A. Thus if A is a simple non-Lie Malcev algebra we have

$$N = 0 \quad \text{and} \quad A = J(A, A, A)\,. \tag{1.12}$$

We shall assume throughout this paper that A is a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field F of characteristic not 2 or 3 containing an element u such that R_u is not a nilpotent linear tansformation. In § 2 the basic multiplicative identities are derived using methods analogous to those of Lie algebras. Decomposing $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\gamma$ into weight spaces relative to R_u [2; page 132] we prove the block multiplication identities $A_\alpha A_\beta \subset A_{\alpha+\beta}$ if $\alpha \neq \beta$, $A_\alpha^2 \subset A_{-\alpha}$ and $A_0^2 = 0$. Further identities are derived in § 3 which lead to the important result that there exists a nonzero weight α such that $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ where $A_0 = A_\alpha A_{-\alpha}$.

In § 4 we show that $R(A_0)$, the set of right multiplications R_{x_0} by elements $x_0 \in A_0$, is a set of commuting linear transformations on the subspaces A_0, A_α and $A_{-\alpha}$. Analogous to Lie algebras we decompose $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ into weight spaces relative to $R(A_0)$ [2; page 133] and thus find a basis of A which simultaneously triangulates the matrices of $R(A_0)$. We now introduce the trace form, $(x, y) = \operatorname{trace} R_x R_y$, in § 5 and assume for the remainder of the paper that the algebraically closed field is of characteristic zero. With this and the results of § 4 we easily show that (x, y) is a nondegenerate invariant form on $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ and $A_0 = uF$.

In § 6 we show that R_u has a diagonal matrix of the form

$$\begin{bmatrix} 0 & & 0 \\ & \alpha I & \\ 0 & & -\alpha I \end{bmatrix}$$

Using this and a few more identities we show in § 7 that the simple Malcev algebra $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ is isomorphic to the seven dimensional algebra A^*.

2. **Basic multiplication identities.** Let R_u $(u \in A)$ be a fixed non-nilpotent linear transformation and decompose the simple Malcev algebra A into the *weight space direct sum* $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\gamma$ relative to R_u where the *weight space* of R_u,

$$A_a = \{x \in A\colon x(aI - R_u)^k = 0 \text{ for some integer } k > 0\} ,$$

is a nonzero R_u-invariant subspace of A corresponding to the *weight* a of R_u. Let $x_\alpha \in A_\alpha$, $x_\beta \in A_\beta$, then using (1.6)

$$J(u, x_\alpha, x_\beta)R_u = J(u, x_\alpha, x_\beta)u = J(u, x_\alpha, ux_\beta) = -J(u, x_\alpha, x_\beta R_u)$$

and therefore

$$J(u, x_\alpha, x_\beta)(\beta I + R_u) = J(u, x_\alpha, x_\beta(\beta I - R_u)) .$$

Now letting $y_\beta = x_\beta(\beta I - R_u) \in A_\beta$ we have

$$\begin{aligned} J(u, x_\alpha, x_\beta(\beta I - R_u)^2) &= J(u, x_\alpha, y_\beta(\beta I - R_u)) \\ &= J(u, x_\alpha, y_\beta)(\beta I + R_u) \\ &= J(u, x_\alpha, x_\beta(\beta I - R_u))(\beta I + R_u) \\ &= (u, x_\alpha, x_\beta)(\beta I + R_u)^2 . \end{aligned}$$

Continuing by induction we obtain

$$J(u, x_\alpha, x_\beta)(\beta I + R_u)^n = J(u, x_\alpha, x_\beta(\beta I - R_u)^n) \tag{2.1}$$

for every integer n. Since $x_\beta \in A_\beta$ there exists an integer N such that $0 = J(u, x_\alpha, x_\beta(\beta I - R_u)^N) = J(u, x_\alpha, x_\beta)(\beta I + R_u)^N$ and this shows $J(u, x_\alpha, x_\beta) \in A_{-\beta}$. Now interchanging the roles of x_β and x_α in (2.1) we also obtain $J(u, x_\alpha, x_\beta) \in A_{-\alpha}$ and thus

$$J(u, A_\alpha, A_\beta) \subset A_{-\alpha} \cap A_{-\beta} . \tag{2.2}$$

From (2.2) we have the following relations

$$J(u, A_\alpha, A_\alpha) \subset A_{-\alpha} \tag{2.3}$$

$$J(u, A_\alpha, A_\beta) = 0 \quad \text{if } \alpha \neq \beta . \tag{2.4}$$

We shall now prove

$$A_\alpha A_\beta \subset A_{\alpha+\beta} \quad \text{if } \alpha \neq \beta . \tag{2.5}$$

For if $\alpha \neq \beta$ and $x_\alpha \in A_\alpha$, $x_\beta \in A_\beta$ we have by (2.4),

$$0 = J(u, x_\alpha, x_\beta) = (x_\alpha x_\beta)Ru - x_\alpha R_u \cdot x_\beta - x_\alpha \cdot x_\beta R_u ;$$

that is, $(x_\alpha x_\beta)R_u = x_\alpha R_u \cdot x_\beta + x_\alpha \cdot x_\beta R_u$ and so R_u is a derivation of

$A_\alpha A_\beta$ into $A_\alpha A_\beta$. This yields

$$(x_\alpha x_\beta)(R_u - (\alpha + \beta)I) = x_\alpha(R_u - \alpha I) \cdot x_\beta + x_\alpha \cdot x_\beta(R_u - \beta I)$$

and in the usual was we prove the Lebnitz rule for derivations which then yields that for some integer N, $(x_\alpha x_\beta)(R_u - (\alpha + \beta)I)^N = 0$ and therefore $x_\alpha x_\beta \in A_{\alpha+\beta}$. In particular we have

$$A_0 A_\alpha \subset A_\alpha \quad \text{if } \alpha \neq 0 . \tag{2.6}$$

We shall now investigate A_0 more closely. Let $x_\alpha \in A_\alpha$, $x_\beta \in A_\beta$ and $x_0 \in A_0$, then by (1.7) $J(x_0, x_\beta, ux_\alpha) + J(u, x_\beta, x_0 x_\alpha) = J(x_0, x_\beta, x_\alpha)u + J(u, x_\beta, x_\alpha)x_0$. Therefore if $0 \neq \alpha \neq \beta$ we have by (2.4) $J(x_0, x_\beta, ux_\alpha) = J(x_0, x_\beta, x_\alpha)u$. This yields $J(x_0, x_\beta, x_\alpha(\alpha I - R_u)) = J(x_0, x_\beta, x_\alpha)(\alpha I + R_u)$ and as in the proof of (2.4) we obtain

$$J(A_0, A_\alpha, A_\beta) = 0 \quad \text{if } 0 \neq \alpha \neq \beta \neq 0 . \tag{2.7}$$

Next let $x_0, y_0 \in A_0$ and $x_\alpha \in A_\alpha$ where $\alpha \neq 0$, then using (1.9), (2.4) and (2.6) we have

$$\begin{aligned} J(x_0 u, y_0, x_\alpha) &= x_0 J(u, y_0, x_\alpha) + J(x_0, y_0, x_\alpha)u - 2J(y_0 x_\alpha, x_0, u) \\ &= J(x_0, y_0, x_\alpha)u \end{aligned}$$

and in general we have $J(x_0 R_u^n, y_0, x_\alpha) = J(x_0, y_0, x_\alpha)R_u^n$ which implies $J(x_0, y_0, x_\alpha) \in A_0$. Now by (1.7), $J(x_0, y_0, ux_\alpha) + J(u, y_0, x_0 x_\alpha) = J(x_0, y_0, x_\alpha)u + J(u, y_0, x_\alpha)x_0$; and using (2.4) and (2.6) we obtain $J(x_0, y_0, x_\alpha R_u) = -J(x_0, y_0, x_\alpha)R_u$ which implies $J(x_0, y_0, x_\alpha(R_u - \alpha I)) = -J(x_0, y_0, x_\alpha)(R_u + \alpha I)$. Thus, as usual, we have $J(x_0, y_0, x_\alpha) \in A_{-\alpha}$ and therefore $J(x_0, y_0, x_\alpha) \in A_0 \cap A_{-\alpha}$ which proves

$$J(A_0, A_0, A_\alpha) = 0 \quad \text{if } \alpha \neq 0 . \tag{2.8}$$

We shall now show $A_0^2 \subset A_0$. From our basic decomposition $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\gamma$ relative to R_u we can find a basis $\{x_1(\tau), \cdots, x_m(\tau)\}$ $(m = m_\tau)$ of A_τ such that

$$x_i(\tau)R_u = \sum_{j=1}^{i-1} a_{ij} x_j(\tau) + \tau x_i(\tau) \tag{2.9}$$

where $\tau, a_{ij} \in F$ and $i = 1, \cdots, m$. In particular let $\{x_1(0), \cdots, x_{n_0}(0)\} \equiv \{x_1, \cdots, x_n\}$ be the above type for A_0. Then $x_1 R_u = 0$ and

$$x_i R_u = \sum_{k=1}^{i-1} a_{ik} x_k \qquad (i = 2, \cdots, n) .$$

Furthermore,

$$\begin{aligned} J(u, x_i, x_j) &= (x_i x_j)R_u + x_j R_u \cdot x_i + x_j \cdot x_i R_u \\ &= (x_i x_j)R_u + \sum_{k=1}^{j-1} a_{jk} x_k x_i + \sum_{k=1}^{i-1} \alpha_{ik} x_j x_k \end{aligned}$$

with the understanding that $a_{10} = 0$.

Using (1.6) and operating on both sides of the previous equation with R_u^n, we obtain

$$\begin{aligned}(-1)^n J(u, x_i, x_j R_u^n) &= J(u, x_i, x_j)R_u^n \\ &= (x_i x_j)R_u^{n+1} + \sum_{k=1}^{j-1} a_{jk}(x_k x_i)R_u^n \\ &\quad + \sum_{k=1}^{i-1} a_{ik}(x_j x_k)R_u^n .\end{aligned}$$

Now by assuming $i < j$ and choosing n large enough, a simple inductive argument yields $x_i x_j \in A_0$ for all i and j. Thus $A_0^2 \subset A_0$.

Using (1.8), $A_0^2 \subset A_0$ and (2.8) we have

$$A_\alpha J(A_0, A_0, A_0) \subset J(A_\alpha, A_0, A_0^2) \subset J(A_\alpha, A_0, A_0) = 0 \quad \text{for } \alpha \neq 0 .$$

Thus, $AJ(A_0, A_0, A_0) \subset \sum_\alpha A_\alpha J(A_0, A_0, A_0) = A_0 J(A_0, A_0, A_0) \subset J(A_0, A_0, A_0)$, or $J(A_0, A_0, A_0)$ is an ideal of A. But since $J(A_0, A_0, A_0) \subset A_0 \neq A$ and A is simple we have

$$J(A_0, A_0, A_0) = 0 . \tag{2.10}$$

Now using (2.8) and (2.10) we have $J(A_0, A_0, A) = \sum_\alpha J(A_0, A_0, A_\alpha) = 0$ and by (1.11) and (1.12),

$$A_0^2 \subset N = 0 . \tag{2.11}$$

In particular this means the kernel of R_u is A_0.

We shall now show $A_\alpha^2 \subset A_{-\alpha}$. Let $x_\alpha, y_\alpha \in A_\alpha$ for $\alpha \neq 0$, then by (2.3) $J(u, x_\alpha, y_\alpha) = (x_\alpha y_\alpha)R_u + y_\alpha R_u \cdot x_\alpha + y_\alpha \cdot x_\alpha R_u = w_{-\alpha} \in A_{-\alpha}$. Therefore $(x_\alpha y_\alpha)R_u = x_\alpha R_u \cdot y_\alpha + y_\alpha \cdot y_\alpha R_u + w_{-\alpha}$ which yields

$$(x_\alpha y_\alpha)(R_u - 2\alpha I) = x_\alpha(R_u - \alpha I) \cdot y_\alpha + x_\alpha \cdot y_\alpha(R_u - \alpha I) + w_{-\alpha}^{(1)} .$$

By induction we obtain

$$(x_\alpha y_\alpha)(R_u - 2\alpha I)^n = w_{-\alpha}^{(n)} + \sum_{r=0}^{n} C_{n,r} x_\alpha(R_u - \alpha I)^{n-r} \cdot y_\alpha(R_u - \alpha I)^r$$

where $w_{-\alpha}^{(n)} \in A_{-\alpha}$. Therefore for large enough N, $(x_\alpha y_\alpha)(R_u - 2\alpha I)^N \in A_{-\alpha}$. Now let $x_\alpha y_\alpha = \sum_\gamma z_\gamma$ where $z_\gamma \in A_\gamma$, then $(x_\alpha y_\alpha)(R_u - 2\alpha I)^N = \sum_\gamma z_\gamma(R_u - 2\alpha I)^N \in A_{-\alpha}$. Therefore by the R_u-invariance of the A_γ and the uniqueness of the decomposition $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\lambda$, $z_\gamma(R_u - 2\alpha I)^N = 0$ if $\gamma \neq -\alpha$. Thus if $\gamma \neq -\alpha$, $z_\gamma \in A_{2\alpha}$. Therefore $x_\alpha y_\alpha = z_{2\alpha} + z_{-\alpha}$ which proves

$$A_\alpha^2 \subset A_{2\alpha} \oplus A_{-\alpha} . \tag{2.12}$$

LEMMA 2.13. $J(u, A_\alpha^2, A_{2\alpha}) = 0$.

Proof. Using (2.12), (2.7) and (2.3) we have

$$J(u, A_\alpha^2, A_{2\alpha}) \subset J(u, A_{-\alpha}, A_{2\alpha}) + J(u, A_{2\alpha}, A_{2\alpha}) \subset J(u, A_{2\alpha}, A_{2\alpha}) \subset A_{-2\alpha} .$$

Now for any $x, y \in A_\alpha$, $z \in A_{2\alpha}$ we have by (1.7) $J(z, u, xy) + J(x, u, zy) = J(z, u, y)x + J(x, u, y)z$ and using (2.4), (2.5) and (2.3) this yields $J(z, u, xy) = J(x, u, y)z \in A_{-\alpha} \cdot A_{2\alpha} \subset A_\alpha$. Combining these results we have $J(u, A_\alpha^2, A_{2\alpha}) \subset A_\alpha \cap A_{-2\alpha} = 0$.

Now let $w \in A_{2\alpha}, x, y \in A_\alpha$ and $xy = z_{2\alpha} + z_{-\alpha}$ where $z_{2\alpha} \in A_{2\alpha}$, $z_{-\alpha} \in A_{-\alpha}$, then using Lemma 2.13 and the fact $J(u, A_{-\alpha}, A_{2\alpha}) = 0$ we have

$$0 = J(u, xy, w) = J(u, z_{2\alpha}, w) + J(u, z_{-\alpha}, w) = J(u, z_{2\alpha}, w) ;$$

that is,

$$J(u, z_{2\alpha}, A_{2\alpha}) = 0 .$$

Now since $z_{2\alpha} \in A_{2\alpha}$ we also have by (2.4) $J(u, z_{2\alpha}, A_\beta) = 0$ if $\beta \neq 2\alpha$. Combining these results, $J(u, z_{2\alpha}, A) = \sum_\beta J(u, z_{2\alpha}, A_\beta) = 0$ and therefore $z_{2\alpha}u \in N = 0$ by (1.11) and (1.12). Thus $0 = z_{2\alpha}R_u$ and therefore $z_{2\alpha} \in A_0 \cap A_{2\alpha} = 0$ and this proves

$$A_\alpha^2 \subset A_{-\alpha} . \tag{2.14}$$

Also note that we now have

$$J(A_0, A_\alpha, A_\alpha) \subset A_{-\alpha} . \tag{2.15}$$

3. More identities.

Let $A = A_0 \oplus A_\alpha \oplus \cdots \oplus A_\gamma$ be the decomposition of A into a weight space direct sum relative to R_u and suppose that for weights α, β, γ of R_u, $\beta \neq \gamma$ and $\beta + \gamma \neq \alpha$. Then for $x \in A_\alpha$, $y \in A_\beta$ and $z \in A_\gamma$ we have by (1.9) and (2.4)

$$J(xu, y, z) = xJ(u, y, z) + J(x, y, z)u - 2J(yz, x, u) = J(x, y, z)u$$

and therefore $J(x(R_u - \alpha I), y, z) = J(x, y, z)(R_u - \alpha I)$. By induction we have $J(x(R_u - \alpha I)^n, y, z) = J(x, y, z)(R_u - \alpha I)^n$ and hence

$$J(A_\alpha, A_\beta, A_\gamma) \subset A_\alpha \quad \text{if } \beta \neq \gamma \text{ and } \beta + \gamma \neq \alpha . \tag{3.1}$$

By the symmetry of the α, β and γ we may also conclude

$$J(A_\beta, A_\gamma, A_\alpha) \subset A_\beta \quad \text{if } \gamma \neq \alpha \text{ and } \gamma + \alpha \neq \beta \tag{3.2}$$

$$J(A_\gamma, A_\alpha, A_\beta) \subset A_\gamma \quad \text{if } \alpha \neq \beta \text{ and } \alpha + \beta \neq \gamma . \tag{3.3}$$

Now assume $\alpha \neq \beta \neq \gamma \neq \alpha$. Suppose $\beta + \gamma = \alpha$. If $\gamma + \alpha = \beta$, then $\gamma = 0$ and therefore $\alpha = \beta$, a contradiction. Therefore $\gamma + \alpha \neq \beta$ and by (3.2) $J(A_\beta, A_\gamma, A_\alpha) \subset A_\beta$. Similarly if $\alpha + \beta = \gamma$, then $\beta = 0$ and $\alpha = \gamma$, a contradiction. Therefore $\alpha + \beta \neq \gamma$ and by (3.3) $J(A_\gamma, A_\alpha, A_\beta) \subset A_\gamma$. Thus we have $J(A_\alpha, A_\beta, A_\gamma) \subset A_\gamma \cap A_\beta = 0$ if $\alpha \neq \beta \neq \gamma \neq \alpha$ and $\beta + \gamma = \alpha$.

With the assumption $\alpha \neq \beta \neq \gamma \neq \alpha$, suppose now that $\beta + \gamma \neq \alpha$. Then by (3.1), $J(A_\alpha, A_\beta, A_\gamma) \subset A_\alpha$. We next note that it is impossible to have $\gamma + \alpha = \beta$ and $\alpha + \beta = \gamma$. So using (3.2) or (3.3) together with $J(A_\alpha, A_\beta, A_\gamma) \subset A_\alpha$ we conclude $J(A_\alpha, A_\beta, A_\gamma) = 0$. Thus we can conclude, using the preceding paragraph,

$$J(A_\alpha, A_\beta, A_\gamma) = 0 \quad \text{if } \alpha \neq \beta \neq \gamma \neq \alpha\,. \tag{3.4}$$

Now assume two weights are equal, that is, $\alpha = \beta$. Suppose $\gamma \neq 0, \alpha, -\alpha$ or 2α, then

$$\begin{aligned} J(A_\alpha, A_\alpha, A_\gamma) &\subset A_\alpha^2 A_\gamma + A_\alpha A_\gamma \cdot A_\alpha + A_\gamma A_\alpha \cdot A_\alpha \\ &\subset A_{-\alpha} A_\gamma + A_{\alpha+\gamma} A_\alpha \\ &\subset A_{-\alpha+\gamma} \oplus A_{\gamma+2\alpha}\,. \end{aligned}$$

However using (3.1) $J(A_\alpha, A_\alpha, A_\gamma) \subset A_\alpha$ and therefore $J(A_\alpha, A_\alpha, A_\gamma) \subset A_\alpha \cap (A_{-\alpha+\gamma} \oplus A_{\gamma+2\alpha}) = 0$. This proves

$$J(A_\alpha, A_\alpha, A_\gamma) = 0 \quad \text{if } \gamma \neq 0, \alpha, \text{ or } -\alpha\ 2\alpha\,. \tag{3.5}$$

For the "exceptional" cases we have

$$J(A_\alpha, A_\alpha, A_\alpha) \subset A_\alpha^2 \cdot A_\alpha \subset A_{-\alpha} A_\alpha \subset A_0\,. \tag{3.6}$$

$$J(A_\alpha, A_\alpha, A_0) \subset A_\alpha^2 A_0 + A_\alpha A_0 \cdot A_\alpha \subset A_{-\alpha}\,. \tag{3.7}$$

$$J(A_\alpha, A_\alpha, A_{-\alpha}) \subset A_\alpha^2 A_{-\alpha} + A_\alpha A_{-\alpha} \cdot A_\alpha \subset A_\alpha\,. \tag{3.8}$$

$$J(A_\alpha, A_\alpha, A_{2\alpha}) = 0\,. \tag{3.9}$$

To prove (3.9) let $x, y \in A_\alpha$, $z \in A_{2\alpha}$, then by (1.9), (2.5) and (2.4)

$$\begin{aligned} J(xu, y, z) &= xJ(u, y, z) + J(x, y, z)u - 2J(yz, x, u) \\ &= J(x, y, z)u \end{aligned}$$

and as usual we have $J(x(R_u - \alpha I)^n, y, z) = J(x, y, z)(R_u - \alpha I)^n$. Therefore $J(x, y, z) \in A_\alpha$. However by (1.7) $J(x, y, uz) + J(u, y, xz) = J(x, y, z)u + J(u, y, z)x$ and using (2.4) we obtain $J(x, y, uz) = J(x, y, z)u$. This yields $J(x, y, z(2\alpha I - R_u)^n) = J(x, y, z)(2\alpha I + R_u)^n$ and therefore $J(x, y, z) \in A_{-2\alpha}$. Combining the above results we have $J(x, y, z) \in A_\alpha \cap A_{-2\alpha} = 0$ if $\alpha \neq 0$.

We shall now show $A_\alpha A_\beta = 0$ if $\alpha \neq 0$ and $\beta \neq 0, \pm\alpha$. Let α and β be fixed weights of R_u and assume $\beta \neq k\alpha$, $k = 0, \pm 1, \pm 2, \cdots$, with

$\alpha \neq 0$. Then for any other weight γ we have by (3.4) $J(A_\beta, A_\alpha, A_\gamma) = 0$ if $\beta \neq \alpha \neq \gamma \neq \beta$. However $\alpha \neq \beta$ and therefore $J(A_\beta, A_\alpha, A_\gamma) =$ if $\alpha \neq \gamma \neq \beta$. Suppose $\gamma = \alpha$, then by (3.5) and the choice of β, $J(A_\beta, A_\alpha, A_\alpha) = 0$. Suppose $\gamma = \beta$, then $J(A_\beta, A_\alpha, A_\beta) = J(A_\beta, A_\beta, A_\alpha) = 0$ if $\alpha \neq 0, \beta, -\beta$ or 2β. We know $\alpha \neq 0, \beta$ or $-\beta$ so if $\alpha = 2\beta$, then by (3.9) $J(A_\beta, A_\beta, A_\alpha) = 0$. Combining all these cases we have shown $J(A_\beta, A_\alpha, A_\gamma) = 0$ for any weight γ and therefore $J(A_\beta, A_\alpha, A) = \sum_\gamma J(A_\beta, A_\alpha, A_\gamma) = 0$. By (1.11) and (1.12) $A_\alpha A_\beta \subset N = 0$. This proves

$$(3.10) \qquad A_\alpha A_\beta = 0 \quad \text{if } \alpha \neq 0 \text{ and } \beta \neq k\alpha, k = 0, \pm 1, \pm 2, \cdots .$$

We now assume $\alpha \neq 0$ and $\beta = k\alpha$ for $k \neq 0, \pm 1$, then $J(A_\alpha, A_\beta, A_\gamma) = J(A_\alpha, A_{k\alpha}, A_\gamma) = 0$ if $\alpha \neq ka \neq \gamma \neq \alpha$, by (3.4). But since $k \neq 1$ we have $J(A_\alpha, A_{k\alpha}, A_\gamma) = 0$ if $\alpha \neq \gamma \neq k\alpha$. Suppose $\gamma = \alpha$, then using (3.5)

$$\begin{aligned} J(A_\alpha, A_\beta, A_\gamma) &= J(A_\alpha, A_{k\alpha}, A_\gamma) \\ &= J(A_\alpha, A_{k\alpha}, A_\alpha) \\ &= J(A_\alpha, A_\alpha, A_{k\alpha}) \\ &= 0 \end{aligned}$$

if $k\alpha \neq 0, \alpha, -\alpha$ or 2α. But by the choice of k we need only consider $k\alpha = 2\alpha$ and in this case $J(A_\alpha, A_\alpha, A_{k\alpha}) = 0$ by (3.9). Now suppose $\gamma = k\alpha$, then

$$\begin{aligned} J(A_\alpha, A_\beta, A_\gamma) &= J(A_\alpha, A_{k\alpha}, A_\gamma) \\ &= J(A_\alpha, A_{k\alpha}, A_{k\alpha}) \\ &= J(A_{k\alpha}, A_{k\alpha}, A_\alpha) \\ &= 0 \end{aligned}$$

if $\alpha \neq 0, k\alpha, -k\alpha$ or $2k\alpha$, by (3.5). Again by the choice of k and α we need only consider $\alpha = 2k\alpha$. In this case $k = 1/2$ and therefore $\gamma = \beta = k\alpha = 1/2\alpha$. This yields $J(A_\alpha, A_\beta, A_\gamma) = J(A_\beta, A_\beta, A_{2\beta}) = 0$ by (3.9). Combining all of these cases we have for any weight γ, $J(A_\alpha, A_{k\alpha}, A_\gamma) = 0$ if $\alpha \neq 0, k \neq 0, \pm 1$ and as before this gives

$$(3.11) \qquad A_\alpha A_{k\alpha} = 0 \quad \text{if } \alpha \neq 0, k \neq 0, \pm 1 .$$

(3.10) and (3.11) yield

$$(3.12) \qquad A_\alpha A_\beta = 0 \quad \text{if } \alpha \neq 0, \beta \neq 0, \pm\alpha .$$

Since R_u is not nilpotent, there exists a weight $\alpha \neq 0$. We shall now show that $-\alpha$ is also a weight of R_u. For suppose $-\alpha$ is not a weight, then by the usual convention $A_{-\alpha} = 0$ and noting that none of the previously derived identities use the fact that $A_{-\alpha} \neq 0$ we have for $\beta \neq 0$ or α, that $A_\alpha A_\beta = 0$ by (3.12). For $\beta = 0$, $A_\alpha A_\beta \subset A_\alpha$ and for

$\beta = \alpha$, $A_\alpha A_\beta \subset A_{-\alpha} = 0$ using (2.14). Therefore A_α is a nonzero ideal of A and so $A = A_\alpha$. But $u \in A$ and $u \notin A_\alpha = A$, a contradiction. Therefore $-\alpha$ is a weight if α is a weight.

Now set $\mathscr{A}_\alpha = A_\alpha A_{-\alpha} \oplus A_\alpha \oplus A_{-\alpha}$ where α is a nonzero weight. Then $\mathscr{A}_\alpha \neq 0$ and for $\beta = 0, \pm\alpha$ we have $\mathscr{A}_\alpha A_\beta \subset \mathscr{A}_\alpha$. For $\beta \neq 0, \pm\alpha$ we have $A_\alpha A_\beta = A_{-\alpha} A_\beta = 0$ by (3.12). Now by (3.4) and (3.12) we have for $x \in A_\alpha$, $y \in A_{-\alpha}$, $z \in A_\beta$ that $0 = J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y = xy \cdot z$ and so $0 = A_\alpha A_{-\alpha} \cdot A_\beta$. Thus in all cases $\mathscr{A}_\alpha A_\beta \subset \mathscr{A}_\alpha$ and therefore $\mathscr{A}_\alpha$ is a nonzero ideal of A and we have $A = \mathscr{A}_\alpha$. This proves

PROPOSITION 3.13. *If A is a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field of characteristic not 2 or 3 and A contains an element u such that R_u is not a nilpotent linear transformation, then there exists an $\alpha \neq 0$ such that $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ where $A_a = \{x \in A : x(aI - R_u)^k = 0$ for some $k > 0\}$ and $A_0 = A_\alpha A_{-\alpha}$.*

4. A decomposition of A relative to A_0.

Let us consider the decomposition of A as given Proposition 3.13; that is,

$$A = A_0 \oplus A_\alpha \oplus A_{-\alpha} .$$

For any $y_0, z_0 \in A_0$ and $x \in A_a (a = 0, \pm\alpha)$, we use (2.8) and (2.11) to see that

$$0 = J(x, y_0, z_0) = x(R_{y_0} R_{z_0} - R_{z_0} R_{y_0}) .$$

Therefore,

$$R(A_0) \equiv \{R_{x_0} : x_0 \in A_0\}$$

is a commuting set of linear transformations acting on A_a. We can find $R(A_0)$-invariant subspaces $M_\lambda(a)$ [2; Chapter 4] such that

$$A_a = \sum_\lambda \oplus M_\lambda(a) \qquad (a = 0, \pm\alpha) ,$$

where on each $M_\lambda(a)$ the transformation R_{x_0}, for any $x_0 \in A_0$, has a matrix of the form

$$\begin{bmatrix} \lambda(x_0) & 0 \\ * & \lambda(x_0) \end{bmatrix} ;$$

that is, $M_\lambda(a)$ has a basis $\{x_1, x_2, \cdots, x_m\}$ $(m = m(\lambda, a))$ such that for any $x_0 \in A_0$, there exists $a_{ij}(x_0) \in F$ for which

$$x_i R_{x_0} = \sum_{j=1}^{i-1} a_{ij}(x_0) x_j + \lambda(x_0) x_i , \tag{4.1}$$

where $\lambda(x_0) \in F$ and, of course, $i = 1, 2, \cdots, m$.

Using the usual terminology we call the function λ defined by $\lambda: x_0 \to \lambda(x_0)$ a *weight of* A_0 *in* A_a or just a *weight* and the corresponding $M_\lambda(\alpha)$ a *weight space of* A_a *corresponding to* λ or just a *weight space of* A_a. It is easily seen [2] that A_a has finitely many weights and the weights are linear functionals on A_0 to F. Also

$$M_\lambda(a) = \{x \in A_a\text{: for all } x_0 \in A_0,\ x(R_{x_0} - \lambda(x_0)I)^k = 0 \text{ for some integer } k > 0\}$$

and for this weight λ we have $\lambda(u) = a$. For suppose $\lambda(u) = b$, then there exists an $x \neq 0$ in $M_\lambda(a)$ such that $bx = xR_u$. But $M_\lambda(a) \subset A_a = \{x \in A\colon x(R_u - aI)^n = 0\}$; therefore $(b-a)x = x(R_u - aI)$ and by induction $(b-a)^n x = x(R_u - aI)^n$ so for some integer N, $(b-a)^N x = x(R_u - aI)^N = 0$ and thus $a = b = \lambda(u)$. We now combine the weight space decompositions of the A_a to form a weight space decomposition of A in

PROPOSITION 4.2. Let $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ be a simple Malcev algebra as determined by Proposition 3.13, then we can write $A = A_0 \oplus \sum_\lambda \oplus M_\lambda(\alpha) \oplus \sum_\mu \oplus M_\mu(-\alpha)$ where all weights are distinct and any nonzero weight ρ of A_0 in A is a weight of A_0 in A_α or $A_{-\alpha}$ but not both.

Proof. The first part is clear noting that in the original weight space decomposition $A_a = \sum_\gamma \oplus M_\gamma(a)$ the weights of A_0 in A_a can be taken to be distinct. Also if λ is a weight of A_0 in A_α and μ a weight of A_0 in $A_{-\alpha}$, then $\lambda(u) = \alpha \neq -\alpha = \mu(u)$ and therefore $\lambda \neq \mu$. Now let $\rho \neq 0$ be any weight of A_0 in A with weight space $M_\rho = \{x \in A\colon x(R_{x_0} - \rho(x_0)I)^k = 0\}$ and let $y = y_0 + y_\alpha + y_{-\alpha} \in M_\rho$ where $y_a \in A_a$ with $a = 0, \pm\alpha$. Then for some integer $N > 0$,

$$\begin{aligned} 0 &= y(R_{x_0} - \rho(x_0)I)^N \\ &= y_0(R_{x_0} - \rho(x_0)I)^N \\ &\quad + y_\alpha(R_{x_0} - \rho(x_0)I)^N + y_{-\alpha}(R_{x_0} - \rho(x_0)I)^N \end{aligned}$$

and by the uniqueness of the decomposition $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ we have $y_a(R_{x_0} - \rho(x_0)I)^N = 0$ for $a = 0, \pm\alpha$. Now by using the binomial theorem and $A_0^2 = 0$ we have $0 = y_0(R_{x_0} - \rho(x_0)I)^N = y_0\rho(x_0)^N$ and since $\rho \neq 0, y_0 = 0$. Thus we have $y_a(R_{x_0} - \rho(x_0)I)^N = 0$, $a = \pm\alpha$, for some integer N and so ρ is a weight of A_0 in A_α and $A_{-\alpha}$. Now suppose y_α and $y_{-\alpha}$ are both nonzero, then since ρ is a weight of A_0 in A_α, $\rho(u) = \alpha$ and since ρ is a weight of A_0 in $A_{-\alpha}$, $\rho(u) = -\alpha$, a contradiction. Thus ρ is a weight of A_0 in either A_α or $A_{-\alpha}$ but not both.

We shall use the usual convention that if ρ is not a weight of A_0 in A, then $M_\rho = 0$. Let $M_\lambda(a)$ and $M_\mu(a)$ be weight spaces of A_0 in A_a

and let $x_0, y_0 \in A_0$ and $x \in M_\lambda(a)$, $y \in M_\mu(a)$, then using (2.8) and (1.7) we have

$$\begin{aligned} J(x, x_0, y_0 y) &= J(y_0, x_0, xy) + J(x, x_0, y_0 y) \\ &= J(y_0, x_0, y)x + J(x, x_0, y)y_0 \\ &= J(x, x_0, y)y_0 \,. \end{aligned}$$

Thus $J(x_0, x, y(R_{y_0} - \mu(y_0)I)) = -J(x_0, x, y)(R_{y_0} + \mu(y_0)I)$ and by induction

$$J(x_0, x, y(R_{y_0} - \mu(y_0)I)^n) = (-1)^n J(x_0, x, y)(R_{y_0} + \mu(y_0)I)^n \,.$$

From this we obtain $J(x_0, x, y) \in M_{-\mu}(-a)$ and interchanging the roles of x and y we see $J(x_0, x, y) \in M_{-\lambda}(-a)$; this proves

$$J(A_0, M_\lambda(a), M_\mu(a)) \subset M_{-\lambda}(-a) \cap M_{-\mu}(-a) \,. \tag{4.3}$$

From (4.3) we obtain

$$J(A_0, M_\lambda(a), M_\lambda(a)) \subset M_{-\lambda}(-a) \tag{4.4}$$

$$J(A_0, M_\lambda(a), M_\mu(a)) = 0 \quad \text{if } \lambda \neq \mu \,. \tag{4.5}$$

We shall next show

$$M_\lambda(a)M_\mu(a) = 0 \quad \text{if } \lambda \neq \mu \,. \tag{4.6}$$

For let $x_0 \in A_0$, $x \in M_\lambda(a)$ and $y \in M_\mu(a)$, then by (4.5) $0 = J(x, y, x_0)$ and therefore $xyR_{x_0} = xR_{x_0} \cdot y + x \cdot yR_{x_0}$ and hence $xy(R_{x_0} - (\mu(x_0) + \lambda(x_0))I) = x(R_{x_0} - \lambda(x_0)I) \cdot y + x \cdot y(R_{x_0} - \mu(x_0)I)$. In the usual way we can prove there exists an integer N such that $xy(R_{x_0} - (\mu(x_0) + \lambda(x_0))I)^N = 0$ and since we know $xy \in A_{-a}$ this shows $xy \in M_{\lambda+\mu}(-a)$ if $\lambda + \mu$ (defined by $(\lambda + \mu)(x_0) = \lambda(x_0) + \mu(x_0)$) is a weight of A_0 in A_{-a}, or $xy = 0$. If $xy \neq 0$, then $\lambda + \mu$ is a weight of A_0 in A_{-a} where λ and μ are weights of A_0 in A_a and therefore $-a = (\lambda + \mu)(u) = \lambda(u) + \mu(u) = a + a$, a contradiction.

Next we have for any weight λ of A_0 in A_a

$$M_\lambda(a)M_\lambda(a) \subset M_{-\lambda}(-a) \tag{4.7}$$

if $-\lambda$ is a weight of A_0 in A_{-a}. For let $x_0 \in A_0$ and $\lambda \equiv \lambda(x_0) \in F$ and let $M_\lambda(a)$ have basis $\{x_1, \cdots, x_m\}$ as in (4.1). Then using (1.2) we obtain

$$\begin{aligned} \lambda^2 x_1 x_2 &= \lambda x_1(\lambda x_2 + a_{21}x_1) \\ &= x_1 R_{x_0} \cdot x_2 R_{x_0} \\ &= (x_0 x_1 \cdot x_2)x_0 + (x_1 x_2 \cdot x_0)x_0 + (x_2 x_0 \cdot x_0)x_1 \\ &= -\lambda x_1 x_2 R_{x_0} + x_1 x_2 R_{x_0}^2 + \lambda^2 x_2 x_1 \end{aligned}$$

and thus

$$0 = x_1 x_2(R_{x_0}^2 - \lambda R_{x_0} - 2\lambda^2 I) = x_1 x_2(R_{x_0} + \lambda I)(R_{x_0} - 2\lambda I) \,.$$

Now since λ is a weight of A_0 in A_a, -2λ is not a weight of A_0 in A_{-a}: $-a = (2\lambda)(u) = 2\lambda(u) = 2a$. Thus the above equation implies $x_1x_2(R_{x_0} + \lambda I) = 0$ and therefore $x_1x_2 \in M_{-\lambda}(-a)$. Next $x_1x_0 \cdot x_3x_0 = \lambda x_1(\lambda x_3 + a_{32}x_2 + a_{31}x_1) = \lambda^2 x_1x_3 + s$ where $s \in M_{-\lambda}(-a)$ and $(x_0x_1 \cdot x_3)x_0 + (x_1x_3 \cdot x_0)x_0 + (x_3x_0 \cdot x_0)x_1 = -\lambda x_1x_3R_{x_0} + x_1x_3R_{x_0}^2 + \lambda^2 x_3x_1 + t$ where $t \in M_{-\lambda}(-a)$. Therefore using (1.2) we obtain $0 = x_1x_3(R_{x_0} + \lambda I)(R_{x_0} - 2\lambda I) + w$ where $w \in M_{-\lambda}(-a)$ and actually $w = 3\lambda a_{31}x_2x_1$. Therefore $0 = x_1x_3(R_{x_0} + \lambda I)^2 (R_{x_0} - 2\lambda I)$ and as before $x_1x_3(R_{x_0} + \lambda I)^2 = 0$ so that $x_1x_3 \in M_{-\lambda}(-a)$. Continuing this process we obtain $x_1x_k \in M_{-\lambda}(-a)$ for $k = 1, 2, \cdots, m$. Next consider the product x_2x_3.

$$\begin{aligned} x_2x_0 \cdot x_3x_0 &= (\lambda x_2 + a_{21}x_1)(\lambda x_3 + a_{32}x_2 + a_{31}x_1) \\ &= \lambda^2 x_2x_3 + s \end{aligned}$$

where $s \in M_{-\lambda}(-a)$ and

$$(x_0x_2 \cdot x_3)x_0 + (x_2x_3 \cdot x_0)x_0 + (x_3x_0 \cdot x_0)x_2 = x_2x_3(R_{x_0}^2 - \lambda R_{x_0} - \lambda^2 I) + t$$

where $t \in M_{-\lambda}(-a)$, therefore $0 = x_2x_3(R_{x_0} + \lambda I)(R_{x_0} - 2\lambda I) + w$ where $w \in M_{-\lambda}(-a)$. Therefore for some integer $k > 0$ such that $w(R_{x_0} + \lambda I)^k = 0$ we have $0 = x_2x_3(R_{x_0} + \lambda I)^{k+1}(R_{x_0} - 2\lambda I)$ and as before $x_2x_3 \in M_{-\lambda}(-a)$. We continue this process showing $x_2x_k \in M_{-\lambda}(-a)$ and in general $x_ix_j \in M_{-\lambda}(-a)$ for $i, j = 1, \cdots, m$. This completes the proof of (4.7).

We now show

$$M_\lambda(a) \cdot M_\mu(-a) = 0 \quad \text{if } \lambda + \mu \neq 0 . \tag{4.8}$$

By (2.7) we have for $x \in M_\lambda(a)$, $y \in M_\mu(-a)$ and $x_0 \in A_0$ that $0 = J(x, y, x_0)$ and as usual we obtain $xy(R_{x_0} - (\lambda(x_0) + \mu(x_0))I)^N = 0$ for some integer $N > 0$. Now $z = xy \in A_0$ and suppose $z \neq 0$, then, since $\lambda + \mu \neq 0, \lambda + \mu$ is a nonzero weight of A_0 in A_0, a contradiction to Proposition 4.2.

Let $x \in M_\rho(a)$, $y \in M_\lambda(a)$ and $z \in M_\mu(-a)$, then using (1.9), (2.7) and (2.8) we have

$$\begin{aligned} J(xx_0, y, z) &= xJ(x_0, y, z) + J(x, y, z)x_0 - 2J(yz, x, x_0) \\ &= J(x, y, z)x_0 \end{aligned}$$

and therefore $J(x(R_{x_0} - \rho(x_0)I), y, z) = J(x, y, z)(R_{x_0} - \rho(x_0)I)$ and as usual we obtain $J(x, y, z) \in M_\rho(a)$. Interchanging x and y we also obtain $J(x, y, z) \in M_\lambda(a)$ and therefore $J(x, y, z) \in M_\lambda(a) \cap M_\rho(a) = 0$ if $\lambda \neq \rho$. Now assume $\lambda \neq \rho$ and assume $\mu = -\lambda$ is a weight of A_0 in A_{-a}, then

$$0 = J(x, y, z) = xy \cdot z + yz \cdot x + zx \cdot y = yz \cdot x ,$$

using (4.6) and (4.8). This proves

$$M_\lambda(a)M_{-\lambda}(-a)\cdot M_\rho(a)=0 \tag{4.9}$$

if $\lambda\neq\rho$ are weights of A_0 in A_a such that $-\lambda$ is a weight of A_0 in A_{-a}.

We shall now show if λ is a nonzero weight of A_0 in A_a with weight space $M_\lambda(a)$, then $-\lambda$ is a nonzero weight of A_0 in A_{-a} with weight space $M_{-\lambda}(-a)$. The proof is similar to that following (3.12): Suppose $-\lambda$ is not a weight of A_0 in A_{-a}, then $M_{-\lambda}(-a)=0$; $M_\lambda(a)M_\lambda(a)=0$; $M_\lambda(a)M_\rho(a)=0$ if $\rho\neq\lambda$; $A_0M_\lambda(a)\subset M_\lambda(a)$ and $M_\lambda(a)M_\mu(-a)=0$ if $\mu+\lambda\neq 0$. Thus $M_\lambda(a)$ is a proper ideal of A, a contradiction.

Set $M_\lambda=M_\lambda(\alpha)M_{-\lambda}(-\alpha)\oplus M_\lambda(\alpha)\oplus M_{-\lambda}(-\alpha)$ for some nonzero weight λ of A_0 in A_α. Then analogous to Proposition 3.13, M_λ can be shown to be a nonzero ideal of A and we have

PROPOSITION 4.10. If $A=A_0\oplus A_\alpha\oplus A_{-\alpha}$ is a simple Malcev algebra as determined by Proposition 3.13, then there exists a nonzero weight λ of A_0 in A with weight space $M_\lambda(\alpha)=A_\alpha$ and such that $-\lambda$ is a weight of A_0 in A with weight space $M_{-\lambda}(-a)=A_{-\alpha}$.

We shall identify α with λ as a weight, that is, use the notation $\alpha(x_0)$ for $\lambda(x_0)$ and also identify $M_\lambda(\alpha)=A_\alpha$, $M_{-\lambda}(-\alpha)=A_{-\alpha}$. Note that Proposition 4.10 implies there exists a basis for A so that for every $x\in A_0$, R_x has a matrix of the form

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \begin{bmatrix} \alpha(x) & & 0 \\ & \ddots & \\ * & & \alpha(x)\end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -\alpha(x) & & 0 \\ & \ddots & \\ * & & -\alpha(x)\end{bmatrix}\end{bmatrix}.$$

5. The trace form. Set $(x,y)=\operatorname{trace} R_xR_y$, then it is shown [3] that this is actually an *invariant form*; that is (x,y) is a bilinear form on A such that for all $x,y,z\in A$, $(xy,z)=(x,yz)$. Also a bilinear form (x,y) is *nondegenerate on* A if $(x,y)=0$ for all $y\in A$ implies $x=0$.

THEOREM 5.1. *If $A=A_0\oplus A_\alpha\oplus A_{-\alpha}$ is a finite dimensional simple non-Lie Malcev algebra over an algebraically closed field of characteristic zero and if A contains an element u such that R_u is not nilpotent, then $(x,y)=$ trace R_xR_y is a nondegenerate invariant form on A and dimension $A_\alpha=$ dimension $A_{-\alpha}$.*

Proof. On $A=A_0\oplus A_\alpha\oplus A_{-\alpha}$ R_u has the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \begin{bmatrix} \alpha & & 0 \\ & \ddots & \\ * & & \alpha \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -\alpha & & 0 \\ & \ddots & \\ * & & -\alpha \end{bmatrix} \end{bmatrix}$$

and since $u \in A = J(A, A, A)$ (by 1.12) we have by [3; 2.12] that $0 = \text{trace } R_u = \alpha(n_\alpha - n_{-\alpha})$ where $n_a = \text{dimension } A_a$, $a = \pm\alpha$.

Now to show (x, y) is nondegenerate, let $T = \{x \in A: (x, A) = 0\}$ where for subsets B, C of A we set $(B, C) = \{(b, c): b \in B, c \in C\}$ and for $x \in A$, $(x, C) = \{(x, c): c \in C\}$. Since (x, y) is an invariant form on A, T is an ideal of A and since A is simple, $T = 0$ or $T = A$. If $T = A$, then $(A, A) = 0$ and from the matrix of R_u we see that

$$0 = (u, u) = \text{trace } R_u^2 = 2n\alpha^2$$

where $n = \text{dimension } A_\alpha$. Since F is of characteristic zero, $\alpha = 0$, a contradiction. Thus $T = 0$ which implies (x, y) is nondegenerate on A.

COROLLARY 5.2. *If $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ is a simple Malcev algebra as above then*

$$(A_0, A_\alpha) = (A_0, A_{-\alpha}) = (A_\alpha, A_\alpha) = (A_{-\alpha}, A_{-\alpha}) = 0 .$$

Proof. Since R_u is nonsingular on A_a, $a \neq 0$, $A_a = A_a R_u$. Therefore $(A_0, A_a) = (A_0, A_a R_u) = (A_0 R_u, A_a) = 0$, the second equality uses (x, y) is an invariant form and the third uses (2.11). Also $(A_a, A_a) = (uA_a, A_a) = (u, A_a A_a) \subset (u, A_{-a}) = 0$.

COROLLARY 5.3. *If A_0^* is the dual space of A_0 consisting of linear functionals on A_0 and $f \in A_0^*$, then $f = c\alpha$ for some $c \in F$.*

Proof. First, (x, y) is nondegenerate on A_0. For if $x_0 \in A_0$ is such that $(x_0, A_0) = 0$, then

$$\begin{aligned}(x_0, A) &= (x_0, A_0 \oplus A_\alpha \oplus A_{-\alpha}) \\ &\subset (x_0, A_0) + (x_0, A_\alpha) + (x_0, A_{-\alpha}) \\ &= 0\end{aligned}$$

by the preceding corollary and therefore $x_0 = 0$ by Theorem 5.1. Now if $f \in A_0^*$, then there exists a unique element [2, page 141] $a_f \in A_0$

such that for all $x \in A_0, f(x) = (x, a_f) = \text{trace } R_x R_{a_f} =$

$$\text{trace} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \begin{bmatrix} \alpha(x) & & 0 \\ & \ddots & \\ * & & \alpha(x) \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -\alpha(x) & & 0 \\ & \ddots & \\ * & & -\alpha(x) \end{bmatrix} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \begin{bmatrix} \alpha(a_f) & & 0 \\ & \ddots & \\ * & & \alpha(a_f) \end{bmatrix} & 0 \\ 0 & 0 & \begin{bmatrix} -\alpha(a_f) & & 0 \\ & \ddots & \\ * & & -\alpha(a_f) \end{bmatrix} \end{bmatrix}$$

$= 2n\alpha(a_f)\alpha(x)$; using the remarks at the end of § 4 to obtain the form of the matrices of R_x and R_{α_f}. Thus $f = c\alpha$ where $c = 2n\alpha(\alpha_f) \in F$.

COROLLARY 5.4. *The dimension of A_0 is one.*

Proof. $0 <$ dimension $A_0 =$ dimension $A_0^* =$ dimension $uF = 1$.

We shall frequently refer to a Malcev algebra A that satisfies Theorem 5.1 as a "usual simple non-Lie Malcev algebra" and for the remainder of this paper we shall assume the algebraically closed field F is of characteristic zero.

6. The diagonalization of R_u. Using Proposition 4.10 and Corollary 5.4 we are able to decompose A relative to $R(A_0)$ into the form

$$A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$$

where $A_0 = uF$. From this the matrix of R_u on $A_a, a = \pm\alpha$, has the form

$$\begin{bmatrix} a & & 0 \\ & \ddots & \\ * & & a \end{bmatrix}.$$

We shall show in this section that R_u can be diagonalized. Put R_u into its Jordan canonical form on A_a, that is, find R_u-invariant subspaces $U_i(a)$ of A_a such that $A_a = U_1(a) \oplus \cdots \oplus U_{m_a}(a)$ and each $U_i(a)$ has a basis $\{x_{i1}, \cdots, x_{im_i}\}$ so that the action of R_u is given by

$$\begin{aligned} x_{i1}R_u &= ax_{i1} \\ x_{ij}R_u &= ax_{ij} + x_{ij-1} \\ j &= 2, \cdots, m_i\,. \end{aligned} \tag{6.1}$$

Thus on $U_i(a)$, R_u has an $m \times m$ matrix of the form

$$\begin{bmatrix} a & & & & 0 \\ 1 & a & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & a \end{bmatrix}$$

where $m =$ dimension $U_i(a)$. We shall now investigate the multiplicative relations between the U's and show that the dimension of all the $U_i(a)$ is one and therefore R_u will have a diagonal matrix.

LEMMA 6.2. $U_i(a)U_i(a) = 0$.

Proof. Let $U_i(a)$ have basis $\{x_1, \cdots, x_m\}$ as given by (6.1). If $m = 1$, we are finished. Suppose $m > 1$, then using (1.6)

$$\begin{aligned} 0 &= -J(u, x_2, x_2)R_u \\ &= J(u, x_2, x_2R_u) \\ &= aJ(u, x_2, x_2) + J(u, x_2, x_1) \\ &= J(u, x_2, x_1) \\ &= x_2x_1 \cdot u + x_1u \cdot x_2 + ux_2 \cdot x_1 \\ &= x_2x_1(R_u - 2aI) . \end{aligned}$$

But we know $A_{2a} = 0$, therefore $x_1x_2 = 0$. Now using (1.6) we have, in general, for any $i = 1, \cdots, m$,

$$\begin{aligned} 0 &= J(u, x_i, x_iR_u) \\ &= J(u, x_i, x_{i-1}) + aJ(u, x_i, x_i) \\ &= J(u, x_i, x_{i-1}) \end{aligned}$$

and again using (1.6),

$$\begin{aligned} 0 &= J(u, x_i, x_{i-1}R_u) \\ &= J(u, x_i, x_{i-2}) + aJ(u, x_i, x_{i-1}) \\ &= J(u, x_i, x_{i-2}) . \end{aligned}$$

Continuing this process we have

$$J(u, x_i, x_k) = 0$$

for all $k \leqq i$. Now if $i < k$, then by the preceding sentence

$$0 = J(u, x_k, x_i) = J(u, x_i, x_k) .$$

Thus

$$J(u, x_i, x_k) = 0 \qquad \text{for all } i, k = 1, \cdots, m .$$

By linearity this implies

$$J(u, x, y) = 0 \qquad \text{for all } x, y \in U_i(a) .$$

Thus

$$xyR_u = xR_u \cdot y + \cdot yR_u$$

and

$$xy(R_u - 2aI) = x(R_u - aI) \cdot y + x \cdot y(R_u - aI)$$

As usual we can find an N large enough so that $xy(R_u - 2aI)^N = 0$. But we know $A_{2a} = 0$, therefore $xy = 0$.

LEMMA 6.3. *Let $x \in A_a$ be such that $xR_u = ax$ and let $U_i(-a) \equiv \{y_1, \cdots, y_m\}$, then $xy_i = 0$ for $i = 1, \cdots, m-1$ and $xy_m = \lambda u$ where $\lambda = -(y_m, x)/2na$.*

Proof. Using the invariant form (x, y) we have $(y_m x, u) = (y_m, xu) = a(y_m, x)$. Since $xy_m \in A_0 = uF$ we may write $xy_m = \lambda u$, then $(y_m x, u) = (-\lambda u, u) = -\lambda(u, u) = -\lambda 2na^2 (a = \pm\alpha)$. Thus $\lambda = -(y_m, x)/2na$.

Now since $x \in A_a$ and $U_i(-a) \subset A_{-a}$, we have by (2.4) and (2.11) that $0 = J(x, y_2, u) = xy_2 \cdot u + y_2 u \cdot x + ux \cdot y_2 = (-ay_2 + y_1)x - axy_2 = y_1 x$. Again $0 = J(x, y_3, u) = xy_3 \cdot u + y_3 u \cdot x + ux \cdot y_3 = (-ay_3 + y_2)x - axy_3 = y_2 x$. Continuing this process we eventually obtain $0 = J(x, y_m, u) = xy_m \cdot u + y_m u \cdot x + ux \cdot y_m = y_{m-1}x$.

THEOREM 6.4. *Let $x \in A_a$ be such that $xR_u = ax$ and let $U_i(-a)$ be such that $xU_i(-a) \neq 0$, then dimension $U_i(-a) = 1$.*

Proof. Let $B = uF \oplus xF \oplus U_i(-a)$, then using the preceding lemmas and their notation we see that B is a subalgebra of A and $xy_m = \lambda u$ where $\lambda \neq 0$. Now by (2.4) we have $J(u, x, y_m) = 0$, therefore by [3; Corollary 4.4] we see that u, x and y_m are contained in a Lie subalgebra, L, of A. However this implies $y_m u = -ay_m + y_{m-1} \in L$ and therefore $y_{m-1} \in L$; again $y_{m-1}u = -ay_{m-1} + y_{m-2} \in L$ and therefore $y_{m-2} \in L$. Continuing this process we obtain $B \subset L$ and so B is a Lie subalgebra of A. Thus for any $z \in B$,

$$\begin{aligned} 0 &= J(z, x, y_m) \\ &= z(R_x R_{y_m} - R_{y_m} R_x - R_{xy_m}) \\ &= z([R_x, R_{y_m}] - \lambda R_u) . \end{aligned}$$

Thus on B we have $\lambda R_u = [R_x, R_{y_m}]$ and therefore the trace of R_u on B is zero. But calculating the trace of R_u from its matrix on B, we obtain that the trace is $0 + a - am$. Thus $m = 1$.

COROLLARY 6.5. *The dimensional of all the $U_i(-a), a = \pm\alpha$, is one.*

Proof. Suppose there exists $U_j(-a) \equiv \{y_1, \cdots, y_m\}$ of dimension $m > 1$. Then for every $U_i(a)$, $y_1 U_i(a) = 0$. For if there exists some

$U_i(a)$ such that $y_1 U_i(a) \neq 0$, then by Theorem 6.4, dimension $U_i(a) = 1$. But this means there exists $x \in A_a$ such that $xR_u = ax$ and $0 \neq xy_1 \in xU_j(-a)$; so again by Theorem 6.4, dimension $U_j(-a) = 1$, a contradiction. Thus $y_1 U_i(a) = 0$ for all i and this implies $y_1 A_a = y_1(U_1(a) \oplus \cdots \oplus U_{m_a}(a)) = 0$. Now from Corollary 5.2 we have, since $y_1 \in A_{-a}$, $(A_0, y_1) = (A_{-a}, y_1) = 0$ and using the preceding sentence

$$(A_a, y_1) = (A_a, y_1 u) = (A_a y_1, u) = 0 .$$

Thus $(A, y_1) = 0$ and since (x, y) is nondegenerate on A, $y_1 = 0$, a contradiction.

7. Proof of the theorem. Let $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ be the usual simple non-Lie Malcev algebra, then we have just seen that A_a is the null space of $R_u - aI$, $a = 0, \pm\alpha$. The choice of $\alpha \neq 0$ is fixed but arbitrary. In particular we want to consider the case $\alpha = -2$, then all we must do is consider $u' = (-2/\alpha)u$ and decompose A relative to $R_{u'}$ (which is also not nilpotent) to obtain $A = A_0 \oplus A_{-2} \oplus A_2$. However we shall work with a fixed α and normalize when necessary.

Let $a, b \in F$ be any characteristic roots (weights) of R_u, that is, $a, b = 0, \pm\alpha$ with characteristic vectors $x, y \in A$; that is, $ax = xR_u$, $by = yR_u$ or $x \in A_a$, $y \in A_b$, then we have

$$J(x, y, u) = xy \cdot u - (a + b)xy \quad \text{where } x \in A_a, y \in A_b . \tag{7.1}$$

Using (2.4) and (7.1) we also have

$$xy \cdot u = (a + b)xy \quad \text{where } y \in A_a, y \in A_b \text{ and } a \neq b . \tag{7.2}$$

Since $xy \in A_{-a}$ if $x, y \in A_a$, we have

$$xy \cdot u = -axy \quad \text{where } x, y \in A_a . \tag{7.3}$$

Combining (7.3) and (7.1) yields

$$J(x, y, u) = -3axy \quad \text{where } x, y \in A_a . \tag{7.4}$$

Let $x, y, z \in A_a$, then using (2.14), (2.4), (1.9) and (7.4) we have

$$\begin{aligned} 0 &= J(xy, z, u) \\ &= xJ(y, z, u) + J(x, z, u)y - 2J(zu, x, y) \\ &= x(-3ayz) + (-3axz)y - 2aJ(z, x, y) . \end{aligned}$$

Therefore

$$\begin{aligned} 2J(x, y, z) &= -3(x \cdot yz + xz \cdot y) \\ &= 3(xy \cdot z + yz \cdot x + zx \cdot y) - 3xy \cdot z \end{aligned}$$

and thus

$$J(x, y, z) = 3xy \cdot z \quad \text{where } x, y, z \in A_a . \tag{7.5}$$

Now $J(x, z, y) = 3xz \cdot y$ and adding this to (7.5) yields $0 = xy \cdot z + xz \cdot y$ and with a slight change of notation we have

$$xy \cdot z = -x \cdot yz \quad \text{where } x, y, z \in A_a . \tag{7.6}$$

From (7.6) with $z = x$ we obtain

$$xy \cdot x = 0 \quad \text{where } x, y \in A_a . \tag{7.7}$$

Now let $x, y \in A_a$, $z \in A_{-a}$, then $-aJ(x, y, z) = J(x, y, zu)$ and $J(z, y, xu) = aJ(z, y, x) = -aJ(x, y, z)$. So

$$\begin{aligned} -2aJ(x, y, z) &= J(z, y, xu) + J(x, y, zu) \\ &= J(z, y, u)x + J(x, y, u)z = J(x, y, u)z , \end{aligned}$$

using (1.7) for the second equality, (2.4) for the third. Thus we have $-2aJ(x, y, z) = J(x, y, u)z = (-3axy)z$ using (7.4) and hence

$$2J(x, y, z) = 3xy \cdot z \quad \text{where } x, y \in A_a, z \in A_{-a} . \tag{7.8}$$

This yields $3xy \cdot z = 2(xy \cdot z + yz \cdot x + zx \cdot y)$ or

$$xy \cdot z = -2(xz \cdot y + x \cdot yz) \quad \text{where } x, y \in A_a, z \in A_{-\alpha} . \tag{7.9}$$

We now use (7.9) to prove the important identity (7.10). Thus let w, x, y, z be elements of A_a and set $v = J(x, y, z)$, $2x' = yz$, $-2y' = xz$ and $2z' = xy$. Then

$$vw = 6(x'w \cdot x + y'w \cdot y + z'w \cdot z) . \tag{7.10}$$

To prove this note that $x', y', z' \in A_{-a}$ and using (7.9) we have $2x'x \cdot w = xw \cdot x' - 2wx' \cdot x$, $2y'y \cdot w = yw \cdot y' - 2wy' \cdot y$, $2z'z \cdot w = zw \cdot z' - 2wz' \cdot z$. Adding these equations and multiplying by 2 yield

$$2vw = 2(xw \cdot x' + yw \cdot y' + zw \cdot z') + 4(x'w \cdot x + y'w \cdot y + z'w \cdot z) .$$

Now using (1.10),

$$\begin{aligned} 2(xw \cdot x' &+ yw \cdot y' + zw \cdot z') = xw \cdot yz + yw \cdot zx + zw \cdot xy \\ &= x(zw \cdot y) + z(wy \cdot x) + w(yx \cdot z) + y(xz \cdot w) + y(xw \cdot z) + x(wz \cdot y) \\ &+ w(zy \cdot x) + z(yx \cdot w) + z(yw \cdot x) + y(wx \cdot z) + w(xz \cdot y) + x(zy \cdot w) \\ &= w(yx \cdot z) + w(zy \cdot x) + w(xz \cdot y) + y(xz \cdot w) + z(yx \cdot w) + x(zy \cdot w) \\ &= -wv + y(-2y'w) + z(-2z'w) + x(-2x'w) \end{aligned}$$

noting some cancellation to obtain the third equality. Thus $2vw = vw + 2(x'w \cdot x + y'w \cdot y + z'w \cdot z) + 4(x'w \cdot x + y'w \cdot y + z'w \cdot z)$ and this proves (7.10).

Since A is simple non-Lie Malcev algebra, we shall use the facts $A^2 = A$ and $A = J(A, A, A)$ to obtain more identities for A. First we have

$$\begin{aligned} A_0 \oplus A_\alpha \oplus A_{-\alpha} = A = J(A, A, A) \\ &\subset J(A_0, A, A) + J(A_\alpha, A, A) + J(A_{-\alpha}, A, A) \\ &\subset J(A_0, A_\alpha, A_\alpha) + J(A_0, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_\alpha, A_\alpha) \\ &+ J(A_{-\alpha}, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_\alpha, A_{-\alpha}) + J(A_\alpha, A_{-\alpha}, A_{-\alpha}) \\ &\subset A_0 \oplus A_\alpha \oplus A_{-\alpha} \end{aligned}$$

and therefore

$$\begin{aligned} A_0 &= J(A_\alpha, A_\alpha, A_\alpha) + J(A_{-\alpha}, A_{-\alpha}, A_{-\alpha}) , \\ A_\alpha &= J(A_0, A_{-\alpha}, A_{-\alpha}) + J(A_\alpha, A_\alpha, A_{-\alpha}) , \\ A_{-\alpha} &= J(A_0, A_\alpha, A_\alpha) + J(A_\alpha, A_{-\alpha}, A_{-\alpha}) . \end{aligned}$$

We now use $A = A^2$ to obtain

$$\begin{aligned} A_0 \oplus A_\alpha \oplus A_{-\alpha} = A = A^2 \\ &= A_0A_\alpha + A_0A_{-\alpha} + A_\alpha^2 + A_\alpha A_{-\alpha} + A_{-\alpha}^2 \end{aligned}$$

and therefore

$$\begin{aligned} A_0 &= A_\alpha A_{-\alpha} , \\ A_\alpha &= A_0A_\alpha + A_{-\alpha}^2 , \\ A_{-\alpha} &= A_0A_{-a} + A_\alpha^2 . \end{aligned}$$

Since $A_0 = uF$ we have $A_0A_a = A_a (a = \pm\alpha)$. Also

$$\begin{aligned} J(A_0, A_{-a}, A_{-a}) \subset A_a = A_0A_a \\ &\subset A_0J(A_0, A_{-a}, A_{-a}) + A_0J(A_a, A_a, A_{-a}) \\ &\subset J(A_0, A_0, A_{-a}^2) + J(A_0, A_{-a}, A_{-a}A_0) + J(A_0, A_{-a}, A_0A_{-a}) \\ &+ J(A_0, A_a, A_aA_{-a}) + J(A_0, A_a, A_{-a}A_a) + J(A_0, A_{-a}, A_a^2) \\ &\subset J(A_0, A_{-a}, A_{-a}) , \end{aligned}$$

obtaining the second inclusion from $A_a = J(A_0, A_{-a}, A_{-a}) + J(A_a, A_a, A_{-a})$ and the third inclusion from (1.8). Thus we have

$$A_a = J(A_0, A_{-a}, A_{-a}) , \quad a \neq 0 .$$

From this and remembering $A_0 = uF$ we obtain

$$A_a = A_{-a}A_{-a} , \quad a \neq 0 .$$

For $A_{-a}A_{-a} \subset A_a = J(A_0, A_{-a}, A_{-a}) \subset A_{-a}A_{-a}$. Also

$$A_0 = J(A_a, A_a, A_a) , \quad a = \pm\alpha .$$

For

$$\begin{aligned} J(A_a, A_a, A_a) \subset A_0 &= A_aA_{-a} \\ &= A_aJ(A_0, A_a, A_a) \\ &\subset J(A_a, A_0, A_a^2) + J(A_a, A_a, A_aA_0) + J(A_a, A_a, A_0A_a) \\ &\subset J(A_a, A_a, A_a) . \end{aligned}$$

We summarize these identities in

PROPOSITION 7.11. Let $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ be the usual simple non-Lie Malcev algebra, then we have for $a = \pm\alpha$,

$$A_a = A_0A_a = A_{-a}A_{-a}$$

and

$$A_0 = A_aA_{-a} = J(A_a, A_a, A_a) .$$

THEOREM 7.12. *Let $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$ be the usual simple non-Lie Malcev algebra, then A is isomorphic to the simple seven dimensional Malcev algebra A^* discussed in the introduction.*

Proof. Since $uF = A_0 = A_\alpha A_{-\alpha} = A_\alpha \cdot A_\alpha A_\alpha$, there exists $x, y, z \in A_\alpha$ such that $x \cdot yz = 2u$. Define $2x' = yz$, $-2y' = xz$ and $2z' = xy$ and form the subspace B generated by $\{u, x, y, z, x', y', z'\}$. First the x, y and z are linearly independent over F. For if $ax + by + cz = 0$ with $a, b, c \in F$ and, for example, $a \neq 0$, then write $x = b'y + c'z$ and therefore using (7.7) $2u = x \cdot yz = b'y \cdot yz + c'z \cdot yz = 0$, a contradiction. Similarly noting $u = xx'$ and assuming a relation of the type $x' = b'y' + c'z'$ and using the definitions of x', y' and z' we see that the x', y' and z' are also linearly independent. Since $A = A_0 \oplus A_\alpha \oplus A_{-\alpha}$, $\{u, x, y, z, x', y', z'\}$ is a linearly independent set of vectors over F. Using identities (1.2), (7.6) and (7.7) we obtain the following multiplication table for B.

	u	x	y	z	x'	y'	z'
u	0	$-\alpha x$	$-\alpha y$	$-\alpha z$	$\alpha x'$	$\alpha y'$	$\alpha z'$
x	αx	0	$2z'$	$-2y'$	u	0	0
y	αy	$-2z'$	0	$2x'$	0	u	0
z	αz	$2y'$	$-2x'$	0	0	0	u
x'	$-\alpha x'$	$-u$	0	0	0	αz	$-\alpha y$
y'	$-\alpha y'$	0	$-u$	0	$-\alpha z$	0	αx
z'	$-\alpha z'$	0	0	$-u$	αy	$-\alpha x$	0

By the remarks at the beginning of this section we can choose $\alpha = -2$

and consequently obtain that B is isomorphic to A^*. It remains to show the dimension of A over F is seven. For this it suffices to show dimension $A_\alpha = 3$, since dimension $A_\alpha =$ dimension $A_{-\alpha}$. Let $0 \neq w \in A_\alpha$, then by (7.5)

$$6u = 3x \cdot yz = -J(x, y, z)$$

and therefore by (7.10),

$$6\alpha w = 6wu = x_0 x + y_0 y + z_0 z$$

where $x_0, y_0, z_0 \in A_0 = uF$. But by the action of u on x, y and z we have $6\alpha w = a_0 x + b_0 y + c_0 z$ where $a_0, b_0, c_0 \in F$. Thus the dimension of A_α is three.

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