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**ON CERTAIN PROJECTIONS IN SPACES OF CONTINUOUS
FUNCTIONS**

S. P. LLOYD

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1. Introduction. Let X be a compact Hausdorff space and let $C(X)$ be the Banach space of continuous real or complex valued functions on X , with supremum norm. We are concerned with the set \mathcal{P} of positive bounded constant decreasing projections in $C(X)$. That is, \mathcal{P} is the set of bounded linear operators $T: C(X) \rightarrow C(X)$ which have the properties $T^2 = T$, $Tf \geq 0$ if $f \geq 0$, $T1 \leq 1$. A great deal is known about the structure of such T when the range of T is a closed self-adjoint subalgebra of $C(X)$ containing constants [1] [4] [5]. In the present paper we develop a corresponding representation theory for members of \mathcal{P} . An application to Markov processes is given.

2. Representation theory. Let \mathcal{X} denote the σ -field of Borel subsets of X . We represent the conjugate space of bounded linear functionals on $C(X)$ as the space of regular real or complex Borel measures in X , with variation norm. In all that follows, the topology in $C^*(X)$ will be the $C(X)$ (weak*) topology.

THEOREM 1. *The members of \mathcal{P} correspond 1-1 to certain $C^*(X)$ valued functions on X , as follows. Suppose $t: X \rightarrow C^*(X)$ corresponds to $T \in \mathcal{P}$. Then t and T are related by (i), and t has properties (ii)-(iv):*

- (i) $Tf(x) = \int f(x')t_x(dx')$, $x \in X$, $f \in C(X)$
- (ii) $t: X \rightarrow C^*(X)$ is continuous (with the $C(X)$ topology in $C^*(X)$).
- (iii) $t_x \geq 0$, $t_x(X) \leq 1$, $x \in X$
- (iv) $t_x = \int t_x t_x(dx')$, $x \in X$.

Proof. Suppose $T \in \mathcal{P}$ is given. Standard representation theory for bounded linear transformations into $C(X)$ gives (i) and (ii) immediately [2, p. 490]. Property (iii) is a consequence of $T \geq 0$, $T1 \leq 1$. It is to be noted that the conditions $T \geq 0$, $T1 \leq 1$, $T \neq 0$ which characterize the nonzero members of \mathcal{P} are equivalent to the conditions $T \geq 0$, $\|T\| = 1$. The function t is simply the restriction of the adjoint $T^*: C^*(X) \rightarrow C^*(X)$ to domain X , regarding X as the set of unit point measures in $C^*(X)$. The adjoint itself has the representation

$$(2) \quad T^*\lambda = \int t_x \lambda(dx), \lambda \in C^*(X),$$

where the integration is in the weak* sense [3]. (That is, for given $\lambda \in C^*(X)$ the value of the integral in (2) is the element of $C^*(X)$ whose values for $f \in C(X)$ are

$$\int \lambda(dx) \int f(x') t_x(dx'), f \in C(X).$$

Condition (iv) is a consequence of $T^2 = T$; the integration again is in the weak* sense. Conversely, any t with properties (ii)–(iv) determines a $T \in \mathcal{G}$ according to (i), and the theorem is proved.

Let φ be the equivalence in X defined by $x_1 \varphi x_2$ if and only if $t_{x_1} = t_{x_2}$. On the quotient space $Y = X/\varphi$ define $\bar{t}: Y \rightarrow C^*(X)$ by $\bar{t}_y = t_x$ if $y = \pi x$, $x \in X$ where $\pi: X \rightarrow Y$ is the quotient mapping. General considerations show that \bar{t} is a homeomorphism of compact Hausdorff Y and the set $K = \{\bar{t}_y: y \in Y\}$ of various distinct values of t . The quotient mapping is closed, so that the decomposition $\{\pi^{-1}y: y \in Y\}$ of X into closed equivalence classes is upper semicontinuous.

Denote by K_1 the closed convex hull of $K \cup \{0\}$, where 0 is the zero measure. Since $K \cup \{0\}$ is compact, K_1 is compact, and is hence the closed convex hull of its extreme points. Denote by Y_0 the set of all $y \in Y$ such that $\bar{t}_y \neq 0$ is an extreme point of K_1 ; all extreme points of K_1 are to be found in $\{\bar{t}_y: y \in Y_0\} \cup \{0\}$ [2, p. 440].

THEOREM 2. *For each $y \in Y_0$ the measure \bar{t}_y lives on $\pi^{-1}y$; that is, $\bar{t}_y(E) = \bar{t}_y(E \cap \pi^{-1}y)$, $E \in X$, $y \in Y_0$. Moreover, $\bar{t}_y(X) = 1$, $y \in Y_0$.*

Proof. Property (1.iv) is

$$(3) \quad \bar{t}_y = \int \bar{t}_{\pi x} \bar{t}_y(dx), y \in Y,$$

in terms of \bar{t} . Fix $y \in Y_0$, and suppose there exists a closed set F disjoint from $\pi^{-1}y$ such that $\bar{t}_y(F) > 0$. Since \bar{t} is one-to-one and continuous, $\bar{t}\pi F = \{\bar{t}_{\pi x}: x \in F\}$ is a compact set which does not contain \bar{t}_y . The closed convex hull of $\bar{t}\pi F$ does not contain \bar{t}_y , either (otherwise $\bar{t}_y \in \bar{t}\pi F$, \bar{t}_y being extreme [2, p. 440]). Thus there exists $f \in C(X)$ which separates \bar{t}_y and $\bar{t}\pi F$ strictly. Expressing (3) as

$$\bar{t}_y = \bar{t}_y(F) \int_F \bar{t}_{\pi x} \frac{\bar{t}_y(dx)}{\bar{t}_y(F)} + \int_{X-F} \bar{t}_{\pi x} \bar{t}_y(dx) + [1 - \bar{t}_y(X)]0,$$

we see that \bar{t}_y is expressed as a proper convex combination of elements of K_1 distinct from \bar{t}_y . This contradicts the assumption that \bar{t}_y is an extreme point of K_1 . The regularity of each \bar{t}_y shows that \bar{t}_y lives

on $\pi^{-1}y$ when $y \in Y_0$. The same sort of argument shows that if $\bar{t}_y(X) \neq 0$, then \bar{t}_y is not an extreme point of K_1 unless $\bar{t}_y(X) = 1$.

THEOREM 3. Y_0 is closed.

Proof. Define $u: Y \rightarrow C^*(Y)$ by $u_y(E) = \bar{t}_y(\pi^{-1}E)$, $E \in Y$, $y \in Y$. The continuity of \bar{t} implies that u is continuous with the $C(Y)$ topology in $C^*(Y)$. From Theorem 2, u_y is for each $y \in Y_0$ the unit point measure at y . Thus for each $f \in C(Y)$ we have

$$(4) \quad f(y) = \int f(y')u_y(dy'), \quad y \in Y_0.$$

Since for each $f \in C(Y)$ the members of (4) are continuous in y , the equality (4) persists for $y \in \bar{Y}_0$. This implies that u_y is the unit point measure at y for each $y \in \bar{Y}_0$. It follows that \bar{t}_y lives on $\pi^{-1}y$ for each $y \in \bar{Y}_0$. It should be clear that each such \bar{t}_y , $y \in \bar{Y}_0$, is necessarily an extreme point of K_1 , and the theorem follows.

THEOREM 4. For each $y \in Y$ the measure \bar{t}_y lives on $\pi^{-1}Y_0$; that is, $\bar{t}_y(E) = \bar{t}_y(E \cap \pi^{-1}Y_0)$, $E \in X$, $y \in Y$.

Proof. Since $\{\bar{t}_y, y \in Y\}$ is in the closed convex hull of compact $\{\bar{t}_y, y \in Y_0\} \cup \{0\}$, for each $y \in Y$ there exists a Borel measure $\nu_y \geq 0$ on compact Y_0 such that

$$(5) \quad \bar{t}_y = \int_{Y_0} \bar{t}_{y'}\nu_y(dy')$$

in the weak* sense. Let F be an arbitrary closed subset of $X - \pi^{-1}Y_0$, and let $f \in C(X)$ satisfy $f(F) = 1$, $f(\pi^{-1}Y_0) = 0$, $0 \leq f \leq 1$. From (5) and Theorem 2 one has $\int f(x)\bar{t}_y(dx) = 0$, $y \in Y$, and hence $\bar{t}_y(F) = 0$, $y \in Y$. Since each \bar{t}_y is regular, the theorem follows.

3. Invariant measures and functions. We now characterize the ranges of T^* and T . From (2), any invariant measure $T^*\lambda$ is contained in the weak* closed subspace spanned by $\{t_x, x \in X\}$. From (1.iv), each t_x is invariant, $x \in X$. Thus range (T^*) is the weak* closed subspace spanned by $\{t_x, x \in X\}$. The extreme points $\{\bar{t}_y, y \in Y_0\}$ constitute a minimal spanning set, clearly.

From (1.i), any invariant function Tf is constant on equivalence classes, and so determines an element of $C(Y)$. Restriction of domain to Y_0 gives an element of $C(Y_0)$. Conversely, let f_0 be an arbitrary element of $C(Y_0)$. Define function f by

$$(6) \quad f(x) = \int_{\pi^{-1}Y_0} f_0(\pi x')t_x(dx'), \quad x \in X.$$

It follows from Theorem 3 and the Tietze extension theorem that $f \in C(X)$ and hence that $Tf = f$. From Theorem 2, the contraction procedure described above applied to f gives f_0 back again. It should then be clear that (6) establishes an isometric order isomorphism of $C(Y_0)$ and range (T) . The isomorphism is algebraic if and only if $Y_0 = Y$ [4].

4. Application to Markov chains. Let (X_1, \mathcal{F}) be a measurable space, and let $p(x, E)$, $x \in X_1$, $E \in \mathcal{F}$, be a transition subprobability. That is, $p(x, \cdot)$ is a measure on \mathcal{F} for each $x \in X_1$ and $0 \leq p(\cdot, E) \leq 1$ is a measurable function for each $E \in \mathcal{F}$. Denote by $B(X_1, \mathcal{F})$ the Banach space of all bounded real or complex measurable functions on X_1 , with supremum norm. Then $P: B(X_1, \mathcal{F}) \rightarrow B(X_1, \mathcal{F})$ defined by

$$Pf(x) = \int f(x')p(x, dx'), \quad x \in X_1, \quad f \in B(X_1, \mathcal{F}),$$

has the properties $P \geq 0$, $\|P\| \leq 1$. Suppose there is an operator T (necessarily unique) in the closed convex hull of $\{P^n, n = 1, 2, \dots\}$ in the weak operator topology with the properties $TP = PT = T$. Then T has the properties $T \geq 0$, $\|T\| \leq 1$, and is the projection onto the subspace of invariant functions of P .

We assume without essential loss of generality that $B(X_1, \mathcal{F})$ separates the points of X_1 . Then there is a totally disconnected compact Hausdorff space X containing X_1 as a dense subset such that each element of $B(X_1, \mathcal{F})$ extends uniquely to an element of $C(X)$ [2, p. 276]. Operator P becomes an operator $P: C(X) \rightarrow C(X)$ with the properties $P \geq 0$, $\|P\| \leq 1$. Such an operator necessarily has the form

$$Pf(x) = \int f(x')p_x(dx'), \quad x \in X, \quad f \in C(X),$$

where $p: X \rightarrow C^*(X)$ is continuous with the $C(X)$ topology in $C^*(X)$ and has the properties $p_x \geq 0$, $p_x(X) \leq 1$, $x \in X$. Clearly, p is the extension of the given transition subprobability to all of X .

The operator T becomes a projection in $C(X)$ to which our results apply. Each set $\pi^{-1}y$, $y \in Y_0$ is an ergodic set and $X - \pi^{-1}Y_0$ is the dissipative set, according to

THEOREM 5. *If $y \in Y_0$ then for almost all $x \in \pi^{-1}y$ with respect to \bar{t}_y the measure p_x lives on $\pi^{-1}y$.*

Proof. From $TP = T$ we obtain

$$\int \bar{t}_y(dx) \int p_x(dx')f(dx') = \int \bar{t}_y(x')f(x'), \quad y \in Y, \quad f \in C(X).$$

Fix $y \in Y_0$ and let F be any closed set disjoint from $\pi^{-1}y$. Let $f \in C(X)$ be such that $f(F) = 1$, $f(\pi^{-1}y) = 0$, $0 \leq f \leq 1$. The right-hand side above vanishes, from Theorem 2, which requires $p_x(F) = 0$ for almost all x with respect to \bar{t}_y . Since p_x is regular, the theorem follows.

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Frantz Woodrow Ashley, Jr., <i>A cone of super-(L) functions</i>	1
Earl Robert Berkson, <i>Some metrics on the subspaces of a Banach space</i>	7
Felix Earl Browder and Walter Strauss, <i>Scattering for non-linear wave equations</i>	23
Edmond Darrell Cashwell and C. J. Everett, <i>Formal power series</i>	45
Frank Sydney Cater, <i>Continuous linear functionals on certain topological vector spaces</i>	65
John Douglas Dixon, <i>General group extensions</i>	73
Robert Pertsch Gilbert, <i>On harmonic functions of four variables with rational p_4-associates</i>	79
Irving Leonard Glicksberg, <i>On convex hulls of translates</i>	97
Simon Hellerstein, <i>On a class of meromorphic functions with deficient zeros and poles</i>	115
Donald William Kahn, <i>Secondary cohomology operations which extend the triple product</i>	125
G. K. Leaf, <i>A spectral theory for a class of linear operators</i>	141
R. Sherman Lehman, <i>Algebraic properties of the composition of solutions of partial differential equations</i>	157
Joseph Lehner, <i>On the generation of discontinuous groups</i>	169
S. P. Lloyd, <i>On certain projections in spaces of continuous functions</i>	171
Fumi-Yuki Maeda, <i>Generalized spectral operators on locally convex spaces</i>	177
Donald Vern Meyer, <i>E^3 modulo a 3-cell</i>	193
William H. Mills, <i>An application of linear programming to permutation groups</i>	197
Richard Scott Pierce, <i>Centers of purity in abelian groups</i>	215
Christian Pommerenke, <i>On meromorphic starlike functions</i>	221
Zalman Rubinstein, <i>Analytic methods in the study of zeros of polynomials</i>	237
B. N. Sahney, <i>On the Nörlund summability of Fourier series</i>	251
Tôru Saitô, <i>Regular elements in an ordered semigroup</i>	263
Lee Meyers Sonneborn, <i>Level sets on spheres</i>	297
Charles Andrew Swanson, <i>Asymptotic estimates for limit point problems</i>	305
Lucien Waelbroeck, <i>On the analytic spectrum of Arens</i>	317
Alvin (Murray) White, <i>Singularities of a harmonic function of three variables given by its series development</i>	321
Kôichi Yamamoto, <i>Decomposition fields of difference sets</i>	337
Chung-Tao Yang, <i>On the action of $SO(3)$ on a cohomology manifold</i>	353