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GENERAL GROUP EXTENSIONS

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Introduction. The aim of this paper is to show how some of the methods useful in studying normal extensions of groups can be used in a problem of more general extensions. The present approach (which might be compared with that of Szep [5]) is made possible because we consider classes of extensions which are still relatively restricted.

If G is an arbitrary subgroup of a group H then the set of all right cosets of G in H forms a mixed group under a naturally defined operation (Loewy [3]). In particular, when G is normal in H then the corresponding mixed group is the ordinary quotient group H/G . This paper is concerned with examining properties of the class of those extensions H of a given group G for which the corresponding mixed group is isomorphic to a given mixed group Γ . As an example of the results, Theorems 2.2 and 2.3 represent analogues of the corresponding theorems of Schreier on factor sets for normal extensions.

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Mixed groups.

1.1 DEFINITION. A *mixed group* is a set Γ on which a product $\alpha\beta \in \Gamma$ is defined for certain pairs $\alpha, \beta \in \Gamma$ such that

(i) a nonempty subset Δ of Γ forms a group under the given product and is called the *nucleus* of Γ ;

(ii) for all $\beta \in \Gamma$, $\alpha\beta$ is defined if and only if $\alpha \in \Delta$; furthermore, $\alpha\beta = \beta$ if and only if $\alpha = 1$, the identity of Δ ;

(iii) if $\alpha, \beta \in \Delta$ and $\gamma \in \Gamma$ then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$. (See Loewy [3] and Bruck [2; page 35]. The general properties of mixed groups are derived in Baer [1].)

In particular, if H is a group with a subgroup G then the set of all right cosets of G in H forms a mixed group when the product of two elements is defined by $(Gx)(Gy) = Gxy$ whenever $x \in N(G; H)$, the normaliser of G in H . In this case we denote the mixed group by H/G and note that its nucleus is the quotient group $N(G; H)/G$ (See Baer [1]).

1.2 DEFINITION. Two mixed groups Γ and Γ' with nuclei Δ and Δ' respectively are *isomorphic* under a mapping τ if τ is a one-to-one

mapping of Γ onto Γ' such that $(\alpha\tau)(\beta\tau)$ is defined in Γ' if and only if $\alpha\beta$ is defined in Γ and that in that case $(\alpha\beta)\tau = (\alpha\tau)(\beta\tau)$.

Because of part (ii) of Definition 1.1, and isomorphism τ of Γ onto Γ' induces on the nucleus Δ of Γ a (group) isomorphism onto the nucleus Δ' of Γ' .

As an example, suppose that H is a group with a subgroup G and let ϕ be a homomorphism of H onto a group H^* . If G^* is the image of G under ϕ and $\ker \phi \cong G$ then it is easy to show that the mixed group H/G is isomorphic to H^*/G^* . The notation $H/G \cong H^*/G^*$ will be used to imply the existence of a homomorphism of H onto H^* with this property.

1.3 DEFINITION. If H is a group with subgroup G and H/G is isomorphic to a mixed group Γ then H/G is a *representation* of Γ and H is an *extension* of G by Γ .

Baer [1; Theorem 3] proves that, except in the case that the mixed group Γ is of order 2 and has unit nucleus, every mixed group Γ has a representation H/G for some groups H and G . (The exceptional case arises because no subgroup G of index 2 in a group H can be its own normaliser.)

1.4. Contrary to the case of normal extensions, not every group G has an extension H by a given mixed group Γ . From the example of 1.2, when H^*/G^* is chosen to be minimal under the quasi-ordering defined there, G^* contains no nontrivial normal subgroup of H^* . In such a case we call H^*/G^* a *cardinal representation* of Γ . (If $|\Gamma| = n$ is finite, and H^*/G^* is a cardinal representation of Γ , then H^* is isomorphic to a permutation group of degree n and G^* corresponds to a subgroup fixing one letter.) Thus a necessary condition that G should have an extension H by Γ is that, for some cardinal representation H^*/G^* of Γ , G^* should be a homomorphic image of G . Examples show, however, that this condition is not sufficient.

2. Extension functions. As a generalisation of the Schreier factor set used in the theory of normal extensions we consider the extension of a group by a mixed group through the medium of a skew product (cf. Redei [4], Szep [5]).

2.1. Let G be a group and Γ be a mixed group with nucleus Δ . We define the skew product $\langle G, \Gamma \rangle$ to be the set $\{\langle a, \alpha \rangle \mid a \in G, \alpha \in \Gamma\}$ on which a binary operation is defined by

$$\langle a, \alpha \rangle \langle b, \beta \rangle = \langle af(\alpha, b, \beta), \phi(\alpha, b, \beta) \rangle$$

with $a, b, f(\alpha, b, \beta) \in G$ and $\alpha, \beta, \phi(\alpha, b, \beta) \in \Gamma$ for some functions f

and ϕ respectively.

We shall denote the identities of G , Γ and $\langle G, \Gamma \rangle$ by $1, 1$ and $\langle 1, 1 \rangle$ respectively. Furthermore we write $\langle G, \alpha \rangle = \{ \langle x, \alpha \rangle \mid x \in G \}$, $\langle a, \Gamma \rangle = \{ \langle a, \xi \rangle \mid \xi \in \Gamma \}$ and identify G with $\langle G, 1 \rangle$ and Γ with $\langle 1, \Gamma \rangle$ under the natural mappings.

2.2 THEOREM. *The skew product $H = \langle G, \Gamma \rangle$ is a group with identity $\langle 1, 1 \rangle$ if and only if the following conditions on f and ϕ hold for all $b, c \in G$ and $\alpha, \beta, \gamma \in \Gamma$:*

- (1) $f(1, b, \beta) = b$ and $\phi(1, b, \beta) = \beta$;
- (2) $f(\alpha, b, \beta)f(\phi(\alpha, b, \beta), c, \gamma) = f(\alpha, bf(\beta, c, \gamma), \phi(\beta, c, \gamma))$;
- (3) $\phi(\phi(\alpha, b, \beta), c, \gamma) = \phi(\alpha, bf(\beta, c, \gamma), \phi(\beta, c, \gamma))$;
- (4) for all $\alpha \in \Gamma$ there exists $\xi \in \Gamma$ such that $\phi(\xi, 1, \alpha) = 1$.

Proof. (1) is equivalent to $\langle 1, 1 \rangle \langle b, \beta \rangle = \langle b, \beta \rangle$, that is that $\langle 1, 1 \rangle$ is a left identity.

(2) and (3) are together equivalent to the associative law

$$\{ \langle a, \alpha \rangle \langle b, \beta \rangle \} \langle c, \gamma \rangle = \langle a, \alpha \rangle \{ \langle b, \beta \rangle \langle c, \gamma \rangle \} .$$

(4) is equivalent to $\langle 1, \alpha \rangle$ having a left inverse $\langle x, \xi \rangle$ (with $x = f(\xi, 1, \alpha)^{-1}$). However, in that case $\langle a, \alpha \rangle = \langle a, 1 \rangle \langle 1, \alpha \rangle$ has a left inverse $\langle x, \xi \rangle \langle a^{-1}, 1 \rangle$ for all $\langle a, \alpha \rangle \in H$.

Thus the stated conditions are necessary and sufficient.

2.3 THEOREM. *If the skew product $H = \langle G, \Gamma \rangle$ is a group with identity $\langle 1, 1 \rangle$ then a necessary and sufficient condition that $\langle G, \Gamma \rangle / \langle G, 1 \rangle = H/G$ should be isomorphic to Γ under the natural mapping*

$$\tau: \langle G, 1 \rangle \langle 1, \alpha \rangle = \langle G, \alpha \rangle \rightarrow \alpha \quad (\alpha \in \Gamma)$$

is that

- (5) $\phi(\alpha, b, 1) = \alpha$ for all $b \in G$ if and only if $\alpha \in \Delta$;
- (6) $\phi(\alpha, 1, \beta) = \alpha\beta$ when $\alpha \in \Delta$.

Proof. Since it is clear that τ is a one-to-one mapping onto Γ the theorem follows from Definition 1.2 when we note:

- (5) is equivalent to $\langle G, \alpha \rangle \tau \in \Delta$ if and only if $\alpha \in \Delta$;
- (6) is equivalent to $\langle G, \alpha \rangle \langle G, \beta \rangle = \langle G, \alpha\beta \rangle$ for $\alpha \in \Delta$.

2.4. A skew product $H = \langle G, \Gamma \rangle$ which satisfies the conditions (1)–(6) will be called an extension of G by Γ with functions f and ϕ . For such an extension it is easily shown that f and ϕ have the

following properties:

- (7) $f(\alpha, 1, 1) = 1$ and $\phi(\alpha, 1, 1) = \alpha$;
 (8) for all $\langle a, \alpha \rangle \in H$ there exists a unique $\xi \in \Gamma$ such that $\phi(\xi, a, \alpha) = 1$;
 (9) the mapping $\gamma \rightarrow \phi(\gamma, a, \alpha)$ permutes the elements of Γ ;
 (10) $\phi(\alpha, b, \beta) = \alpha\beta$ when $\alpha \in \Delta$.

2.5 DEFINITION. Two extensions $H_1 = \langle G, \Gamma \rangle_1$ and $H_2 = \langle G, \Gamma \rangle_2$ with functions f_1, ϕ_1 and f_2, ϕ_2 respectively are called *equivalent* if there is an isomorphism of H_1 onto H_2 which leaves each element of G fixed. If, moreover, the coset $\langle G, \alpha \rangle_1$ is mapped onto the coset $\langle G, \alpha \rangle_2$ for each $\alpha \in \Gamma$ then H_1 is said to be *biequivalent* to H_2 .

Thus if θ is a one-to-one mapping of H_1 onto H_2 then in order for this mapping to be a biequivalence we must have

- (i) $\langle a, 1 \rangle_1 \theta = \langle a, 1 \rangle_2$;
 (ii) $\langle a, \alpha \rangle_1 \langle b, \beta \rangle_1 \theta = \langle a, \alpha \rangle_1 \theta \langle b, \beta \rangle_1 \theta$
 (iii) $\langle 1, \alpha \rangle_1 \theta = \langle x_\alpha, \alpha \rangle_2$ for some $x_\alpha \in G$.

2.6 THEOREM. If $H_1 = \langle G, \Gamma \rangle_1$ and $H_2 = \langle G, \Gamma \rangle_2$ are two extensions of G by Γ with functions f_1, ϕ_1 and f_2, ϕ_2 respectively then H_1 is biequivalent to H_2 if and only if there is a function $\alpha \rightarrow x_\alpha$ of Γ into G such that

- (11) $x_\alpha f_2(\alpha, bx_\beta, \beta) = f_1(\alpha, b, \beta) x_{\phi_1(\alpha, b, \beta)}$;
 (12) $\phi_2(\alpha, bx_\beta, \beta) = \phi_1(\alpha, b, \beta)$.

Proof. If θ is a biequivalence of H_1 onto H_2 then define x_α by 2.5 (iii). Then $\langle a, \alpha \rangle_1 \theta = \langle ax_\alpha, \alpha \rangle_2$. Therefore

$$\langle a, \alpha \rangle_1 \langle b, \beta \rangle_1 \theta = \langle af_1(\alpha, b, \beta) x_{\phi_1(\alpha, b, \beta)}, \phi_1(\alpha, b, \beta) \rangle_2$$

and $\langle a, \alpha \rangle_1 \theta \langle b, \beta \rangle_1 \theta = \langle ax_\alpha f_2(\alpha, bx_\beta, \beta), \phi_2(\alpha, bx_\beta, \beta) \rangle_2$ and so (11) and (12) are together implied by 2.5(ii).

Conversely, if we are given (11) and (12) and define θ as the one-to-one mapping of H_1 onto H_2 given by $\langle a, \alpha \rangle_1 \theta = \langle ax_\alpha, \alpha \rangle_2$ then 2.5 (i), (ii) and (iii) follow, so θ is the required biequivalence.

2.7. Let $H = \langle G, \Gamma \rangle$ be an extension of G by a mixed group Γ with functions f and ϕ . The kernel Δ of Γ is isomorphic to $N(G; H)/G$ (by 1.1). Therefore $G = \langle G, 1 \rangle$ is a normal subgroup (and H is a normal extension of G) if and only if $\Delta = \Gamma$, and Γ is a group. Alternatively, using (5) and (6) we have that $\phi(\alpha, b, 1) = \alpha$ (for all $\alpha \in \Gamma, b \in G$) as a necessary and sufficient condition that H be a normal extension of G .

A second important case is when H is a splitting extension (i.e.

$\Gamma = \langle 1, \Gamma \rangle$ is a subgroup of H) so that $H = G\Gamma$ and $G \cap \Gamma = 1$. In terms of the extension functions, H is a splitting extension if and only if $f(\alpha, 1, \beta) = 1$ for all $\alpha, \beta \in \Gamma$. To prove this we note that, since $\langle 1, \alpha \rangle \langle 1, \beta \rangle = \langle f(\alpha, 1, \beta), \phi(\alpha, 1, \beta) \rangle$, the condition is certainly necessary. It is also sufficient because when it holds we also have

$$\langle 1, \alpha \rangle^{-1} = \langle f(\xi, 1, \alpha)^{-1}, \xi \rangle = \langle 1, \xi \rangle \in \Gamma$$

where ξ is defined as in Theorem 2.2.

2.8 THEOREM. *An extension $H = \langle G, \Gamma \rangle$ of G by Γ with functions f and ϕ is a splitting extension if and only if for some function $\alpha \rightarrow x_\alpha$ of Γ into G we have*

$$(13) \quad f(\alpha, x_\beta, \beta) = x_\alpha^{-1} x_{\phi(\alpha, x_\beta, \beta)}.$$

Proof. Apply Theorem 2.6 with $f_1(\alpha, 1, \beta) = 1$.

COROLLARY. *If the conditions of the theorem are satisfied then $\Gamma^* = \{ \langle x_\alpha, \alpha \rangle \mid \alpha \in \Gamma \}$ is a subgroup of H such that $H = G\Gamma^*$ and $G \cap \Gamma^* = 1$.*

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Frantz Woodrow Ashley, Jr., <i>A cone of super-(L) functions</i>	1
Earl Robert Berkson, <i>Some metrics on the subspaces of a Banach space</i>	7
Felix Earl Browder and Walter Strauss, <i>Scattering for non-linear wave equations</i>	23
Edmond Darrell Cashwell and C. J. Everett, <i>Formal power series</i>	45
Frank Sydney Cater, <i>Continuous linear functionals on certain topological vector spaces</i>	65
John Douglas Dixon, <i>General group extensions</i>	73
Robert Pertsch Gilbert, <i>On harmonic functions of four variables with rational p_4-associates</i>	79
Irving Leonard Glicksberg, <i>On convex hulls of translates</i>	97
Simon Hellerstein, <i>On a class of meromorphic functions with deficient zeros and poles</i>	115
Donald William Kahn, <i>Secondary cohomology operations which extend the triple product</i>	125
G. K. Leaf, <i>A spectral theory for a class of linear operators</i>	141
R. Sherman Lehman, <i>Algebraic properties of the composition of solutions of partial differential equations</i>	157
Joseph Lehner, <i>On the generation of discontinuous groups</i>	169
S. P. Lloyd, <i>On certain projections in spaces of continuous functions</i>	171
Fumi-Yuki Maeda, <i>Generalized spectral operators on locally convex spaces</i>	177
Donald Vern Meyer, <i>E^3 modulo a 3-cell</i>	193
William H. Mills, <i>An application of linear programming to permutation groups</i>	197
Richard Scott Pierce, <i>Centers of purity in abelian groups</i>	215
Christian Pommerenke, <i>On meromorphic starlike functions</i>	221
Zalman Rubinstein, <i>Analytic methods in the study of zeros of polynomials</i>	237
B. N. Sahney, <i>On the Nörlund summability of Fourier series</i>	251
Tôru Saitô, <i>Regular elements in an ordered semigroup</i>	263
Lee Meyers Sonneborn, <i>Level sets on spheres</i>	297
Charles Andrew Swanson, <i>Asymptotic estimates for limit point problems</i>	305
Lucien Waelbroeck, <i>On the analytic spectrum of Arens</i>	317
Alvin (Murray) White, <i>Singularities of a harmonic function of three variables given by its series development</i>	321
Kôichi Yamamoto, <i>Decomposition fields of difference sets</i>	337
Chung-Tao Yang, <i>On the action of $SO(3)$ on a cohomology manifold</i>	353