

# Pacific Journal of Mathematics

## ON UNIMODULAR MATRICES

ISIDORE HELLER AND ALAN JEROME HOFFMAN

# ON UNIMODULAR MATRICES

I. HELLER AND A. J. HOFFMAN

**1. Introduction and summary.** For the purpose of this note a matrix is called unimodular if every minor determinant equals 0, 1 or  $-1$ .

I. Heller and C. B. Tompkins [1] have considered a set

$$S = \{u_i, v_j, u_i + v_j, u_i - u_{i*}, v_j - v_{j*}\}$$

where the  $u_1, u_2, \dots, u_m, v_1, v_2, \dots, v_n$  are linearly independent vectors in  $m + n = k$ -dimensional space  $E$ , and have shown that in the coordinate representation of  $S$  with respect to an arbitrary basis in  $E$  every nonvanishing determinant of  $k$  vectors of  $S$  has the same absolute value, and that, with respect to a basis in  $S$ , the vectors of  $S$  or of any subset of  $S$  are the columns of a unimodular matrix. For the purpose of this note the class of unimodular matrices obtained in this fashion shall be denoted as the class  $T$ .

A. J. Hoffman and J. B. Kruskal [4] have considered incidence matrices  $A$  of vertices versus directed paths of an oriented graph  $G$ , and proved that:

(i) if  $G$  is alternating, then  $A$  is unimodular;

(ii) if the matrix  $A$  of *all* directed paths of  $G$  is unimodular, then  $G$  is alternating. The terms are defined as follows. A graph  $G$  is oriented if it has no circular edges, at most one edge between any given two vertices, and each edge is oriented. A path is a sequence of distinct vertices  $v_1, v_2, \dots, v_k$  of  $G$  such that, for each  $i$  from 1 to  $k - 1$ ,  $G$  contains an edge connecting  $v_i$  with  $v_{i+1}$ ; if the orientation of these edges is from  $v_i$  to  $v_{i+1}$ , the path is directed; if the orientation alternates throughout the sequence, the path is alternating. A loop is a sequence of vertices  $v_1, v_2, \dots, v_k$ , which is a path except that  $v_k = v_1$ . A loop is alternating if successive edges are oppositely oriented and the first and last edges are oppositely oriented. The graph is alternating if every loop is alternating. The incidence matrix  $A = (a_{ij})$  of the vertices  $v_i$  of  $G$  versus a set of directed paths  $p_1, p_2, \dots, p_k$  of  $G$  is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is in } p_j \\ 0 & \text{otherwise.} \end{cases}$$

The class of unimodular matrices thus associated with alternating graphs shall be denoted by  $K$ .

I. Heller [2] and [3] has considered unimodular matrices obtained

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by representing the edges (interpreted as vectors) of an  $n$ -simplex in terms of a basis chosen among the edges (in graph theoretical terms: the edges and vertices of the simplex form a complete graph  $G$ ; a basis is a maximal tree in  $G$ , that is, a tree containing all vertices of  $G$ ), and has shown that:

(i) the matrix representing all edges of the simplex is unimodular and maximal (i.e., will not remain unimodular when a new column is adjoined);

(ii) the columns of every unimodular matrix of  $n$  rows and  $n(n + 1)$  columns represent the edges of an  $n$ -simplex.

The class of (unimodular) matrices whose columns are among the edges of a simplex shall be denoted by  $H$ .  $H$  can also be defined as a class of incidence matrices: A matrix  $A$  belongs to  $H$  if there is some oriented graph  $F$  without loops such that  $A$  is the incidence matrix of the edges of  $F$  versus a set of path in  $F$ . That is,

$$a_{ij} = \begin{cases} 1 & \text{if edge } e_i \text{ is in path } p_j \\ -1 & \text{if } -e_i \text{ is in } p_j \\ 0 & \text{otherwise .} \end{cases}$$

In [2] it has further been shown that:

- (iii) there exist unimodular matrices which do not belong to  $H$ ;
- (iv) the classes  $H$  and  $T$  are identical.

The purpose of the present note is to show that the class  $K$  is identical with the set of nonnegative matrices of  $H$ .

2. THEOREM. *If a matrix  $A$  of  $n$  rows and  $m$  columns belongs to  $K$  (i.e.,  $A$  is the incidence matrix of the  $n$  vertices of some alternating graph  $G$  versus a set of  $m$  directed paths in  $G$ ), then  $A$  belongs to  $H$  (i.e, there is some  $n$ -simplex  $S$  and a basis  $B$  among its edges such that the columns of  $A$  represent edges of  $S$  in terms of  $B$ ). Conversely, every non-negative matrix of  $H$  belongs to  $K$ .*

3. NOTATION. An oriented graph is viewed as a set

$$(3.1) \quad R = V \cup E ,$$

where  $V$  is the set of vertices  $A_1, A_2, \dots, A_n$ , and  $E$  is the set of oriented edges  $e$ , that is certain ordered pairs  $(A_i, A_j)$  with  $j \neq i$  of elements of  $V$ , such that at most one of the two pairs  $(A_i, A_j), (A_j, A_i)$  is in  $E$ .

For brevity of notation we define

$$(3.2) \quad [A_i, A_j] = \{(A_i, A_j), (A_j, A_i)\} .$$

The origin and endpoint of an edge  $e$  are denoted by  $\rho e$  and  $\sigma e$ :

$$(3.3) \quad \rho(A, B) = A , \quad \sigma(A, B) = B ,$$

If  $A$  and  $B$  are vertices of  $R$ , the relation  $A < B$  ( $A$  is immediate predecessor of  $B$ ), also written as  $B > A$ , is defined by

$$(3.4) \quad A < B \iff (A, B) \in R .$$

Similarly, if  $a, b$  are edges of  $R$ ,

$$(3.5) \quad a < b \iff \sigma a = \rho b .$$

A subset  $V'$  of vertices of  $R$  defines a subgraph of  $R$

$$(3.6) \quad R(V') = V' \cup E'$$

where  $(A, B) \in E' \iff A \in V', B \in V', (A, B) \in E$ .

4. *Proof.* Using the graph-theoretical definition of the class  $H$ , the first half of the theorem shall be proved by showing that to each alternating graph  $G$  there is an oriented loopless graph  $F$  such that the  $K$ -matrices associated with  $G$  are among the  $H$ -matrices associated with  $F$ .

A column of a  $K$ -matrix is the incidence column  $K_p$  of the vertices of  $G$  versus a directed path  $p$  in  $G$ ; a column of an  $H$ -matrix is the incidence column  $H_q$  of the edges of  $F$  versus a path  $q$  in  $F$ . For given  $G$  it will therefore be sufficient to show the existence of an  $F$  such that

$$(4.1) \quad \begin{array}{l} \text{to each directed path } p \text{ in } G \text{ there is a path} \\ q = \varphi(p) \text{ in } F \text{ such that } K_p = H_q . \end{array}$$

This will be shown by constructing an  $F$  and a mapping  $\varphi$  of the set of vertices of  $G$  onto the set of edges of  $F$  in such a way that  $\varphi$  satisfies (4.1), or equivalently, that  $\varphi$  preserves the relation defined in (3.4) and (3.5), that is, for any two distinct vertices  $A, B$  of  $G$ ,

$$(4.2) \quad A < B \text{ (in } G) \implies \varphi(A) < \varphi(B) \text{ (in } F) .$$

The construction of  $F$  and  $\varphi$  shall now be carried out under the assumption that  $G$  is connected. If  $G$  is not connected, the same construction can be applied to each component of  $G$ , yielding an  $F$  with an equal number of components.

If  $G$  has  $n$  vertices, take as the vertices of  $F$  a set of  $n + 1$  distinct elements  $P_0, P_1, \dots, P_n$ .

The  $n$  edges  $e_1, e_2, \dots, e_n$  of  $F$  are defined successively as follows.

First, choose an arbitrary vertex  $A_1$  in  $G$ , define

$$(4.3) \quad \varphi(A_1) = e_1 = (P_0, P_1) ,$$

and note that:

(i) the subgraph  $G_1 = G(A_1)$ , consisting of the one vertex  $A_1$  of  $G$ , is, trivially, connected;

- (ii) the graph  $F_1 = \{P_0, P_1, (P_0, P_1)\}$  is connected;
- (iii) with respect to  $G_1$  and  $F_1$ ,  $\varphi$  trivially satisfies (4.2).

Then, assuming  $A_\nu \in G$  already chosen and  $e_\nu = \varphi(A_\nu)$  defined for  $\nu = 1, 2, \dots, k$  in such a manner that  $G_k = G\{A_1, A_2, \dots, A_k\}$  and  $F_k = \{P_0, P_1, \dots, P_k, e_1, \dots, e_k\}$  are each connected and  $\varphi$  satisfies (4.2) with respect to  $G_k$  and  $F_k$ , choose  $A_{k+1} \in G$  such that

$$(4.4) \quad [A_i, A_{k+1}] \cap G \neq 0$$

for some  $i \leq k$  and define

$$(4.5) \quad \varphi(A_{k+1}) = e_{k+1} = \begin{cases} (\sigma e_i, P_{k+1}) & \text{when } (A_i, A_{k+1}) \in G \\ (P_{k+1}, \rho e_i) & \text{when } (A_{k+1}, A_i) \in G, \end{cases}$$

noting that this definition depends on the choice of  $i$  since more than one  $i$  may satisfy (4.4).

Obviously,  $G_{k+1}$  and  $F_{k+1}$  are each connected.

To show that  $\varphi$  satisfies (4.2) with respect to  $G_{k+1}$  and  $F_{k+1}$ , let  $A_r < A_s$  in  $G_{k+1}$ .

If  $r \leq k$  and  $s \leq k$ , (4.2) is satisfied according to the induction's hypothesis.

For  $\{r, s\} = \{i, k + 1\}$ , (4.2) is satisfied by definition (4.5). Namely: for  $r = i$ ,  $s = k + 1$ , (4.5) defines  $e_{k+1} = (\sigma e_i, P_{k+1})$ , hence  $\sigma e_i = \rho e_{k+1}$ , which by (3.5) means  $e_i < e_{k+1}$ ; similarly for  $s = i$ ,  $r = k + 1$ , (4.5) defines  $e_{k+1} = (P_{k+1}, \rho e_i)$ , hence  $\sigma e_{k+1} = \rho e_i$ , which means  $e_{k+1} < e_i$ .

There remains the case  $\{r, s\} = \{j, k + 1\}$ ,  $j \neq i$ ,  $j \leq k$ , with

$$(4.6) \quad [A_j, A_{k+1}] \cap G_{k+1} \neq 0,$$

that is either  $A_j < A_{k+1}$  or  $A_{k+1} < A_j$  in  $G_{k+1}$ .

In this case  $A_{k+1}$ , which by (4.4) has an edge in common with  $A_i$ , now also has an edge in common with  $A_j \neq A_i$ , thus connecting these two distinct vertices of  $G_k$  by the path

$$(4.7) \quad A_i, A_{k+1}, A_j$$

in  $G_{k+1}$  but outside  $G_k$ .

On the other hand, by the induction's hypothesis,  $G_k$  is connected. Hence  $A_i$  and  $A_j$  are connected by a path in  $G_k$

$$(4.8) \quad A_i, A_{t_1}, A_{t_2}, \dots, A_{t_\lambda}, A_j$$

( $\lambda = 0$  not a priori excluded).

The paths (4.7) and (4.8) combine to the loop

$$(4.9) \quad A_{k+1}, A_i, A_{t_1}, A_{t_2}, \dots, A_{t_\lambda}, A_j, A_{k+1}$$

in  $G_{k+1}$ , which is obviously also a loop in  $G$ .

Since  $G$  is alternating, the loop (4.9) must be alternating. This implies that the number of vertices is even, hence  $\lambda = 2\nu + 1$ , and that the orientation is either

$$(4.10) \quad A_{k+1} < A_i > A_{t_1} < A_{t_2} > \cdots < A_{t_{2\nu}} > A_{t_{2\nu+1}} < A_j > A_{k+1}$$

or the opposite.

Now assume first

$$(4.11) \quad A_{k+1} < A_j,$$

which implies the orientation (4.10), and consider that part of the loop which is in  $G_k$ , namely the path (4.8)

(4.10) and the induction's hypothesis that, relative to  $G_k$  and  $F_k$ ,  $\varphi$  satisfies (4.2), imply

$$(4.12) \quad e_i > e_{t_1} < e_{t_2} > \cdots < e_{t_{2\nu}} > e_{t_{2\nu+1}} < e_j,$$

hence

$$(4.13) \quad \rho e_i = \sigma e_{t_1} = \rho e_{t_2} = \sigma e_{t_3} = \cdots = \rho e_{t_{2\nu}} = \sigma e_{t_{2\nu+1}} = \rho e_j.$$

The definition (4.5) of  $e_{k+1}$ , in conjunction with  $A_{k+1} < A_i$  from (4.10), implies

$$(4.14) \quad \sigma e_{k+1} = \rho e_i.$$

This together with (4.13) yields

$$(4.15) \quad \sigma e_{k+1} = \rho e_j, \quad \text{that is } e_{k+1} < e_j,$$

which proves that assumption (4.11) implies (4.15).

Similarly, the assumption  $A_{k+1} > A_j$  yields  $e_{k+1} > e_j$ , by reversing the relation  $<$  and interchanging  $\rho$  and  $\sigma$  in the above argument.

This completes the proof that to any connected alternating graph  $G$  there exists a connected oriented graph  $F$  and a mapping  $\varphi$  satisfying (4.2)

That  $F$  has no loops (and hence is a tree) is obvious from the fact that its  $n + 1$  vertices are connected by  $n$  edges. Hence, the incidence matrices of  $F$  certainly belong to class  $H$ .

If  $G$  consists of  $k$  components, the construction will yield an  $F$  consisting of  $k$  trees.

This completes the proof of the theorem's first half, namely that every  $K$ -matrix is an  $H$ -matrix.

The second half of the theorem, namely that each nonnegative  $H$ -matrix is a  $K$ -matrix, is due to J. Edmonds. It will be proved by showing that to each loopless oriented  $F$  there is an alternating  $G$  and a mapping  $\psi$  of the edges of  $F$  onto the vertices of  $G$  that preserves the relation  $<$ , that is, for any two edges  $a, b$  of  $F$

$$(4.16) \quad a < b \implies \psi(a) < \psi(b).$$

This is achieved by the following simple construction.

If  $F$  has  $n$  edges  $e_1, e_2, \dots, e_n$ , choose a set of  $n$  elements  $A_1, A_2, \dots, A_n$  as the vertices of  $G$ , define  $\psi$  by

$$(4.17) \quad \psi e_i = A_i,$$

and define the edges of  $G$  by

$$(4.18) \quad (A_i, A_j) \in G \iff e_i < e_j,$$

that is,  $G$  shall have an edge oriented from  $A_i$  to  $A_j$  if and only if  $\sigma e_i = \rho e_j$ .

Obviously  $\psi$  preserves the relation  $<$ , since (4.18) is equivalent to

$$(4.19) \quad A_i < A_j \iff e_i < e_j.$$

Note that  $<$  is also preserved by the inverse of  $\psi$ , that is, in the transition from  $G$  to  $F$ .

Note further that  $G$  is oriented (in the sense of the definition given in [4] and cited in §1 of present note), that is:

(a) each edge of  $G$  is oriented, since the edges of  $G$  have been defined by (4.18) as oriented edges;

(b)  $G$  has no circular edge, since  $(A_i, A_i) \in G$  for some  $i$  would imply  $e_i < e_i$ , or equivalently  $\sigma e_i = \rho e_i$ , that is,  $e_i$  a circular edge in  $F$ , contradicting the assumption on  $F$ ;

(c)  $G$  has at most one edge between any given two vertices:  $(A_i, A_j) \in G$  and  $(A_j, A_i) \in G$  for some pair  $i, j$ , would imply  $e_i < e_j$  and  $e_j < e_i$ , that is  $\sigma e_i = \rho e_j$  and  $\sigma e_j = \rho e_i$ , hence  $e_i$  and  $e_j$  would form a 2-loop (with the vertices  $\rho e_i, \sigma e_i$ ), again contradicting the assumption on  $F$ .

Finally, to show that  $G$  is alternating, note that, by (4.17) and (4.19),  $G, F$  and  $\varphi = \psi^{-1}$  satisfy the condition (4.1). Thus the incidence matrices (of vertices versus directed paths) associated with  $G$  are among the incidence matrices (edges versus paths) associated with  $F$ , and hence unimodular. Especially then, the incidence matrix of the vertices versus *all* the directed paths of  $G$  is unimodular, which, by the Hoffman-Kruskal Theorem (Theorem 4 in [4], cited in §1 of this note), implies that  $G$  is necessarily alternating.

This completes proof of the theorem.

It is worth noting that the last part of the proof (namely that  $G$  is alternating) can easily be established without using the result of [4] (which contains more than is needed here).

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