

# Pacific Journal of Mathematics

**A NOTE ON ABELIAN GROUP EXTENSIONS**

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# A NOTE ON ABELIAN GROUP EXTENSIONS

R. J. NUNKE

In Exercise 21 page 248 of his book *Abelian Groups* L. Fuchs asks for a proof of the following

**THEOREM.** *If  $A$  is a torsion-free and  $C$  a torsion group, then  $\text{Ext}(A, C)$  is either 0 or contains an element of infinite order.*

Unfortunately the hint given with the exercise leads only to the conclusion that every countable subgroup of  $A$  is free. Professor Fuchs has informed me that he meant to assume  $A$  countable. The purpose of this note is to prove this theorem.

**LEMMA.** *If  $C_1, C_2, \dots$  is a sequence of abelian groups,  $\Pi C_i$  their direct product and  $\Sigma C_i$  their direct sum, then  $\text{Ext}(A, \Pi C_i / \Sigma C_i) = 0$  for all torsion-free groups  $A$ .*

*Proof.* A special case of this lemma with all the  $C_i = Z$  the group of integers is a consequence of Theorem 1 of [1]. The proof of the special case given in [4] makes no use of the fact that  $C_i = Z$ . This proof will be sketched here. It is enough to prove the case in which  $A$  is the rational numbers. Since  $\text{Ext}(A, \Pi C_i / \Sigma C_i)$  is a homomorphic image of  $\text{Ext}(A, \Pi C_i)$  we must show that each extension  $0 \rightarrow \Pi C_i \rightarrow E \rightarrow A \rightarrow 0$  splits over  $\Pi C_i / \Sigma C_i$ , i.e., that there is a map  $f: E \rightarrow \Pi C_i / \Sigma C_i$  whose restriction to  $\Pi C_i$  is the canonical projection. With  $A$  the rationals we choose elements  $e^1, e^2, \dots$  in  $E$  such that  $e^n$  maps onto  $1/n!$  modulo  $\Pi C_i$ . Then  $E$  is generated by  $\Pi C_i$  and the  $e$ 's with relations

$$e^n = (n + 1)e^{n+1} + c^n \qquad n = 1, 2, \dots$$

where  $c^n \in \Pi C_i$ . We choose  $b^n \in \Sigma C_i$  such that the first  $n$  coordinates of  $c^n + b^n$  are 0 and put

$$x^n = \sum_{k \geq n} (k!/n!)(c^k + b^k).$$

Then

$$x^n = (n + 1)x^{n+1} + c^n + b^n$$

and we can define  $f$  to be the projection on  $\Pi C_i$  and by  $f(e^n) = x^n + \Sigma C_i$ .

**PROPOSITION.** *If  $C$  is the direct sum of infinitely many copies of*

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$D$  and if  $A$  is torsion-free with  $\text{Ext}(A, D) \neq 0$ , then  $\text{Ext}(A, C)$  has an element of infinite order.

*Proof.* Since  $D$  is a direct summand of  $C$  we have  $\text{Ext}(A, C) \neq 0$ . The sequence

$$\text{Ext}(A, \Sigma C) \rightarrow \text{Ext}(A, \Pi C) \rightarrow \text{Ext}(A, \Pi C / \Sigma C) \rightarrow 0$$

is exact where  $\Sigma C$  is the direct sum and  $\Pi C$  the direct product of countably many copies of  $C$ . By the lemma  $\text{Ext}(A, \Pi C / \Sigma C) = 0$  so that the left-most map in the sequence is an epimorphism. Since  $A$  is torsion-free  $\text{Ext}(A, C)$  is divisible and hence has elements of arbitrarily large finite order if it has nonzero elements of finite order at all. Hence  $\text{Ext}(A, \Pi C) \cong \Pi \text{Ext}(A, C)$  has an element of infinite order. It follows that  $\text{Ext}(A, \Sigma C)$  also has an element of infinite order. Since  $C$  is the direct sum of infinitely many copies of  $D$  we have  $\Sigma C \cong C$  so that  $\text{Ext}(A, \Sigma C) \cong \text{Ext}(A, C)$  proving the proposition.

Now to prove the theorem we suppose that  $A$  is torsion-free,  $C$  is torsion and that  $\text{Ext}(A, C)$  is a nonzero torsion group. Then  $\text{Ext}(A, C)$  has a nonzero  $p$ -primary component for some prime  $p$ . Since  $C = C' \oplus E$  where  $C'$  is the  $p$ -primary component of  $C$  and  $E$  is the sum of the other primary components we have

$$\text{Ext}(A, C) = \text{Ext}(A, C') \oplus \text{Ext}(A, E).$$

Multiplication by  $p$  is an automorphism of  $E$ , hence also an automorphism of  $\text{Ext}(A, E)$ . It follows that  $\text{Ext}(A, C')$  is a nonzero torsion group. Hence in proving the theorem we may assume that  $C$  is  $p$ -primary.

In [3] it was shown that, for  $A$  torsion-free and  $C$   $p$ -primary,

$$\text{Ext}(A, C) \cong \text{Ext}(A, M)$$

where  $M$  is a direct sum of copies of  $\Sigma Z / p^n Z$ , the number of copies being equal to the final rank of  $C$ . If  $C$  has bounded order, then  $\text{Ext}(A, C) = 0$  for all torsion-free groups  $A$ . Otherwise the final rank of  $C$  is infinite. This last case is the one to be considered. Then  $M$  is the direct sum of countably many copies of itself and the proposition shows that  $\text{Ext}(A, M)$  is either 0 or has an element of infinite order.

The referee has pointed out that a stronger form of the lemma in this paper has been proved by A. Hulanicki (Bull. Acad. Pol. Sci. Ser. Sci. Math. Astr. Phys., 10 (1962), 77-80.) He showed that each element of infinite height in  $\Pi C_i / \Sigma C_i$  is in the maximal divisible subgroup, hence this group is algebraically compact.

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Tsuayoshi Andô, <i>On fundamental properties of a Banach space with a cone</i> .....	1163
Sterling K. Berberian, <i>A note on hyponormal operators</i> .....	1171
Errett Albert Bishop, <i>Analytic functions with values in a Frechet space</i> .....	1177
(Sherman) Elwood Bohn, <i>Equicontinuity of solutions of a quasi-linear equation</i> .....	1193
Andrew Michael Bruckner and E. Ostrow, <i>Some function classes related to the class of convex functions</i> .....	1203
J. H. Curtiss, <i>Limits and bounds for divided differences on a Jordan curve in the complex domain</i> .....	1217
P. H. Doyle, III and John Gilbert Hocking, <i>Dimensional invertibility</i> .....	1235
David G. Feingold and Richard Steven Varga, <i>Block diagonally dominant matrices and generalizations of the Gerschgorin circle theorem</i> .....	1241
Leonard Dubois Fountain and Lloyd Kenneth Jackson, <i>A generalized solution of the boundary value problem for <math>y'' = f(x, y, y')</math></i> .....	1251
Robert William Gilmer, Jr., <i>Rings in which semi-primary ideals are primary</i> .....	1273
Ruth Goodman, <i>K-polar polynomials</i> .....	1277
Israel Halperin and Maria Wonenburger, <i>On the additivity of lattice completeness</i> .....	1289
Robert Winship Heath, <i>Arc-wise connectedness in semi-metric spaces</i> .....	1301
Isidore Heller and Alan Jerome Hoffman, <i>On unimodular matrices</i> .....	1321
Robert G. Heyneman, <i>Duality in general ergodic theory</i> .....	1329
Charles Ray Hobby, <i>Abelian subgroups of p-groups</i> .....	1343
Kenneth Myron Hoffman and Hugo Rossi, <i>The minimum boundary for an analytic polyhedron</i> .....	1347
Adam Koranyi, <i>The Bergman kernel function for tubes over convex cones</i> .....	1355
Pesi Rustom Masani and Jack Max Robertson, <i>The time-domain analysis of a continuous parameter weakly stationary stochastic process</i> .....	1361
William Schumacher Massey, <i>Non-existence of almost-complex structures on quaternionic projective spaces</i> .....	1379
Deane Montgomery and Chung-Tao Yang, <i>A theorem on the action of SO(3)</i> .....	1385
Ronald John Nunke, <i>A note on Abelian group extensions</i> .....	1401
Carl Mark Percy, <i>A complete set of unitary invariants for operators generating finite <math>W^*</math>-algebras of type I</i> .....	1405
Edward C. Posner, <i>Integral closure of rings of solutions of linear differential equations</i> .....	1417
Duane Sather, <i>Asymptotics. III. Stationary phase for two parameters with an application to Bessel functions</i> .....	1423
J. Śladkowska, <i>Bounds of analytic functions of two complex variables in domains with the Bergman-Shilov boundary</i> .....	1435
Joseph Gail Stampfli, <i>Hyponormal operators</i> .....	1453
George Gustave Weill, <i>Some extremal properties of linear combinations of kernels on Riemann surfaces</i> .....	1459
Edward Takashi Kobayashi, <i>Errata: "A remark on the Nijenhuis tensor"</i> .....	1467