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**ON MEROMORPHIC STARLIKE FUNCTIONS**

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**1. Introduction.** Let  $\mathfrak{S}(\alpha)$  ( $0 \leq \alpha < 1$ ) denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^{-n}$$

that are analytic in  $1 < |z| < \infty$  and satisfy

$$\operatorname{Re} z \frac{f'(z)}{f(z)} \geq \alpha.$$

The class  $\mathfrak{S}(0)$  is formed by the meromorphic starlike functions  $f(z) = z + \dots$ , that is by the functions that map  $\{|z| > 1\}$  onto a region whose compact complement is starlike with respect to the origin.

Apart from the case  $\alpha = 0$  the most interesting case is  $\alpha = 1/2$ . We shall see (Corollary 1) that every  $f \in \mathfrak{S}(\frac{1}{2})$  can be approximated by the roots of polynomials with zeros in the unit disk, that is by functions of the form

$$(1.1) \quad f(z) = \left( \prod_{\nu=1}^n (z - z_{\nu}) \right)^{1/n} \quad (|z_{\nu}| \leq 1).$$

These functions belong to  $\mathfrak{S}(\frac{1}{2})$ .

The main results will be: As  $r \rightarrow 1$ ,

$$(1.2) \quad \max_{f \in \mathfrak{S}(\alpha)} \max_{|z| \leq r} |f'(z)| \sim \frac{2^{2(1-\alpha)} e^{-1}}{(1 - r^{-1}) \log 1/(1 - r^{-1})}.$$

If  $f \in \mathfrak{S}(\alpha)$  then

$$(1.3) \quad |f'(z)| \geq \alpha |z^{-1} f(z)| + (1 - \alpha) |z^{-1} f(z)|^{-\alpha/(1-\alpha)} (1 - |z|^{-2}) \geq (1 - |z|^{-2})^{1-\alpha}.$$

For each fixed  $f \in \mathfrak{S}(\alpha)$ ,

$$|f'(z)| \geq \kappa (1 - |z|^{-1})^{\max(1-2\alpha, \varepsilon)}$$

for every  $\varepsilon > 0$  and some constant  $\kappa = \kappa(f, \varepsilon) > 0$ .

The coefficients of  $f(z)$  satisfy

$$(1.4) \quad |a_n| \leq \frac{2(1-\alpha)}{n+1},$$

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$$(1.5) \quad \inf_{n \rightarrow \infty} \sup |a_n| < 2(1 - \alpha) .$$

All these inequalities are best possible.

Inequality (1.3) is well-known for the case  $\alpha = 0$  whereas (1.2) seems to be new even for  $\alpha = 0$ . Inequality (1.4) has been proved by Clunie [1] for  $\alpha = 0$ . Some of the problems can also be studied by the variation method developed by Royster [9].

We shall also prove inequalities analogous to (1.4) and (1.5) for functions  $g(\zeta)$  that are analytic and bounded in  $|\zeta| < 1$  and satisfy  $\operatorname{Re} \zeta g'(\zeta)/g(\zeta) \geq \alpha$ . (For  $\alpha = 0$ , see [2] and [7]).

## 2. A representation formula.

**THEOREM 1.** *Let  $f \in \mathfrak{S}(\alpha)$  ( $0 \leq \alpha < 1$ ). Then*

$$(2.1) \quad f(z) = z \exp \left( 2(1 - \alpha) \int_0^{2\pi} \log(1 - e^{it} z^{-1}) d\gamma(t) \right)$$

where  $\gamma(t)$  increases and  $\gamma(t + 2\pi) - \gamma(t) = 1$ . Also  $\arg f(re^{it}) \rightarrow \pi t + 2\pi(1 - \alpha)\gamma(t)$  as  $r \rightarrow 1$ , and

$$(2.2) \quad z \frac{f'(z)}{f(z)} = \alpha + (1 - \alpha) \int_0^{2\pi} \frac{1 + e^{it} z^{-1}}{1 - e^{it} z^{-1}} d\gamma(t) .$$

Conversely, every function of the form (2.1) belongs to  $\mathfrak{S}(\alpha)$ .

*Proof.* We shall reduce (2.1) in the general case to the case  $\alpha = 0$  where (2.1) is a known formula. If  $f \in \mathfrak{S}(\alpha)$  let

$$g(z) = z^{-\alpha/(1-\alpha)} f(z)^{1/(1-\alpha)} = z + \dots .$$

Then  $\operatorname{Re} z g'(z)/g(z) = -\alpha/(1 - \alpha) + 1/(1 - \alpha) \cdot \operatorname{Re} z f'(z)/f(z) \geq 0$ . Hence  $g(z)$  is starlike and (compare [7, Lemma 1])

$$g(z) = z \exp \left( \frac{1}{\pi} \int_0^{2\pi} \log(1 - e^{it} z^{-1}) dV(t) \right) ,$$

$$z g'(z)/g(z) = \frac{1}{2\pi} \int_0^{2\pi} (1 + e^{it} z^{-1}) / (1 - e^{it} z^{-1}) dV(t)$$

where  $V(t) = \lim_{r \rightarrow 1} \arg g(re^{it})$  is monotone increasing and  $V(t + 2\pi) - V(t) = 2\pi$ . Putting  $\gamma(t) = V(t)/(2\pi)$  we see that  $f(z) = z^\alpha g(z)^{1-\alpha}$  satisfies (2.1) and (2.2). Direct computation shows that every function of the form (2.1) belongs to  $\mathfrak{S}(\alpha)$ .

**COROLLARY 1.** *For given  $f \in \mathfrak{S}(\alpha)$ , there is a sequence of functions  $f_n \in \mathfrak{S}(\alpha)$  of the form*

$$f_n(z) = z \prod_1^n (1 - z_\nu z^{-1})^{2(1-\alpha)/n} \quad (|z_\nu| \leq 1)$$

that converges to  $f(z)$  locally uniformly in  $|z| > 1$ . All functions of this form belong to  $\mathfrak{S}(\alpha)$ .

This corollary follows immediately from (2.1) by approximating  $\gamma(t)$  by step-functions with  $n$  steps of height  $1/n$ . The last assertion of the corollary is established by computation. For  $\alpha = \frac{1}{2}$ , the corollary shows that  $f$  can be approximated by functions of the form (1.1).

### 3. Estimates of the function-values.

**THEOREM 2.** Let  $f(z) = z + a_0 + a_1 z^{-1} + \dots \in \mathfrak{S}(\alpha)$  and  $|z| = r > 1$ . Then

$$(3.1) \quad \left| \frac{f(z)}{z} \right| \leq \left( 1 + \frac{|a_0|}{1-\alpha} r^{-1} + r^{-2} \right)^{1-\alpha} \leq (1 + r^{-1})^{2(1-\alpha)},$$

$$(3.2) \quad \left| \frac{f(z)}{z} \right| \geq (1 - r^{-1})^{1-\alpha + |a_0|^{1/2}} (1 + r^{-1})^{1-\alpha - |a_0|^{1/2}} \geq (1 - r^{-1})^{2(1-\alpha)}.$$

Equality can be attained in all inequalities.

For  $\alpha = 0$  we find  $(1 + r^{-1})^2 \geq |z^{-1}f(z)| \geq (1 - r^{-1})^2$ , and these are well-known inequalities which hold for all functions  $f(z) = z + \dots$  univalent and  $\neq 0$  in  $|z| > 1$ .

*Proof of (3.1).* Using (2.1) and the fact that the geometric mean is not greater than the arithmetic mean we see that

$$(3.3) \quad \begin{aligned} |z^{-1}f(z)|^{1/(1-\alpha)} &= \exp \left( \int_0^{2\pi} \log |1 - e^{it} z^{-1}|^2 d\gamma(t) \right) \\ &\leq \int_0^{2\pi} |1 - e^{it} z^{-1}|^2 d\gamma(t) \\ &= \int_0^{2\pi} (1 + r^{-2}) d\gamma(t) - 2Re \left[ z^{-1} \int_0^{2\pi} e^{-it} d\gamma(t) \right]. \end{aligned}$$

Also by (2.1)

$$a_0 = -2(1-\alpha) \int_0^{2\pi} e^{it} d\gamma(t).$$

Hence  $|a_0| \leq 2(1-\alpha)$ , and by (3.3)

$$|z^{-1}f(z)|^{1/(1-\alpha)} \leq 1 + r^{-2} + |a_0| r^{-1}/(1-\alpha) \leq (1 + r^{-1})^2$$

from which inequalities (3.1) follow. We have equality for the function

$$f(z) = z \left( 1 + \frac{a_0}{1-\alpha} z^{-1} + z^{-2} \right)^{1-\alpha} = z + a_0 + \dots \in \mathfrak{S}(\alpha)$$

where  $0 \leq a_0 \leq 2(1-\alpha)$ , and  $z = r > 1$ .

*Proof of (3.2).* Since  $zf'(z)/f(z) = 1 - a_0z^{-1} + \dots$  and  $\operatorname{Re} zf'(z)/f(z) \geq \alpha$  the function

$$(3.4) \quad \phi(z) = z \frac{zf'(z)/f(z) - 1}{zf'(z)/f(z) + (1 - 2\alpha)} = -\frac{a_0}{2(1 - \alpha)} + \dots$$

is analytic in  $|z| > 1$  and satisfies  $|\phi(z)| < 1$ . Then

$$(3.5) \quad \frac{\partial}{\partial r} \log |z^{-1}f(z)| = -\frac{1}{r} + \operatorname{Re} \left[ \frac{z}{r} \frac{f'(z)}{f(z)} \right] \leq 2(1 - \alpha) \left| \frac{z^{-1}\phi(z)}{r(1 - z^{-1}\phi(z))} \right|.$$

If  $b = |\phi(\infty)| = |a_0|/2(1 - \alpha)$  then [3, p. 287]

$$|\phi(z)| \leq (br + 1)/(b + r),$$

hence by (3.5)

$$\frac{\partial}{\partial r} \log |z^{-1}f(z)| \leq 2(1 - \alpha) \frac{br + 1}{r(r^2 - 1)}.$$

Integration over  $[r, +\infty]$  gives

$$\log |z^{-1}f(z)| \geq (1 - \alpha)(1 + b) \log(1 - r^{-1}) + (1 - \alpha)(1 - b) \log(1 + r^{-1}),$$

and inequalities (3.2) follow because  $b = |a_0|/2(1 - \alpha) = |\phi(\infty)| \leq 1$ .

Equality is attained for  $z = r > 1$  by the function

$$f(z) = z(1 - z^{-1})^{1-\alpha+a_0/2}(1 + z^{-1})^{1-\alpha-a_0/2} = z - a_0 + \dots$$

with  $0 \leq a_0 \leq 2(1 - \alpha)$ .

We could have proved (3.1) by the same method as (3.2). But the proof directly from the representation formula seemed to be more interesting.

**4. Upper estimate of the derivative.** The best possible upper estimate of the derivative for the functions  $f(z) = z + \dots$  that are meromorphic and univalent in  $|z| > 1$  is  $|f'(re^{i\theta})| \leq 1/(1 - r^{-2})$  (see for instance [3, p. 120]). We shall show that this inequality can be improved for starlike functions and small  $r > 1$ .

**THEOREM 3.** *Let  $0 \leq \alpha < 1$ . Then as  $r \rightarrow 1$*

$$\max_{f \in \mathcal{S}(\alpha)} \max_{|z| \leq r} |f'(z)| \sim \frac{2^{2(1-\alpha)} e^{-1}}{(1 - r^{-1}) \log 1/(1 - r^{-1})}.$$

The proof will show that the function  $f(z)$  for which  $|f'(z_0)|$  becomes maximal for a given  $z_0$  has the form

$$(4.1) \quad f(z) = z(1 - z_1 z^{-1})^{2(1-\alpha)\gamma_1} (1 - z_2 z^{-1})^{2(1-\alpha)\gamma_2}$$

with  $0 \leq \gamma_1 \leq 1$ ,  $\gamma_1 + \gamma_2 = 1$  and  $|z_1| = |z_2| = 1$ .

*Proof.* 1. Let  $n \geq 2$  and different  $z_\nu$  ( $\nu = 1, \dots, n$ ) with  $|z_\nu| = 1$  be given. We consider the functions

$$(4.2) \quad f(z) = z \prod_{\nu=1}^n (1 - z_\nu z^{-1})^{2(1-\alpha)\gamma_\nu}$$

with  $0 \leq \gamma_\nu \leq 1$ ,  $\gamma_1 + \dots + \gamma_n = 1$ . For fixed  $z_0$  ( $|z_0| > 1$ ) let  $f(z)$  be the function of this form for which  $|f'(z_0)|$  becomes maximal. We shall show that only two of the  $\gamma_\nu$  can be  $\neq 0$  for this maximal function. We shall use an elementary variation method (compare [5]).

2. Suppose this were false, and  $\gamma_\nu > 0$  ( $\nu = 1, 2, 3$ ). If  $\beta_1 + \beta_2 + \beta_3 = 0$ ,  $\beta_4 = \dots = \beta_n = 0$  then  $\gamma_\nu^* = \gamma_\nu \pm \delta\beta_\nu \geq 0$  for sufficiently small  $\delta > 0$ , and  $\sum \gamma_\nu^* = 1$ . Let  $f_*(z)$  be the function of the form (4.2) where  $\gamma_\nu$  has been replaced by  $\gamma_\nu^*$ . Let  $\xi = z_0^{-1}$ . Then

$$\begin{aligned} |f'_*(z_0)|^2 &= |z_0 f'_*(z_0)/f_*(z_0)|^2 \cdot |z_0^{-1} f_*(z_0)|^2 \\ &= \left| \sum_{\nu=1}^n (\gamma_\nu \pm \delta\beta_\nu) \frac{1 + (1 - 2\alpha)z_\nu \xi}{1 - z_\nu \xi} \right|^2 \prod_{\nu=1}^n |1 - z_\nu \xi|^{4(1-\alpha)(\gamma_\nu \pm \delta\beta_\nu)}. \end{aligned}$$

For abbreviation let

$$c = \sum_{\nu=1}^n \gamma_\nu \frac{1 + (1 - 2\alpha)z_\nu \xi}{1 - z_\nu \xi}, \quad b = \sum_{\nu=1}^3 \beta_\nu \frac{1 + (1 - 2\alpha)z_\nu \xi}{1 - z_\nu \xi}$$

and

$$l = \sum_{\nu=1}^3 \beta_\nu \log |1 - z_\nu \xi|.$$

Since  $\beta_4 = \dots = \beta_n = 0$  it follows that

$$\begin{aligned} |f'_*(z_0)|^2 &= (|c|^2 \pm 2\delta \operatorname{Re}[b\bar{c}] + \delta^2 |b|^2) \cdot |z_0^{-1} f(z_0)|^2 \\ &\quad \cdot (1 \pm 4\delta(1 - \alpha)l + 8\delta^2(1 - \alpha)^2 l^2 + 0(\delta^3)) \\ &= |z_0^{-1} f(z_0)|^2 [|c|^2 \pm 2\delta \operatorname{Re}[b\bar{c}] + 2(1 - \alpha)|c|^2 l \\ &\quad + \delta^2(|b|^2 + 8(1 - \alpha)l \operatorname{Re}[b\bar{c}] + 8(1 - \alpha)^2 |c|^2 l^2) + 0(\delta^3)]. \end{aligned}$$

Since  $|f'_*(z_0)|^2 \leq |f'(z_0)|^2 = |z_0^{-1} f(z_0)|^2 |c|^2$  by the maximal choice of  $f$ , it follows that

$$\operatorname{Re}[b\bar{c}] + 2(1 - \alpha)|c|^2 l = 0$$

and

$$|b|^2 + 8(1 - \alpha)l \operatorname{Re}[b\bar{c}] + 8(1 - \alpha)^2 |c|^2 l^2 \leq 0.$$

Eliminating  $\operatorname{Re}[b\bar{c}]$  we obtain

$$|b|^2 \leq 8(1 - \alpha) |c|^2 l^2.$$

In particular,  $l = 0$  implies  $b = 0$ .

The two homogeneous linear equations

$$\begin{aligned} l &= \sum_1^3 \beta_\nu \log |1 - z_\nu \zeta| = 0 \\ \beta_1 + \beta_2 + \beta_3 &= 0 \end{aligned}$$

have a nontrivial solution  $\beta_1, \beta_2, \beta_3$ . With these values it follows, as we have proved, that

$$b = \sum_1^3 \beta_\nu \frac{1 + (1 - 2\alpha)z_\nu \zeta}{1 - z_\nu \zeta} = 0.$$

Hence the three different points  $w_\nu = (1 + (1 - 2\alpha)z_\nu \zeta)/(1 - z_\nu \zeta)$  ( $\nu = 1, 2, 3$ ) lie on a straight line, which is impossible because  $|z_\nu| = 1$  and  $|\zeta| < 1$ . We have therefore proved that in the maximal function all exponents  $\gamma_\nu$  are 0 except possibly two.

3. We can write this maximal function in the form (4.1). Thus, by (2.1) the function  $f \in \mathfrak{S}(\alpha)$  for which  $|f'(z_0)|$  is maximal also has the form (4.1).

We may therefore assume that  $f(z)$  has the form (4.1) and also that  $\gamma_1 \leq \gamma_2$ , hence  $\gamma_1 \leq \frac{1}{2} \leq \gamma_2$ . Let  $\rho = |z|^{-1}$ . Then

$$\begin{aligned} |f'(z)| &\leq 2^{2(1-\alpha)} + 2(1 - \alpha)\gamma_1(1 - \rho)^{2(1-\alpha)\gamma_1-1} 2^{2(1-\alpha)\gamma_2} \\ &\quad + 2(1 - \alpha)\gamma_2 \cdot 2^{2(1-\alpha)\gamma_1} \max [2^{2(1-\alpha)\gamma_2-1}, (1 - \rho)^{2(1-\alpha)\gamma_2-1}]. \end{aligned}$$

Since

$$(4.3) \quad \max_{\gamma \geq 0} \gamma(1 - \rho)^{2(1-\alpha)\gamma} = 1/[2(1 - \alpha)e \log 1/(1 - \rho)]$$

it follows that

$$(4.4) \quad |f'(z)| \leq 4 + 2^{2(1-\alpha)} e^{-1} [(1 - \rho) \log 1/(1 - \rho)]^{-1} + 16(1 - \rho)^{-\alpha}.$$

4. We finally consider the special function

$$f(z) = z(1 - z^{-1})^{2(1-\alpha)\gamma}(1 + z^{-1})^{2(1-\alpha)(1-\gamma)}$$

with  $\gamma = 1/[2(1 - \alpha) \log 1/(1 - \rho)]$  for  $\rho < 1$  near to 1. Then  $f \in \mathfrak{S}(\alpha)$ , and by (4.3)

$$f'(\rho^{-1}) \sim 2^{2(1-\alpha)} e^{-1} [(1 - \rho) \log 1/(1 - \rho)]^{-1} \quad (\rho \rightarrow 1).$$

Together with (4.4) this proves Theorem 3.

**5. Lower estimates of the derivative.** The best possible lower estimate of the derivative for functions  $f(z) = z + \dots$  meromorphic and univalent in  $|z| > 1$  is  $|f'(z)| \geq 1 - |z|^{-2}$ , with equality for  $f(z) = z + z^{-1}$ . This function belongs to  $\mathfrak{S}(0)$ . For  $0 < \alpha < 1$  we can prove more.

**THEOREM 4.** *If  $f \in \mathfrak{S}(\alpha)$  ( $0 \leq \alpha < 1$ ) then*

$$(5.1) \quad |f'(z)| \geq \alpha \left| \frac{f(z)}{z} \right| + (1 - \alpha) \left| \frac{z}{f(z)} \right|^{\alpha/(1-\alpha)} \left( 1 - \frac{1}{|z|^2} \right) \geq \left( 1 - \frac{1}{|z|^2} \right)^{1-\alpha}.$$

*Equality can be attained in all inequalities.*

*Proof.* By (2.1) we have

$$\left| \frac{z}{f(z)} \right|^{1/(1-\alpha)} = \exp \left( \int_0^{2\pi} \log \frac{1}{|1 - e^{it}z^{-1}|^2} d\gamma(t) \right)$$

with  $d\gamma(t) \geq 0$  and  $\int_0^{2\pi} d\gamma(t) = 1$ . Since the geometric mean is not greater than the arithmetic mean it follows that

$$\left| \frac{z}{f(z)} \right|^{1/(1-\alpha)} \leq \int_0^{2\pi} \frac{1}{|1 - e^{it}z^{-1}|^2} d\gamma(t).$$

Hence by (2.2)

$$\begin{aligned} \left| \frac{z}{f(z)} \right|^{1/(1-\alpha)} (1 - |z|^{-2}) &\leq \int_0^{2\pi} \frac{1 - |z|^{-2}}{|1 - e^{it}z^{-1}|^2} d\gamma(t) \\ &= \frac{1}{1 - \alpha} \left( \operatorname{Re} z \frac{f'(z)}{f(z)} - \alpha \right) \end{aligned}$$

and therefore

$$\left| z \frac{f'(z)}{f(z)} \right| \geq \operatorname{Re} z \frac{f'(z)}{f(z)} \geq \alpha + (1 - \alpha) \left| \frac{z}{f(z)} \right|^{1/(1-\alpha)} (1 - |z|^{-2})$$

from which (5.1) follows. Equality is attained by

$$f(z) = z(1 - 2r^{-1}z^{-1} + z^{-2})^{1-\alpha} \in \mathfrak{S}(\alpha)$$

for  $z = r > 1$ .

Though (5.1) is best possible for the whole class  $\mathfrak{S}(\alpha)$  more is true for each fixed function in  $\mathfrak{S}(\alpha)$  if  $\alpha > 0$ .

**THEOREM 5.** *Let  $f \in \mathfrak{S}(\alpha)$ . (i) If  $0 \leq \alpha < \frac{1}{2}$  then there is a  $\kappa = \kappa(f) > 0$  such that*

$$|f'(re^{i\theta})| \geq \kappa(1 - r^{-1})^{1-2\alpha}$$



The exponent cannot be decreased. (ii) If  $\frac{1}{2} \leq \alpha < 1$  then

$$|f'(re^{i\theta})| \geq \kappa(\varepsilon)(1 - r^{-1})^\varepsilon$$

for every  $\varepsilon > 0$  where  $\kappa(\varepsilon) = \kappa(f, \varepsilon) > 0$ .

*Proof.* We may take  $\alpha > 0$ . Suppose Theorem 5 were false. Then there would be a sequence  $z_k = r_k e^{i\theta_k}$  such that

$$(5.2) \quad f'(z_k) = o((1 - r_k^{-1})^{1-2\alpha})$$

and also

$$(5.3) \quad f'(z_k) = o((1 - r_k^{-1})^{\varepsilon_0})$$

for some  $\varepsilon_0 > 0$ . We may assume that  $\theta_k \rightarrow 0$ . We have to distinguish two cases:

*Case 1.* The point  $t = 0$  is a discontinuity of the function  $\gamma(t)$  of Theorem 1. Then let  $\lambda_0$  be the height of the jump of  $\gamma(t)$  at 0, and let  $\lambda_j$  ( $j = 1, \dots, m$ ) be the jumps of height  $\geq \lambda_0$ , occurring at  $t_j \neq 0$ . Let  $\lambda^*$  be the highest jump  $< \lambda_0$ , and let  $\delta = \lambda_0 - \lambda^*$ . Then  $\delta > 0$ . If  $\sigma(t)$  is the function that is constant except for jumps of height 1 at 0,  $\pm 2\pi, \dots$  let

$$(5.4) \quad \gamma^*(t) = \gamma(t) - \lambda_0 \sigma(t) - \sum_{j=1}^m \lambda_j \sigma(t - t_j).$$

This function increases and has highest jump  $\lambda^*$ . We see from Theorem 1 that

$$(5.5) \quad z_k \frac{f'(z_k)}{f(z_k)} = \alpha + (1 - \alpha)\lambda_0 \frac{1 + z_k^{-1}}{1 - z_k^{-1}} + (1 - \alpha) \sum_{\mu=1}^m \lambda_\mu \frac{1 + e^{it_\mu} z_k^{-1}}{1 - e^{it_\mu} z_k^{-1}} + F^*(z_k)$$

where

$$F^*(z) = (1 - \alpha) \int_0^{2\pi} \frac{1 + e^{it} z^{-1}}{1 - e^{it} z^{-1}} d\gamma^*(t)$$

is a function of positive real part. Hence [4, Theorem 2]

$$(5.6) \quad |F^*(z_k)| \leq (1 - \alpha)(\lambda^* + \frac{1}{3}\delta)/(1 - r_k^{-1})$$

for large  $k$ . The third term on the right side of (5.5) is bounded as  $k \rightarrow \infty$  because  $\theta_k \rightarrow 0 \neq t_j$  ( $j = 1, \dots, m$ ). Hence by (5.5) and (5.6)

$$\begin{aligned} |z_k f'(z_k)/f(z_k)| &\geq (1 - \alpha)(\lambda_0 - \frac{1}{3}\delta - \lambda^* - \frac{1}{3}\delta)/(1 - r_k^{-1}) \\ &= \frac{1}{3}\delta/(1 - r_k^{-1}) \end{aligned}$$

for large  $k$ . On the other hand, Theorem 2 implies  $|z_k^{-1}f(z_k)| \geq (1 - r_k^{-1})^{2-2\alpha}$ . Therefore  $|f'(z_k)| \geq \frac{1}{3}\delta(1 - r_k^{-1})^{1-2\alpha}$ , in contradiction to (5.2).

*Case 2.* The function  $\gamma(t)$  is continuous at  $t = 0$ . Let now  $\lambda_j$  ( $j = 1, \dots, m$ ) be the jumps of height  $\geq \varepsilon_0/4(1 - \alpha)$  occuring at  $t_j \neq 0$ . Let  $\gamma^*(t)$  be defined as in (5.4) (with  $\lambda_0 = 0$ ). Then we see from Theorem 1 that

$$(5.7) \quad f(z) = \prod_{j=1}^m (1 - e^{it_j} z^{-1})^{2(1-\alpha)\lambda_j} f^*(z)$$

where

$$f^*(z) = z \exp \left( 2(1 - \alpha) \int_0^{2\pi} \log(1 - e^{it} z^{-1}) d\gamma^*(t) \right)$$

is a starlike function. Hence  $1/f^*(\zeta^{-1})$  is starlike in  $|\zeta| < 1$ . Since all jumps of  $\gamma^*(t)$  are  $< \varepsilon_0/4(1 - \alpha)$  it follows [7, Theorem 1] that

$$1/f^*(\zeta^{-1}) = o((1 - |\zeta|)^{-\varepsilon_0}).$$

Since  $\theta_k \rightarrow 0 \neq t_j$  as  $k \rightarrow \infty$  we therefore obtain from (5.7) that

$$|z_k^{-1}f(z_k)| \geq (1 - r_k^{-1})^{\varepsilon_0}$$

for large  $k$ . Because  $\operatorname{Re} z f'(z)/f(z) \geq \alpha > 0$  it follows that

$$|f'(z_k)| \geq \alpha |z_k^{-1}f(z_k)| \geq \alpha(1 - r_k^{-1})^{\varepsilon_0}$$

in contradiction to (5.3).

Finally, that function  $f(z) = z(1 - z^{-1})^{2-2\alpha} \in \mathfrak{S}(\alpha)$  shows that  $1 - 2\alpha$  cannot be made smaller.

**6. Estimates of the coefficients** If  $f(z) = z + a_0 + a_1 z^{-1} + \dots$  is a starlike function then, as Clunie [1] has proved,  $|a_n| \leq 2/(n + 1)$ . To generalise this inequality to the class  $\mathfrak{S}(\alpha)$  we first prove the following lemma, using Clunie's method.

**LEMMA 1.** *If  $f \in \mathfrak{S}(\alpha)$  ( $0 \leq \alpha < 1$ ) and  $n = 0, 1, \dots$  then*

$$(n + 1)^2 |a_n|^2 \leq 4(1 - \alpha)^2 - 4(1 - \alpha) \sum_{\nu=0}^{n-1} (\nu + \alpha) |a_\nu|^2.$$

*Proof.* As in (3.4) let again

$$\phi(z) = z \frac{zf'(z) - f(z)}{zf'(z) + (1 - 2\alpha)f(z)}.$$

Then it follows that

$$-\sum_{\nu=0}^{\infty} (\nu+1)a_{\nu}z^{-\nu} = \phi(z)\left[2(1-\alpha) - \sum_{\nu=0}^{\infty} (\nu-1+2\alpha)a_{\nu}z^{-\nu-1}\right],$$

hence for  $n = 0, 1, \dots$

$$\begin{aligned} & -\sum_{\nu=0}^n (\nu+1)a_{\nu}z^{-\nu} - \sum_{\nu=n+1}^{\infty} (\nu+1)a_{\nu}z^{-\nu} + \phi(z) \sum_{\nu=n}^{\infty} (\nu-1+2\alpha)a_{\nu}z^{-\nu-1} \\ & = \phi(z)\left[2(1-\alpha) - \sum_{\nu=0}^{n-1} (\nu-1+2\alpha)a_{\nu}z^{-\nu-1}\right]. \end{aligned}$$

Since the second and third term on the left side involve only powers  $z^{-\mu}$  with  $\mu \geq n+1$  and since  $|\phi(z)| < 1$  Parseval's formula gives

$$\sum_{\nu=0}^n (\nu+1)^2 |a_{\nu}|^2 \leq 4(1-\alpha)^2 + \sum_{\nu=0}^{n-1} (\nu-1+2\alpha)^2 |a_{\nu}|^2,$$

and Lemma 1 follows at once.

**COROLLARY 2.** *Let  $f \in \mathfrak{S}(\alpha)$ ,  $0 < \alpha < 1$ , and let  $A$  be the area of the compact complement of the image region  $\{f(z): |z| > 1\}$ . Then*

$$\pi \geq A > \pi\alpha,$$

*and these inequalities are best possible.*

*Proof.* The inequality  $\pi \geq A$  is of course classical. Lemma 1 implies

$$\sum_{\nu=0}^{\infty} (\nu+\alpha) |a_{\nu}|^2 \leq 1-\alpha,$$

hence

$$A = \pi\left(1 - \sum_{\nu=1}^{\infty} \nu |a_{\nu}|^2\right) \geq \pi\left(\alpha + \alpha \sum_{\nu=0}^{\infty} |a_{\nu}|^2\right) \geq \pi\alpha.$$

Equality could only hold if  $a_{\nu} = 0$  for  $\nu = 0, 1, \dots$ . But then  $A = \pi$ .

To show that  $A > \pi\alpha$  is best possible we consider

$$(6.1) \quad f(z) = z(1 + z^{-n-1})^{2(1-\alpha)/(n+1)} = z + \frac{2(1-\alpha)}{n+1}z^{-n} + \dots \in \mathfrak{S}(\alpha).$$

The function  $w = f(z)$  maps  $|z| = 1$  onto a set contained in

$$\begin{aligned} \{ |w| & \leq 2^{2(1-\alpha)/(n+1)}, |\arg w - 2\pi k/(n+1)| \\ & \leq \pi\alpha/(n+1), k = 0, \dots, n \}. \end{aligned}$$

Therefore  $A$  is smaller than the area  $\pi\alpha \cdot 2^{4(1-\alpha)/(n+1)}$  of this last set which tends  $\rightarrow \pi\alpha$  as  $n \rightarrow \infty$ .

**THEOREM 6.** *Let  $f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n}$  be in  $\mathfrak{S}(\alpha)$  ( $0 \leq \alpha < 1$ ).*

Then

$$(6.2) \quad |a_n| \leq \frac{2(1-\alpha)}{n+1}$$

for  $n = 0, 1, \dots$ , with equality for the functions (6.1).

Inequality (6.2) follows immediately from Lemma 1. The next theorem will show that (6.2) cannot be improved much even for a fixed function in  $\mathfrak{S}(\alpha)$ . In particular, there is a starlike function with  $a_n \neq o(n^{-1})$ .

**THEOREM 7.** *If  $f \in \mathfrak{S}(\alpha)$  then*

$$(6.3) \quad \limsup_{n \rightarrow \infty} n |a_n| < 2(1-\alpha).$$

For every  $\varepsilon > 0$  there is a function  $f \in \mathfrak{S}(\alpha)$  such that

$$\limsup_{n \rightarrow \infty} n |a_n| > 2(1-\alpha) - \varepsilon$$

*Proof.* It follows from Lemma 1 that

$$\limsup_{n \rightarrow \infty} (n+1)^2 |a_n|^2 \leq 4(1-\alpha)^2 - 4(1-\alpha) \sum_{\nu=0}^{\infty} (\nu+\alpha) |a_\nu|^2 < 4(1-\alpha)^2$$

except when  $a_\nu = 0$  for  $\nu = 0, 1, \dots$ . But then (6.3) is trivial. The last assertion will be proved at the end of this paper.

We shall now consider bounded starlike functions that are analytic in  $|\zeta| < 1$ .

**THEOREM 8.** *Let  $g(\zeta) = \sum_{n=1}^{\infty} b_n \zeta^n$  be analytic in  $|\zeta| < 1$  and satisfy*

$$(6.4) \quad \operatorname{Re} \zeta g'(\zeta)/g(\zeta) \geq \alpha.$$

and  $|g(\zeta)| < 1$ . Then

$$(6.5) \quad \sum_{n=1}^{\infty} (n-\alpha) |b_n|^2 \leq 1-\alpha.$$

For  $n = 2, 3, \dots$

$$(6.6) \quad |b_n| < \frac{2(1-\alpha)}{n+1-2\alpha}.$$

The factor  $2(1-\alpha)$  cannot be replaced by a smaller factor independent of  $n$ . For every  $\varepsilon > 0$  there is a function  $g(\zeta)$  such that

$$(6.7) \quad \limsup_{n \rightarrow \infty} n |b_n| > 2(1-\alpha) - \varepsilon.$$

In the case  $\alpha = 0$  inequality (6.6) has been proved (in a slightly

weaker form) by Clunie and Keogh [2], and in [7, Theorem 3]. Clunie and Keogh also gave an example of a starlike bounded function with  $b_n \neq o(n^{-1})$ .

*Proof.* For  $0 < \rho < 1$

$$(6.8) \quad \sum_{n=1}^{\infty} (n - \alpha) |b_n|^2 \rho^{2n} = \frac{1}{2\pi} \int_0^{2\pi} |g(\rho e^{i\theta})|^2 (Re[\rho e^{i\theta} g'/g] - \alpha) d\theta.$$

Since  $|g(\rho e^{i\theta})| < 1$  it follows by (6.4) that

$$\sum_{n=1}^{\infty} (n - \alpha) |b_n|^2 \rho^{2n} \leq \frac{1}{2\pi} \int_0^{2\pi} (Re[\rho e^{i\theta} g'/g] - \alpha) d\theta = 1 - \alpha$$

which implies (6.5)

As in the proof of Lemma 1 we obtain

$$(n + 1 - 2\alpha)^2 |b_n|^2 \leq 4(1 - \alpha) \sum_{\nu=1}^n (\nu - \alpha) |b_\nu|^2.$$

By (6.5) this expression is  $\leq 4(1 - \alpha)^2$ . If we had equality it would follow from (6.5) that  $g(\zeta) = b_1 \zeta + \dots$  ( $b_1 \neq 0$ ) is a polynomial of degree  $n > 1$ . Then  $\zeta g'(\zeta)/g(\zeta)$  is continuous and  $|g(\zeta)| < 1$  almost everywhere on  $|\zeta| = 1$ , and (6.8) shows that (6.5) holds with strict inequality.

Let  $g_0(\zeta) = n^{1/(n-1)}(\zeta + 2n^{-1}\zeta^n + \dots)$  be the starlike function of Example 1 in [7]. It satisfies  $|g_0(\zeta)| < 1$ . Then

$$g(\zeta) = \zeta^\alpha g_0(\zeta)^{1-\alpha} = n^{-(1-\alpha)/(n-1)}(\zeta + 2(1-\alpha)n^{-1}\zeta^n + \dots)$$

satisfies (6.4) and  $|g(\zeta)| < 1$ . Hence the factor  $2(1 - \alpha)$  in (6.6) cannot be made smaller. The existence of a function with (6.7) will be proved in the following.

**7. Construction of examples.** We shall now complete the proofs of Theorems 7 and 8 by showing: For every  $\varepsilon > 0$  there exist functions

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^{-n} \quad (|z| > 1)$$

in  $\mathfrak{S}(\alpha)$  and

$$g(\zeta) = \sum_{n=1}^{\infty} b_n \zeta^n \quad (|\zeta| < 1)$$

with  $Re \zeta g'(\zeta)/g(\zeta) \geq \alpha$  and  $|g(\zeta)| < 1$  such that

$$\limsup_{n \rightarrow \infty} n |a_n| > 2(1 - \alpha) - \varepsilon, \quad \limsup_{n \rightarrow \infty} n |b_n| > 2(1 - \alpha) - \varepsilon.$$

We shall need the following lemma.

LEMMA 2. Given  $\delta > 0$  and  $\eta > 0$  there exists a function

$$h(\zeta) = 1 + \sum_{n=1}^{\infty} c_n \zeta^n$$

with  $c_n \geq 0$  that is analytic and has positive real part in  $|\zeta| < 1$  such that

$$(7.1) \quad \limsup_{n \rightarrow \infty} c_n > 2 - \delta ,$$

$$(7.2) \quad \sum_1^{\infty} \frac{c_n}{n} < \eta .$$

This lemma was first proved by F. Riesz [8] but only with  $\limsup c_n \geq 1$ . It seems not to be known in the general form given here. It will be seen that the weaker form is not sufficient to prove the existence of an  $f \in \mathfrak{S}(0)$  with  $a_n \neq o(n^{-1})$ . But the weaker form is sufficient to prove the existence of a bounded starlike function  $g(\zeta)$  with  $b_n \neq o(n^{-1})$  [2].

*Proof.* Let  $1 > \lambda > 1 - \frac{1}{2}\delta$  and let  $p$  be so large that

$$\begin{aligned} P(\theta) &= 1 + 2\lambda \cos \theta + \cdots + 2\lambda^p \cos p\theta \\ &= \operatorname{Re}[1 + 2\lambda e^{i\theta} + \cdots + 2(\lambda e^{i\theta})^p] > 0 . \end{aligned}$$

Let  $q$  be such that

$$(7.3) \quad (2p+1)2^{-q} < \min(\frac{1}{3}\eta, \frac{1}{2}) .$$

For  $m = 1, 2, \dots$  let

$$(7.4) \quad Q_m(\theta) = \prod_{k=1}^m P(2^{qk}\theta) .$$

Then  $Q_m(\theta) > 0$ . Because of (7.3) induction shows that

$$Q_m(\theta) = 1 + \sum_{k=1}^{2p \cdot 2^{qm}} c_n \cos n\theta$$

where  $c_n$  is independent of  $m$  and  $0 \leq c_n \leq 2$ ,  $c_{2^qm} = 2\lambda$ , and  $c_n > 0$  only if

$$(7.5) \quad n = \sum \mu_k 2^{qk} \quad (-p \leq \mu_k \leq p) .$$

It follows that

$$\sum_1^{\infty} \frac{c_n}{n} \leq \sum' \frac{2}{n}$$

where the last sum is taken over all  $n$  of the form (7.5). For  $j = 1, 2, \dots$ , we group together all those  $n$  for which  $\mu_j \neq 0$  but  $\mu_k = 0$  for  $k > j$ .

Then  $n \geq 2^{qj} - p2^{q(j-1)} - \dots - p2^q > 2^{qj}(1 - 2p \cdot 2^{-q}) > 2^{qj-1}$  by (7.3). The number of these  $n$  is  $< (2p + 1)^j$ . Hence again by (7.3)

$$\sum_1^\infty \frac{c_n}{n} \leq 2 \sum_{j=1}^\infty (2p + 1)^j 2^{-qj+1} < 8(2p + 1)2^{-q} < \eta.$$

The function

$$h(\zeta) = 1 + \sum_{n=1}^\infty c_n \zeta^n$$

is analytic in  $|\zeta| < 1$ , and  $\operatorname{Re} h(\zeta) \geq 0$  by (7.4). Also  $c_{3^q m} = 2\lambda > 2 - \delta$ , and Lemma 2 is proved.

We shall now construct the starlike functions. Let  $0 < \varepsilon < 1$  and  $\delta = \varepsilon/6$ ,  $e^{(1-\alpha)\eta} = 1 + \varepsilon/6$ . If  $h(\zeta)$  is the function of Lemma 2 let

$$H(\zeta) = \int_0^\zeta s^{-1}(h(s) - 1)ds = \sum_{n=1}^\infty \frac{c_n}{n} \zeta^n.$$

Let  $\beta = 1 - \alpha$  and

$$(7.6) \quad f(z) = z \exp[-\beta H(z^{-1})], \quad g(\zeta) = \zeta \exp[\beta(H(\zeta) - \eta)].$$

Then  $zf'(z)/f(z) = 1 + \beta z^{-1}H'(z^{-1}) = \alpha + (1 - \alpha)h(z^{-1})$ , hence  $f(z) = z + \dots \in \mathfrak{S}(\alpha)$ . Also,  $g(\zeta)$  satisfies (6.4), and by (7.2)

$$|g(\zeta)| \leq \exp\left[\beta\left(\sum_1^\infty n^{-1}c_n - \eta\right)\right] < 1.$$

By (7.6),

$$(7.7) \quad \zeta^{-1}g(\zeta)e^{\beta\eta} - \zeta f(\zeta^{-1}) = 2\left(\beta H(\zeta) + \frac{1}{3!}\beta^3 H(\zeta)^3 + \dots\right).$$

Since the coefficients  $n^{-1}c_n$  of  $H(\zeta)$  are nonnegative the function on the right side of (7.7) has coefficients  $\geq 2\beta n^{-1}c_n$ . Hence

$$-a_{n-1} + b_{n+1}e^{\beta\eta} \geq 2\beta n^{-1}c_n$$

and therefore

$$(7.8) \quad n|a_{n-1}| \geq 2\beta c_n - n|b_{n+1}|e^{\beta\eta},$$

$$(7.9) \quad n|b_{n+1}| \geq (2\beta c_n - n|a_{n-1}|)e^{-\beta\eta}.$$

By Theorems 6 and 8 we have  $n|a_{n-1}| \leq 2\beta$  and  $n|b_{n+1}| \leq 2\beta$ . Therefore (7.8) and (7.9) together with (7.1) give

$$\limsup_{n \rightarrow \infty} n|a_n| \geq 2\beta(2 - \delta - e^{\beta\eta}) = 2\beta(1 - \varepsilon/3) > 2\beta - \varepsilon,$$

$$\limsup_{n \rightarrow \infty} n|b_n| \geq 2\beta(1 - \delta)e^{-\beta\eta} \geq 2\beta(1 - \varepsilon/6)/(1 + \varepsilon/6) > 2\beta - \varepsilon.$$

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