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**A CONE OF SUPER- $(L)$  FUNCTIONS**

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# A CONE OF SUPER-( $L$ ) FUNCTIONS

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**1. Introduction.** Bonsall [2] introduced the following generalization of the concept of a real-valued concave function of one real variable on a closed interval  $[a, b]$ , where  $a < b$ :

**DEFINITION 1.1.** Let  $y_1$  and  $y_2$  be arbitrary real numbers, and let  $x_1$  and  $x_2$  be real numbers such that  $a \leq x_1 < x_2 \leq b$ . Let  $L(y) = d^2y/dx^2 + p(x)dy/dx + q(x)y = 0$  be such that there exists a unique solution  $F$  on  $[a, b]$  (where the appropriate one-sided derivatives are used at the end-points) for which  $F(x_i) = y_i$ ,  $i = 1, 2$ . Then a real-valued function  $f$  is *super-( $L$ )* on  $[a, b]$  if  $f(x) \geq F(f, x_1, x_2; x)$  for all  $x, x_1$ , and  $x_2$  such that  $a \leq x_1 < x_2 \leq b$  and  $x_1 \leq x \leq x_2$ , where  $F(f, x_1, x_2; x)$  is the solution of  $L(y) = 0$  such that  $F(f, x_1, x_2; x_i) = f(x_i)$ ,  $i = 1, 2$ .

This definition is a special case of the generalized concave function introduced by Beckenbach [1].

In this paper, it will be shown that the set of non-negative continuous super-( $L$ ) functions on  $[a, b]$  is a convex cone, and the extremal structure of the cone will be characterized. A result due to Choquet [4] will then be used to prove the existence of a type of integral representation for the elements of the cone in terms of the extremal elements of the cone. It will be assumed throughout this paper that the functions considered are continuous on  $[a, b]$ .

**DEFINITION 1.2.** Let  $A$  be a set in a real linear space. Then  $A$  is a *convex cone* if

(1) for every  $f$  and  $g$  in  $A$  and every nonnegative real number  $k$ ,  $f + g$  and  $kf$  belong to  $A$ , and

(2)  $f$  in  $A$  and  $-f$  in  $A$  imply  $f = 0$ , the origin of the real linear space.

It is easy to check that if  $f$  and  $g$  are super-( $L$ ) functions on  $[a, b]$  and  $k$  is a nonnegative real number, then  $kf$  and  $f + g$  are super-( $L$ ) functions on  $[a, b]$ , and hence it follows that the set  $C$  of nonnegative super-( $L$ ) functions on  $[a, b]$  forms a convex cone.

**2. Extremal structure of  $C$ .** McLachlan [5] has characterized the extremal structure of the convex cone of nonnegative concave functions on  $[a, b]$ . It will be shown in this section that the extremal structure of  $C$  is analogous to that obtained by McLachlan.

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DEFINITION 2.1. A real-valued function  $f$  on  $[a, b]$  is ( $L$ )-linear on  $[x_1, x_2]$  if  $f(x) = F(f, x_1, x_2; x)$  for all  $x$  in  $[x_1, x_2]$ , where  $a \leq x_1 < x_2 \leq b$ .

LEMMA 2.1. If  $f, f_1$ , and  $f_2$  are super- $(L)$  functions on  $[a, b]$  such that  $f(x) = f_1(x) + f_2(x)$  for all  $x$  in  $[x_1, x_2]$ , where  $a \leq x_1 < x_2 \leq b$ , and  $f$  is ( $L$ )-linear on  $[x_1, x_2]$ , then  $f_1$  and  $f_2$  are ( $L$ )-linear on  $[x_1, x_2]$ .

The proof is straightforward and will be omitted.

DEFINITION 2.2. A real-valued function  $f$  on  $[a, b]$  is an ( $L$ )-conical function with its vertex over  $w$  in  $[a, b]$  if

(1)  $f(w) > 0$ ,

(2)  $f(a) = f(b) = 0$  if  $w \neq a, b$ ;  $f(a) = 0$  if  $w = b$ ; or  $f(b) = 0$  if  $w = a$ ; and

(3)  $f$  is ( $L$ )-linear on  $[a, w]$  and on  $[w, b]$ .

LEMMA 2.2 [2, p. 101]. If  $f$  is super- $(L)$  on  $[a, b]$ , then  $f(x) \leq F(f, x_1, x_2; x)$  for all  $x$  in  $[a, x_1]$  and  $[x_2, b]$ .

LEMMA 2.3. If  $f$  in  $C$  is such that  $f(x_0) = 0$  for some  $x_0$  in  $(a, b)$ , then  $f = 0$ .

*Proof.* Suppose there exists an  $x'$  in  $[a, b]$  such that  $f(x') > 0$ . There is no loss in generality in assuming that  $x' < x_0$ . Since  $f$  is super- $(L)$ ,  $F(f, a, b; x_0) \leq f(x_0) = 0$ . If  $F(f, a, b; x_0) < 0$ , then  $F(f, a, b; b) \geq 0$  and  $F(f, a, b; a) \geq 0$  imply  $F(f, a, b; x)$  has zero function value at two distinct points, and hence is zero on  $[a, b]$ , a contradiction. Thus  $F(f, a, b; x_0) = 0 = f(x_0)$ , and it follows that  $F(f, a, x_0; x) = F(f, a, b; x) = F(f, x_0, b; x)$  on  $[a, b]$ . Since  $f$  is super- $(L)$ ,  $f(x) \geq F(f, a, x_0; x)$  on  $[a, x_0]$  and  $f(x) \geq F(f, x_0, b; x)$  on  $[x_0, b]$ . By Lemma 2.2,  $f(x) \leq F(f, a, x_0; x)$  on  $[x_0, b]$  and  $f(x) \leq F(f, x_0, b; x)$  on  $[a, x_0]$ . Thus  $f(x) = F(f, a, b; x)$  on  $[a, b]$ . Let  $x_1$  and  $x_2$  be such that  $a \leq x_1 < x_0 < x_2 \leq b$ , and let  $y_1$  and  $y_2$  be positive real numbers. Since  $F(f, a, b; x') > 0$ , it follows that  $F(f, a, b; x) > 0$  for all  $x \neq x_0$  in  $[a, b]$ . Then there exist real numbers  $r_1$  and  $r_2$  such that  $y_i = r_i F(f, a, b; x_i)$ ,  $i = 1, 2$ . Let  $G$  be the solution of  $L(y) = 0$  such that  $G(x_i) = y_i$ ,  $i = 1, 2$ . Assume  $r_1 \geq r_2$ , since the proof for the other case is similar. Then  $r_1 F(f, a, b; x_2) \geq r_2 F(f, a, b; x_2) = y_2 = G(x_2)$ . If  $G(x_0) < 0$ ,  $G$  is zero at two distinct points and hence is identically zero on  $[a, b]$ , a contradiction. Then  $G(x_0) \geq F(f, a, b; x_0)$  and  $G(x_2) \leq r_1 F(f, a, b; x_2)$  imply the existence of an  $x_3$  in  $[x_0, x_2]$  for which  $G(x_3) = r_1 F(f, a, b; x_3)$ . Hence  $G(x) = r_1 F(f, a, b; x)$  on  $[a, b]$  since  $G(x_1) = r_1 F(f, a, b; x_1)$ . This contradicts the existence of solutions of  $L(y) = 0$  taking arbitrary positive function values at  $x_0$  and  $x_1 \neq x_0$ .

**DEFINITION 2.3.** Let  $A$  be a convex cone. An element  $f$  of  $A$  is an *extremal element* of  $A$  if for every pair of elements  $f_1$  and  $f_2$  of  $A$  such that  $f = f_1 + f_2$  there exists a real number  $k$  such that  $f_1 = kf$ .

**THEOREM 2.1.** A function  $f (\neq 0)$  is an extremal element of  $C$  if and only if  $f$  is an  $(L)$ -conical function.

*Proof.* It is easy to check that an  $(L)$ -conical function is super- $(L)$ . The result follows in a straightforward fashion upon applying Lemma 2.1.

If  $f$  in  $C$  is not  $(L)$ -linear on  $[a, b]$  and is such that either  $f(a) > 0$  or  $f(b) > 0$ , then  $f$  is not an extremal element of  $C$  since a nonproportional decomposition for  $f$  is  $F(f, a, b; x)$  and  $f(x) - F(f, a, b; x)$ .

If  $f$  in  $C$  is  $(L)$ -linear on  $[a, b]$  and  $f(a) > 0$  and  $f(b) > 0$ , then  $f$  is not an extremal element of  $C$  since a nonproportional decomposition for  $f$  is the  $(L)$ -linear function  $f_1$  such that  $f_1(a) = f(a)$  and  $f_1(b) = 0$  and the  $(L)$ -linear function  $f_2$  such that  $f_2(a) = 0$  and  $f_2(b) = f(b)$ .

To complete the proof of the theorem, let  $f \neq 0$  be an element of  $C$  which is not  $(L)$ -conical and is such that  $f(a) = f(b) = 0$ . Let  $x_0$  be such that  $a < x_0 < b$ . Assume  $f$  is not  $(L)$ -linear on  $\{x_0, b\}$ , since the proof for the other case is similar. Let  $g(x) = f(x) - F(f, x_0, b; x)$  on  $[a, b]$ . For each positive real number  $y$ , let  $F_y$  be the  $(L)$ -linear function determined by  $(a, 0)$  and  $(b, y)$ . Let  $u = \inf \{y: F_y(x) > g(x) \text{ for all } x \text{ in } [a, b]\}$ . Clearly  $u$  exists and is positive. The assumption that  $F_u(x) > g(x)$  for all  $x$  in  $[a, b]$  leads to a contradiction, so  $\{x: F_u(x) = g(x)\}$  is not empty. Let  $\bar{x} = \sup \{x: F_u(x) = g(x)\}$ . Since  $F_u(b) > 0$ , there exists an  $x'$  in  $(\bar{x}, b)$  such that  $F_u(x) = F(f, a, x'; x)$  on  $[a, b]$ . Let  $f_1(x) = F(f, a, x'; x)$  on  $[a, \bar{x}]$  and  $f_1(x) = f(x) - F(f, x_0, b; x)$  on  $[\bar{x}, b]$ . Let  $f_2 = f - f_1$ . It will be shown that  $f_1$  and  $f_2$  form a nonproportional decomposition of  $f$ . Clearly  $f_1$  and  $f_2$  are nonnegative. Let  $x_1$  and  $x_2$  be in  $[a, b]$ . Since  $f_1$  is super- $(L)$  on  $[a, \bar{x}]$  and on  $[\bar{x}, b]$ , it will be assumed that  $a \leq x_1 < \bar{x}$  and  $\bar{x} < x_2 \leq b$ . If  $f_1(x_3) < F(f_1, x_1, x_2; x_3)$  for some  $x_3$  in  $(x_1, x_2)$ , then  $F(f_1, x_1, x_2; x)$  must intersect  $F(f, a, x'; x)$  for some  $x$  in  $[\bar{x}, x_2)$ , a contradiction. Thus  $f_1$  is super- $(L)$  on  $[a, b]$ . By observing that  $f_2(x) \leq F(f, x_0, b; x)$  on  $[a, \bar{x}]$ , a similar argument may be used to prove  $f_2$  is super- $(L)$  on  $[a, b]$ . Suppose there exists a real number  $k$  such that  $f_1 = kf$ . Then  $f$  is  $(L)$ -linear on  $[a, \bar{x}]$  since  $f_1$  is and  $k \neq 0$ . Since  $f = f_1 + f_2$  and  $f_2 \neq 0$ , it follows that  $k \neq 1$ , and so  $f_2 = (1 - k)f$  implies  $f$  is  $(L)$ -linear on  $[\bar{x}, b]$ . This contradicts the assumption that  $f$  is not  $(L)$ -conical.

**3. Integral representation.** The existence of an integral representation (Radon measure) for the elements of the cone  $C$  in terms of its extremal elements will be based on the following theorem due to Choquet:

**THEOREM 3.1** [4, p. 237]. *If the linear space  $L$  is a locally convex Hausdorff space, and if  $A$  is a convex compact subset of  $L$ , then for each  $x$  in  $A$  there exists a nonnegative Radon measure on the closure of the set of extreme points of  $A$  whose center of gravity is  $x$ .*

The theorem will be applied in the following way: First, it is known that  $C - C$  is a real linear space such that the vertex of  $C$  is the origin of  $C - C$  [3]. It is also known that when  $C - C$  is topologized with the topology of simple convergence (the induced product topology of  $R^{[a, b]}$ ), it is a locally convex Hausdorff topological linear space [4]. It will be shown that  $B = \{f : f \text{ is in } C, f(x_0) = 1\}$ , where  $x_0$  is a fixed real number in  $(a, b)$ , is a convex compact subset of  $C - C$  which meets each ray of  $C$  once and only once and does not contain 0, the origin of  $C - C$ , and that the set of extreme points of  $B$  is closed in  $C - C$ . Then by the theorem there will exist an integral representation for each element of  $B$  in terms of extreme points of  $B$ . It will then follow that there exists an integral representation for each element of  $C$  in terms of extremal elements of  $C$  since  $B$  meets each ray of  $C$  once and only once and does not contain 0.

**LEMMA 3.1.** *Let  $F_1$  be the  $(L)$ -linear function determined by the points  $(x_0, 1)$  and  $(b, 0)$ , and let  $F_2$  be the  $(L)$ -linear function determined by the points  $(a, 0)$  and  $(x_0, 1)$ . Then  $\{f(x) : f \text{ is in } B\} = [F_2(x), F_1(x)]$  for each  $x$  in  $[a, x_0]$  and  $\{f(x) : f \text{ is in } B\} = [F_1(x), F_2(x)]$  for each  $x$  in  $[x_0, b]$ .*

*Proof.* Clearly  $F_1$  and  $F_2$  belong to  $B$ ,  $F_1(x) > F_2(x)$  on  $[a, x_0]$ , and  $F_1(x) < F_2(x)$  on  $(x_0, b]$ . Let  $f$  belong to  $B$ . The assumption that there exists an  $x_1$  in  $[a, x_0]$  such that  $f(x_1) > F_1(x_1)$  leads to a contradiction through an application of Lemma 2.3. Similarly,  $f(x) \leq F_2(x)$  on  $[x_0, b]$ . The assumption that  $f(x_2) < F_2(x_2)$  for some  $x_2$  in  $[a, x_0]$  or  $f(x_3) < F_1(x_3)$  for some  $x_3$  in  $(x_0, b]$  either contradicts  $f$  being super- $(L)$  or  $f$  being nonnegative. Therefore  $\{f(x) : f \text{ is in } B\} \subset [F_2(x), F_1(x)]$  for each  $x$  in  $[a, x_0]$  and  $\{f(x) : f \text{ is in } B\} \subset [F_1(x), F_2(x)]$  for each  $x$  in  $[x_0, b]$ . Given any  $x$  in  $[a, b]$  and  $y$  between  $F_1(x)$  and  $F_2(x)$ , there exists an  $(L)$ -linear function in  $B$  passing through that point. Hence  $\{f(x) : f \text{ is in } B\} = [F_2(x), F_1(x)]$  for each  $x$  in  $[a, x_0]$  and  $\{f(x) : f \text{ is in } B\} = [F_1(x), F_2(x)]$  for each  $x$  in  $[x_0, b]$ .

The Tychonoff product theorem may now be applied to show that  $B$  is a subset of a compact set in  $R^{[a, b]}$ , so that to prove  $B$  is compact it is only necessary to prove it is closed in  $R^{[a, b]}$ .

**LEMMA 3.2.** *The convex cone  $C$  is closed in  $R^{[a, b]}$  for the topology of simple convergence.*

*Proof.* Let  $f$  belong to the complement of  $C$ . If there exists an  $x_1$  in  $[a, b]$  such that  $f(x_1) < 0$ , let  $\varepsilon = -f(x_1)/2$ . Let  $g$  belong to  $U(f; x_1; \varepsilon)$ , a neighborhood of  $f$  in the topology of simple convergence. Then  $|g(x_1) - f(x_1)| < \varepsilon$  implies  $g(x_1) - f(x_1) < -f(x_1)/2$ , so that  $g(x_1) < f(x_1)/2 < 0$ . Thus  $g$  is not in  $C$ , and hence  $U(f; x_1; \varepsilon)$  is in the complement of  $C$ .

If  $f$  is nonnegative, then  $f$  is not super-(L). Hence there exist  $x_1, x_2$ , and  $x_3$  such that  $x_1 < x_3 < x_2$  and  $f(x_3) < F(f, x_1, x_2; x_3)$ . Let  $k_i = f(x_i) [F(f, x_1, x_2; x_3) - f(x_3)] / [f(x_i) + F(f, x_1, x_2; x_3)]$ ,  $i = 1, 2$ . Take  $\varepsilon = \min \{k_1, k_2\}/2$  if  $k_1 > 0$  and  $k_2 > 0$ ,  $\varepsilon = k_1/2$  if  $k_2 = 0$ , or  $\varepsilon = k_2/2$  if  $k_1 = 0$ . Then  $U(f; x_1, x_2, x_3; \varepsilon)$  is in the complement of  $C$ .

**THEOREM 3.2.** *The set  $B$  is a convex compact subset of  $C - C$  which meets each ray of  $C$  once and only once and which does not contain 0.*

*Proof.* Let  $f_1$  and  $f_2$  belong to  $B$ , and let  $k$  be any real number such that  $0 < k < 1$ . Then  $kf_1 + (1-k)f_2$  belongs to  $C$  and  $(kf_1 + (1-k)f_2)(x_0) = 1$ , so that  $kf_1 + (1-k)f_2$  belongs to  $B$ . Thus  $B$  is convex.

Let  $f$  be in the complement of  $B$  relative to  $C$ . Since  $f$  is in  $C$ ,  $f(x_0) \neq 1$ . Let  $\varepsilon = |f(x_0) - 1|$ . Then  $U(f; x_0; \varepsilon) \cap C$  is in the complement of  $B$  relative to  $C$ , and hence  $B$  is closed in  $R^{[a, b]}$  by Lemma 3.2. It now follows that  $B$  is a compact subset of  $C - C$ .

Let  $H$  be any ray in  $C$ . Let  $f$  in  $C$  be such that  $H = \{kf : k \text{ is a nonnegative real number}\}$ . By Lemma 2.3,  $f(x_0) \neq 0$  since  $x_0$  is in  $(a, b)$ , so that  $k_1 = 1/f(x_0)$  is such that  $k_1 f$  belongs to  $B$ . Thus the intersection of  $B$  with  $H$  exists and is unique. Obviously 0 is not in  $B$ .

**THEOREM 3.3.** *The set  $e(B)$  of extreme points of  $B$  is closed in  $C - C$  for the topology of simple convergence.*

*Proof.* By Theorem 3.2,  $B$  is closed relative to  $C - C$ , so that it is only necessary to prove  $e(B)$  is closed relative to  $B$ . Let  $f$  be in the complement of  $e(B)$  relative to  $B$ . Then clearly there exists an  $x_1$  in  $(a, b)$  such that  $f(x_1) \neq F_1(x_1)$  and  $f(x_1) \neq F_2(x_1)$ , where  $F_1$  and  $F_2$  are the functions defined in Lemma 3.1. It will be assumed that  $x_1$  is in  $(a, x_0)$  since the proof for the other case is similar. Let  $G_1$  be the (L)-conical function in  $e(B)$  determined by the points  $(a, 0)$  and  $(x_1, f(x_1))$ . Let  $\bar{x}$  be the  $x$ -coordinate of the vertex of  $G_1$ , and observe that  $\bar{x} < x_0$ . Suppose  $f(x) = G_1(x)$  on  $[a, \bar{x}]$ . Then  $f$  super-(L) and  $F(f, \bar{x}, x_0; x) = F_1(x)$  on  $[a, b]$  imply  $f(x) \geq F_1(x)$  on  $[\bar{x}, x_0]$  and  $f(x) \leq F_1(x)$  on  $[x_0, b]$  by Lemma 2.2. By Lemma 3.1,  $f(x) \leq F_1(x)$  on  $[\bar{x}, x_0]$  and  $f(x) \geq F_1(x)$  on  $[x_0, b]$ . Thus  $f = G_1$ , which contradicts  $f$  being in the complement of  $e(B)$ . Therefore there exists an  $x_2$  in  $[a, \bar{x}]$  such

that  $f(x_2) \neq G_1(x_2)$ . Let  $G_2$  be the  $(L)$ -conical function in  $e(B)$  determined by the points  $(a, 0)$  and  $(x_2, [f(x_2) + G_1(x_2)]/2)$ . Since  $G_2(a) = G_1(a)$  and  $G_2(x_2) \neq G_1(x_2)$ , it follows that  $G_2(x_1) \neq G_1(x_1) = f(x_1)$ . Let  $\varepsilon = (1/2) \min \{|f(x_1) - G_2(x_1)|, |f(x_2) - G_1(x_2)|\}$ . Then  $U(f; x_1, x_2; \varepsilon) \cap B$  is in the complement of  $e(B)$  relative to  $B$ , and hence  $e(B)$  is closed relative to  $B$ .

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