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ON THE NÖRLUND SUMMABILITY OF FOURIER SERIES

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1. Let $f(x)$ be a function integrable $-L$ over the interval $(-\pi, \pi)$ and periodic with period 2π, outside this interval.

Let

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s(x)\}\,, \tag{1.1}$$

and

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \operatorname{Sin} nx) \tag{1.2}$$

be the Fourier Series of the function $\phi(t)$.

Nörlund Summability of Fourier Series (1.2) has been considered by Woronoi [6] and later on by Nörlund [4]. These results have been extended by Hille and Tamarkin [2], [3], and later on by Astrachan [1]. Recently, extending a result due to Hille and Tamarkin [3], Varshney [5] has proved the proved the following:

THEOREM. V. *If the sequence* $\{p_n\}$ *satisfies the following conditions*:

$$\frac{n\,|\,pn\,|}{\log n} < c\,|\,P_n\,|\,, \tag{1.3}$$

$$\sum_{k=0}^{n}\frac{k\,|\,p_k - p_{k-1}\,|}{\log(k+1)} < c\,|\,P_n\,| \tag{1.4}$$

and

$$\sum_{k=0}^{n}\frac{P_k}{k\log(k+1)} < c\,|\,P_n\,| \tag{1.5}$$

and also if

$$\bar{\Phi}_1(t) = \int_0^t |\,\phi(u)\,|\,du = 0\left(t/\log\frac{1}{t}\right) \tag{1.6}$$

then the Fourier Series (1.2) *associated with the function* $\phi(t)$ *is summable by Nörlund means i.e. summable* (N, p_n) *to the sum zero at the point* $t = x$.

The object here is to prove the following:

THEOREM. *If the sequence* $\{p_n\}$ *satisfies the following conditions*

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$$\frac{n\,|\,p_n\,|}{(\log n)^r} < c\,|\,P_n\,| \tag{1.7}$$

and

$$\sum_{k=0}^{n} \frac{k\,|\,p_k - p_{k-1}\,|}{\{\log (k+1)\}^r} < c\,|\,P_n\,| \tag{1.8}$$

and also if

$$\bar{\Phi}_1(t) = \int_0^t |\,\phi(u)\,|\,du = 0\left\{t\Big/\left(\log\frac{1}{t}\right)^r\right\} \tag{1.9}$$

and

$$\frac{1}{P(n)}\int_{\pi/n}^{\delta} \frac{|\,\phi(t+\pi/n) - \phi(t)\,|}{t} P\left(\frac{1}{t}\right)dt = 0(1) \tag{1.10}$$

then the Fourier Series (1.2) *associated with the function* $\phi(t)$ *is summable by Nörlund means i.e. summable* (N, p_n) *to the sum zero at the point* $t = x$ *for all* $0 \leqq r \leqq 1$.

2. The following notations will be used in the sequel.

We write $S_n(x)$ as the nth partial sum of the series (1.2) and the Nörlund transform of the partial sum of the series (1.2) we denote by $\sigma_n(x)$.

Also we write ; where $P_n \equiv P(n)$,

$$N_n(t) = \frac{1}{\pi P_n}\sum_{k=0}^{n} p_{n-k}\frac{\text{Sin}\,(k+1/2)t}{t}\,. \tag{2.1}$$

We recall that the conditions of regularity of the method of summation are

$$\sum_{k=0}^{n} r_k = \sum_{k=0}^{n} |\,p_k\,| < c\,|\,P_n\,| \tag{2.2}$$

and

$$\{p_n/P_n\} \to 0 \quad \text{as} \quad n \to \infty\,. \tag{2.3}$$

3. If we write

$$S_n(x) = \frac{1}{\pi}\int_0^{\pi} \phi(t)\frac{\text{Sin}\,(n+1/2)t}{t}\,dt \tag{3.1}$$

then we have

$$\sigma_n(x) = \frac{1}{\pi P(n)} \int_0^\pi \phi(t) \Big(\sum_{k=0}^{n} p_{n-k} \frac{\operatorname{Sin}(k+1/2)t}{t} \Big) dt$$

$$= \int_0^\pi \phi(t) N_n(t) dt$$

$$= \Big(\int_0^{\pi/n} + \int_{\pi/n}^{\delta} + \int_\delta^\pi \Big) \phi(t) N_n(t) dt$$

$$= I_1 + I_2 + I_3 \,, \quad \text{say} \tag{3.2}$$

where δ is fixed.

Hence

$$I_2 = \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \sum_{k=0}^{n} (p_k \operatorname{Sin}(n-k+\tfrac{1}{2})t) dt$$

$$= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \sum_{k=0}^{n} p_k \{ \operatorname{Sin}(n+\tfrac{1}{2})t \cdot \cos kt - \cos(n+\tfrac{1}{2})t \cdot \operatorname{Sin} kt \} dt$$

$$= I_{2,1} - I_{2,2} \quad \text{say.} \tag{3.3}$$

Now, if we write

$$I_{2,1} = \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n+\tfrac{1}{2})t \cdot \Big\{ \sum_{kt \leqq 1} + \sum_{kt>1} \Big\} p_k \cos kt \cdot dt$$

$$= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n+\tfrac{1}{2})t \sum_{kt \leqq 1} p_k \cos ktdt$$

$$+ \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \cdot \operatorname{Sin}(n+\tfrac{1}{2})t \sum_{kt>1} p_k \cos kt \cdot dt \tag{3.4}$$

$$= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n+\tfrac{1}{2})t \Sigma_1 dt$$

$$+ \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n+\tfrac{1}{2})t \Sigma_2 dt$$

$$= I_{2,1,1} + I_{2,1,2} \,, \quad \text{say.} \tag{3.5}$$

4. We shall require the following lemma.

LEMMA. *If we write*

$$|p_n| = r_n \,, \qquad R_n = r_0 + r_1 + r_2 + \cdots + r_n \tag{4.1}$$

and

$$r(u) = r_{[u]} \,, \qquad R(u) = R_{[u]} \tag{4.2}$$

where $[u]$ *denotes the integer (largest)* $\leqq u$, *and*

$$V_0 \equiv 0 \,, \qquad V_n \equiv \sum_{k=0}^{n} |p_k - p_{k-1}| \,, \qquad V_u \equiv V_{[u]} \tag{4.3}$$

then we have, from (3.4) ,

$$\Sigma_1 > P_t \cos 1 > \tfrac{1}{2} P(\tfrac{1}{t}) \tag{4.4}$$

and

$$|\Sigma_2| = \frac{A}{t}\left\{r\left(\frac{1}{t}\right) + r(n) + V(n) - V\left(\frac{1}{t} - 1\right)\right\} . \tag{4.5}$$

This is known Hille and Tamarkin [3].

5. Now we shall prove the theorem.

Proof. Since

$$\begin{aligned} I_1 &= \frac{1}{\pi}\int_0^{\pi/n} \phi(t) N_n(t) dt \\ &= \frac{1}{\pi}\int_0^{\pi/n} |\phi(t)|\, O(n) dt \\ &= O(n)[\bar{\Phi}_1(t)]_0^{\pi/n} \qquad \text{by (1.9)} \\ &= O(n)\left[0\left\{t\Big/\left(\log\frac{1}{t}\right)^r\right\}\right]_0^{\pi/n} \\ &= 0(1) , \qquad \text{as } n \to \infty . \end{aligned} \tag{5.1}$$

From (3.5) and Lemma, above, we have

$$\begin{aligned} I_{2,1,1} &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n + \tfrac{1}{2})tP\left(\frac{1}{t}\right)dt \\ &= \frac{1}{\pi P(n)}\int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right)dt + o(1) , \end{aligned}$$

by the regularity of the method of summation.

$$\begin{aligned} &= -\frac{1}{\pi P(n)}\int_0^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)} P\left(\frac{1}{t+\pi/n}\right) \operatorname{Sin} nt\, dt + o(1) \\ &= \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right)dt - \frac{1}{2\pi P(n)}\int_0^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{t+\pi/n} \operatorname{Sin} nt \\ &\quad \cdot P\left(\frac{1}{t+\pi/n}\right)dt + o(1) \\ &= \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n} \frac{\phi(t)}{t} \operatorname{Sin} nt\, P\left(\frac{1}{t}\right)dt + \frac{1}{2\pi P(n)}\int_{\delta-\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin} nt \\ &\quad \cdot P\left(\frac{1}{t}\right)dt - \frac{1}{2\pi P(n)}\int_0^{\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right)dt \\ &\quad - \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n} \frac{\phi(t+\pi/n)}{(t+\pi/n)} \operatorname{Sin} nt\, P\left(\frac{1}{t+\pi/n}\right)dt + o(1) \end{aligned}$$

$$= \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n}\left\{\frac{\phi(t)}{t}P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t+\pi/n}P\left(\frac{1}{t+\pi/n}\right)\right\}\operatorname{Sin} nt\,dt$$

$$- \frac{1}{2\pi P(n)}\int_0^{\pi/n}\frac{\phi(t+\pi/n)}{t+\pi/n}\operatorname{Sin} nt\,P\left(\frac{1}{t+\pi/n}\right)dt$$

$$+ \frac{1}{2\pi P(n)}\int_{\delta-\pi/n}^{\delta}\frac{\phi(t)}{t}P\left(\frac{1}{t}\right)\operatorname{Sin} nt\,dt + o(1)$$

$$= \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n}\left[\left\{\frac{\phi(t)}{t}P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t}P\left(\frac{1}{t}\right)\right\}\right.$$

$$+ \left\{\frac{\phi(t+\pi/n)}{t}P\left(\frac{1}{t}\right) - \frac{\phi(t+\pi/n)}{t}P\left(\frac{1}{t+\pi/n}\right)\right\}$$

$$\left.+ \left\{\frac{\phi(t+\pi/n)}{t}P\left(\frac{1}{t+\pi/n}\right) - \frac{\phi(t+\pi/n)}{t+\pi/n}P\left(\frac{1}{t+\pi/n}\right)\right\}\right]\operatorname{Sin} nt\,dt$$

$$- \frac{1}{2\pi P(n)}\int_0^{\pi/n}\frac{\phi(t+\pi/n)}{t+\pi/n}\operatorname{Sin} nt\,P\left(\frac{1}{t+\pi/n}\right)dt$$

$$+ \frac{1}{2\pi P(n)}\int_{\delta-\pi/n}^{\delta}\frac{\phi(t)}{t}\operatorname{Sin} nt\,P\left(\frac{1}{t}\right)dt + o(1)$$

$$= \frac{1}{2\pi P(n)}\left(\int_{\pi/n}^{\delta-\pi/n}\left\{\frac{[\phi(t) - \phi(t+\pi/n)]}{t}P\left(\frac{1}{t}\right)dt\right.\right.$$

$$+ \left[P\left(\frac{1}{t}\right) - P\left(\frac{1}{t+\pi/n}\right)\right]\frac{\phi(t+\pi/n)}{t}$$

$$\left.+ P\left(\frac{1}{t+\pi/n}\right)\phi(t+\pi/n)\frac{\pi/n}{t(t+\pi/n)}\right\}\operatorname{Sin} nt\,dt$$

$$- \int_0^{\pi/n}\frac{\phi(t+\pi/n)}{t+\pi/n}\operatorname{Sin} nt\,P\left(\frac{1}{t+\pi/n}\right)dt$$

$$\left.+ \int_{\delta-\pi/n}^{\delta}\frac{\phi(t)}{t}\operatorname{Sin} nt\,P\left(\frac{1}{t}\right)dt\right)$$

$$+ o(1)$$

(5.2) $= (P_1 + P_2 + P_3) + P_4 + P_5$, say.

By virtue of (1.10) we have

(5.3) $$P_1 = o(1)\,, \quad \text{as} \quad n \to \infty\,.$$

Also

$$P_2 = \frac{1}{2\pi P(n)}\int_{\pi/n}^{\delta-\pi/n}\frac{\phi(t+\pi/n)}{t}\left\{P\left(\frac{1}{t}\right) - P\left(\frac{1}{t+\pi/n}\right)\right\}\operatorname{Sin} nt\,dt\,.$$

Since, for all $0 < 1/\alpha < 1/\beta$, we have

$$P\left(\frac{1}{\beta}\right) - P\left(\frac{1}{\alpha}\right) = \int_{1/\alpha}^{1/\beta}p(\beta)ds + O\left\{p\left(\frac{1}{\alpha}\right) + p\left(\frac{1}{\beta}\right)\right\}.$$

Hence

$$P_2 = O\Big(\frac{1}{P(n)}\Big)\Big\{\int_{\pi/n}^{\delta} |\phi(t+\pi/n)| \frac{dt}{t}\int_{1/(t+\pi/n)}^{1/t} r(s)ds\Big\}$$
$$+ O\Big(\frac{1}{P(n)}\int_{\pi/n}^{\delta} |\phi(t+\pi/n)| \, r\Big(\frac{1}{t}\Big)\frac{dt}{t}\Big)$$
$$+ O\Big(\frac{1}{P(n)}\int_{\pi/n}^{\delta} |\phi(t+\pi/n)| \, r\Big(\frac{1}{t+\pi/n}\Big)\frac{dt}{t}\Big)$$

(5.4) $$= P_{2,1} + P_{2,2} + P_{2,3} \quad \text{say.}$$

Now

$$P_{2,1} = O\Big(\frac{1}{P(n)}\Big)\Big\{\bar{\Phi}_1(t+\pi/n)\frac{1}{t}\int_{1/(t+\pi/n)}^{1/t} r(s)ds\Big\}_{\pi/n}^{\delta}$$
$$+ O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} \bar{\Phi}_1(t+\pi/n)\frac{dt}{t^2}\int_{1/(t+\pi/n)}^{1/t} r(s)ds$$
$$+ O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} \bar{\Phi}_1(t+\pi/n)\frac{1}{t} r\Big(\frac{1}{t}\Big)\frac{dt}{t^2}$$

(5.5) $$= P_{2,1,1} + P_{2,1,2} + P_{2,1,3}\,, \quad \text{say.}$$

We have

$$P_{2,1,1} = O\Big(\frac{1}{P(n)}\Big)\Big\{o\Big(\frac{1}{(\log 1/t)^r}\Big)\int_{1/(t+\pi/n)}^{1/t} r(s)ds\Big\}_{\pi/n}^{\delta} \quad \text{by (1.9)}$$
$$= o\Big(\frac{1}{P(n)}\Big)\Big\{\frac{1}{(\log n)^r}\int_{n/2\pi}^{n/\pi} r(s)ds\Big\}$$
$$+ o\Big(\frac{1}{P(n)}\Big)\Big\{\frac{1}{(\log 1/\delta)^r}\int_{1/(\delta+\pi/n)}^{1/\delta} r(s)ds\Big\}$$

(5.6) $$= o(1)\,, \quad \text{as} \quad n\to\infty\,, \quad \text{by (1.8).}$$

And

$$P_{2,1,2} = O\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} o\Big(\frac{t}{(\log 1/t)^r}\Big)\frac{dt}{t^2}\int_{1/(t+\pi/n)}^{1/t} r(s)ds \quad (1.9)$$
$$= o\Big(\frac{1}{P(n)}\Big)\int_{\pi/n}^{\delta} \frac{dt}{t(\log 1/t)^r}\int_{1/(t+\pi/n)}^{1/t} r(s)ds$$
$$= o\Big(\frac{1}{P(n)}\Big)\int_{1/(\delta+\pi/n)}^{n/\pi} r(s)ds\int_{(1/s)-\pi/n}^{1/s} \frac{dt}{t(\log 1/t)^r} + o(1)\,,$$

by change of order of integration.

$$= o\Big(\frac{1}{P(n)}\Big)\int_{1/(\delta+\pi/n)}^{n/\pi} \frac{r(s)ds}{(\log 1/s)^r}\int_{(1/s)-\pi/n}^{1/s} \frac{dt}{t} + o(1)$$
$$= o\Big(\frac{1}{P(n)}\Big)\int_{1/(\delta+\pi/n)}^{n/\pi} \frac{r(s)ds}{(\log 1/s)^r}(s\,\pi/n) + o(1)$$

$$= o\left(\frac{1}{P(n)}\right)\sum_{k=0}^{n}\frac{r(k)}{(\log(k+1))^r} + o(1)$$

$$= o(1)\,, \quad \text{as} \quad n \to \infty\,, \quad \text{by (1.8).} \tag{5.7}$$

Finally, considering $P_{2,1,3}$, we have

$$P_{2,1,3} = O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right)\frac{dt}{t}\, r\left(\frac{1}{t}\right)\frac{1}{t^2}\,, \qquad \text{by (1.9)}$$

$$= o\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta}\frac{r(1/t)}{(\log 1/t)^r}\frac{dt}{t^2}$$

$$= o\left(\frac{1}{P(n)}\right)\int_{n/\pi}^{1/\delta}\frac{r(s)}{(\log s)^r}ds$$

$$= o\left(\frac{1}{P(n)}\right)\sum_{k=0}^{n} r(k)/\{\log(k+1)\}^r$$

$$= o(1) \quad \text{as} \quad n \to \infty\,, \quad \text{by (1.8).} \tag{5.8}$$

Thus from (5.5), (5.6), (5.7) and (5.8) we see that

$$P_{2,1} = o(1) \quad \text{as} \quad n \to \infty\,. \tag{5.9}$$

Estimating $P_{2,2}$ we find that

$$P_{2,2} = O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} |\phi(t+\pi/n)|\, r\left(\frac{1}{t}\right)dt$$

$$= O\left(\frac{1}{P(n)}\right)\left\{\left[\bar{\Phi}_1(t) r\left(\frac{1}{t}\right)\frac{1}{t}\right]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta}\bar{\Phi}_1(t) r\left(\frac{1}{t}\right)\frac{dt}{t^2}\right.$$

$$\left. - \int_{\pi/n}^{\delta}\bar{\Phi}_1(t)\frac{1}{t}\,dr\left(\frac{1}{t}\right)\right\} + o(1)\,.$$

Here, the integrated part is $o(1)$, by virtue of (1.7) and the fact that $P(n) \to \infty$ as $n \to \infty$. The second part is

$$o\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} o\left(\frac{1}{(\log 1/t)^r}\right) r\left(\frac{1}{t}\right)\frac{dt}{t}\,, \qquad \text{by (1.9)}$$

$$= o\left(\frac{1}{P(n)}\right)\int_{n/\pi}^{1/\delta}\frac{r(s)}{s(\log s)^r}ds$$

$$= o(1)\,, \quad \text{by (1.8).}$$

The third term is

$$O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{\delta} o\left(\frac{t}{(\log 1/t)^r}\right)\frac{1}{t}\,dr\left(\frac{1}{t}\right)$$

$$= o\left(\frac{1}{P(n)}\right)\sum_{k=0}^{n}\frac{|r_k - r_{k-1}|}{(\log(k+1))^r}$$

$$= o(1) \quad \text{as} \quad n \to \infty\,, \quad \text{by (1.8).}$$

Thus we see that

(5.10) $$P_{2,2}=o(1) \quad \text{as} \quad n\rightarrow\infty.$$

Similarly, we can show that

(5.11) $$P_{2,3}=o(1) \quad \text{as} \quad n\rightarrow\infty .$$

Hence by (5.4), (5.9), (5.10) and (5.11) we get

(5.12) $$P_{2}=o(1) \quad \text{as} \quad n\rightarrow\infty .$$

Evaluating P_3 we have

$$\begin{aligned}
P_3 &= O\left(\frac{\pi}{nP(n)}\right)\int_{\pi/n}^{\delta}|\phi(t+\pi/n)|\,P\left(\frac{1}{t+\pi/n}\right)\frac{dt}{t(t+\pi/n)}\\
&= O\left(\frac{1}{n}\right)\int_{\pi/n}^{\delta}|\phi(t+\pi/n)|\frac{dt}{t^2}\\
&= O\left(\frac{1}{n}\right)\left[\left\{\bar{\Phi}_1(t+\pi/n)\frac{1}{t^2}\right\}_{\pi/n}^{\delta}+2\int_{\pi/n}^{\delta}\bar{\Phi}_1(t+\pi/n)\frac{dt}{t^3}\right]\\
&= o\left(\frac{1}{n}\right)+o\left(\frac{1}{(\log n)^r}\right) \quad \text{by (1.9)}
\end{aligned}$$

(5.13) $$= o(1)\,, \quad \text{as} \quad n\rightarrow\infty .$$

And

$$\begin{aligned}
P_4 &= \frac{1}{2\pi P(n)}\int_{o}^{\pi/n}\phi(t+\pi/n)\,\text{Sin}\,nt\,P\left(\frac{1}{t+\pi/n}\right)\frac{dt}{t+\pi/n}\\
&= -\frac{1}{2\pi P(n)}\int_{\pi/n}^{2\pi/n}\phi(t)\,\text{Sin}\,nt\,P\left(\frac{1}{t}\right)\frac{dt}{t}\\
&= O\left(\frac{1}{P(n)}\right)\int_{\pi/n}^{2\pi/n}|\phi(t)|\,O(nt)P\left(\frac{1}{t}\right)\frac{dt}{t}\\
&= O(n)\int_{\pi/n}^{2\pi/n}|\phi(t)|\,dt\\
&= o\left(\frac{1}{(\log n)^r}\right), \quad \text{by (1.9)}
\end{aligned}$$

(5.14) $$= o(1) \quad \text{as} \quad n\rightarrow\infty.$$

Also

$$P_5 = O\left(\frac{1}{P(n)}\right)\int_{\delta-\pi/n}^{\delta}|\phi(t)|\,P\left(\frac{1}{t}\right)\frac{dt}{t}$$

(5.15) $$= o(1)\,,$$

by the regularity of the method of summation and since the interval $(\delta-\pi/n,\delta)$ tends to zero as $n\rightarrow\infty$.

Consequently from (5.2), (5.3), (5.12), (5.13), (5.14) and (5.15) we have

$$I_{2,1,1} = o(1)\ , \quad \text{as} \quad n \to \infty\ . \tag{5.16}$$

Now

$$\begin{aligned} I_{2,1,2} &= \frac{1}{\pi P(n)} \int_{\pi/n}^{\delta} \frac{\phi(t)}{t} \operatorname{Sin}(n+\tfrac{1}{2})t\Sigma_2 dt \\ &= O\Big(\frac{1}{P(n)}\Big) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t} \Big\{ \frac{r(1/t)}{t} + \frac{r(n)}{t} \\ &\quad + \frac{1}{t}\Big[V(n) - V\Big(\frac{1}{t} - 1\Big)\Big]\Big\} dt\ , \quad \text{by (4.5)} \\ &= Q_1 + Q_2 + Q_3\ , \quad \text{say.} \end{aligned} \tag{5.17}$$

We have

$$\begin{aligned} Q_1 &= O\Big(\frac{1}{P(n)}\Big) \int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2} r\Big(\frac{1}{t}\Big) dt \\ &= O\Big(\frac{1}{P(n)}\Big) \Big\{ \Big[o\Big(\frac{t}{\{\log 1/t\}^r}\Big) r\Big(\frac{1}{t}\Big) \frac{1}{t^2} \Big]_{\pi/n}^{\delta} + \int_{\pi/n}^{\delta} o\Big(\frac{t}{\{\log 1/t\}^r}\Big) r\Big(\frac{1}{t}\Big) \frac{dt}{t^3} \\ &\quad + \int_{\pi/n}^{\delta} o\Big(\frac{t}{\{\log 1/t\}^r}\Big) \frac{1}{t^2} dr\Big(\frac{1}{t}\Big) \Big\} \quad \text{by (1.9).} \end{aligned}$$

Here the integrated part is $o(1)$, by (1.7) and the fact that $P(n) \to \infty$ as $n \to \infty$. Also the second term is

$$\begin{aligned} o\Big(\frac{1}{P(n)}\Big) & \int_{\pi/n}^{\delta} r\Big(\frac{1}{t}\Big) \frac{dt}{t^2} \frac{1}{(\log 1/t)^r} \\ &= o\Big(\frac{1}{P(n)}\Big) \int_{n/\pi}^{1/\delta} \frac{r(s)}{(\log s)^r} ds \\ &= o\Big(\frac{1}{P(n)}\Big) \sum_{k=0}^{n} \frac{k\,|p_k - p_{k-1}|}{(\log (k+1))^r} \\ &= o(1) \quad \text{by (1.8).} \end{aligned}$$

The third part is

$$\begin{aligned} o\Big(\frac{1}{P(n)}\Big) & \int_{n/\pi}^{1/\delta} \frac{s\,dr(s)}{(\log s)^r} \\ &= o\Big(\frac{1}{P(n)}\Big) \sum_{k=0}^{n} \frac{k\,|p_k - p_{k-1}|}{(\log (k+1))^r} \\ &= o(1) \quad \text{as} \quad n \to \infty\ , \quad \text{by (1.8).} \end{aligned}$$

Thus we see that

$$Q_1 = o(1) \quad \text{as} \quad n \to \infty\ . \tag{5.18}$$

Now

$$Q_2 = O\left(\frac{r(n)}{P(n)}\right)\int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2}\,dt$$

$$= O\left(\frac{r(n)}{P(n)}\right)\left\{\left(\frac{\bar{\Phi}_1(t)}{t^2}\right)_{\pi/n}^{\delta} + 2\int_{\pi/n}^{\delta} \bar{\Phi}_1(t)\frac{dt}{t^3}\right\}$$

$$= o\left(\frac{r(n)}{P(n)}\right) + o\left(\frac{nr(n)}{P(n)(\log n)^r}\right)$$

$$+ o\left(\frac{r(n)}{P(n)}\right)\int_{\pi/n}^{\delta} \frac{dt}{t^2(\log 1/t)^r}$$

$$= o\left\{\frac{(\log n)^r}{n}\right\} + o(1) + o\left\{\frac{r(n)n}{P(n)(\log n)^r}\right\}$$

$$= o(1) + o\left(\frac{1}{n}\right)$$

$$(5.19) \qquad = o(1) \quad \text{as} \quad n \to \infty .$$

Lastly

$$Q_3 = O\left(\frac{1}{P(n)}\int_{\pi/n}^{\delta} \frac{|\phi(t)|}{t^2}\left[V(n) - V\left(\frac{1}{t} - 1\right)\right]dt\right.$$

$$= O\left(\frac{1}{P(n)}\right)\left\{\left[\frac{\bar{\Phi}_1(t)}{t^2}\left(V(n) - V\left(\frac{1}{t} - 1\right)\right)\right]_{\pi/n}^{\delta}\right.$$

$$+ 2\int_{\pi/n}^{\delta} \frac{\bar{\Phi}_1(t)}{t^3}\left[V(n) - V\left(\frac{1}{t} - 1\right)\right]dt$$

$$\left. - \int_{\pi/n}^{\delta} \frac{\bar{\Phi}_1(t)}{t^2}\,dV\left(\frac{1}{t} - 1\right)\right\} .$$

The integrated part is $o(1)$, by (1.7) and the fact that

$$W_n \equiv \sum_{k=0}^{n} \frac{k\,|p_k - p_{k-1}|}{(\log (k+1))^r} ; \qquad W_0 \equiv 0 .$$

Then, by (1.8) we have

$$V_n = \sum_{k=0}^{n} |p_k - p_{k-1}|$$

$$= \sum_{k=0}^{n} \frac{\{\log (k+1)\}^r}{k}(W_k - W_{k-1})$$

$$= \sum_{k=0}^{n-1} W_k\left\{\Delta\left[\frac{\{\log (k+1\}^r}{k}\right]\right\} + \frac{W_n\{\log (n+1)\}}{n}$$

$$= o\{R(n)\} .$$

Now the second term is

$$o\left[\frac{1}{P(n)}\int_{\pi/n}^{\delta}\left\{V(n)-V\left(\frac{1}{t}-1\right)\right\}\frac{dt}{t^2(\log 1/t)^r}\right]$$
$$=o\left[\frac{1}{P(n)}\int_{1/\delta}^{n/\pi}\frac{ds}{(\log s)^r}[V(n)-V(s-1)]\right]$$
$$=o\left\{\frac{1}{P(n)}\int_0^n\frac{s}{\{\log(s+1)\}^r}dV(s)\right\}$$
$$+o\left[\frac{1}{R(n)}\{V(n)-V(s-1)\}\frac{s}{\{\log(s+1)\}^r}\right]_0^n$$
$$=o\left\{\frac{1}{P(n)}\sum_{k=0}^{n}\frac{k\,|\,p_k-p_{k-1}\,|}{\{\log(k+1)\}^r}\right\}+o\left\{\frac{1}{P(n)}\frac{n\,|\,p_n-p_{n-1}\,|}{\{\log(n+1)\}^r}\right\}$$

which is $o(1)$, by virtue of (1.7), (1.8) and the fact that $V_n=o\{P(n)\}$.

The third term is

$$o\left\{\frac{1}{P(n)}\int_{\pi/n}^{\delta}\frac{1}{t(\log 1/t)^r}\left|dV\left(\frac{1}{t}-1\right)\right|\right\},\quad \text{by (1.9)}$$
$$=o\left\{\frac{1}{P(n)}\right\}\int_0^n\frac{s\,|\,dV(s-1)\,|}{\{\log(s+1)\}^r}$$

which is $o(1)$, as in the case of second term.

Thus we have

$$Q_3=o(1)\quad\text{as}\quad n\to\infty\,. \tag{5.20}$$

From (5.17), (5.18), (5.19) and (5.20), we have

$$I_{2,1,2}=o(1)\quad\text{as}\quad n\to\infty\,. \tag{5.21}$$

From (3.4) (3.5), (3.16) and (5.21), we see that

$$I_{2,1}=o(1)\quad\text{as}\quad n\to\infty\,. \tag{5.22}$$

Similarly, we can show that

$$I_{2,2}=o(1)\quad\text{as}\quad n\to\infty\,. \tag{5.23}$$

From (3.3), (5.22) and (5.23) we get

$$I_2=o(1)\quad\text{as}\quad n\to\infty\,. \tag{5.24}$$

Lastly by Riemann Lebesgue Theorem and the regularity of the method of summation, we have, as $n\to\infty$

$$I_3=o(1)\,. \tag{5.25}$$

Collection of (3.2), (5.24) and (5.25) as $n\to\infty$, completes the proof of the theorem.

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