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SOLUTION OF LOOP EQUATIONS BY ADJUNCTION

R. ARTZY

Solutions of integral equations over groups by means of adjunction of new elements have been studied by B. H. Neumann [3] and F. Levin [2]. Here an analogous question for loops will be dealt with, and the results will prove to be useful also for groups.

Let $(L, .)$ be a loop with neutral element e, x an indeterminate. Let w be a word whose letters are x and elements of L. Let n be the number of times that x appears in w. Form $f(x)$ from w by inserting parentheses between its letters so as to make it into a uniquely defined expression if juxtaposition means loop multiplication. The equation $f(x)=r$, r in L, will be called *an integral loop equation in x of degree n.*

An integral loop equation $f(x) = r$ is *monic* if $f(x)$ is a product of two factors both containing x. Every integral loop equation can be made monic by a finite number of left or right divisions by elements of L.

Not every integral loop equation has a solution as indicated by the monic example $x^2 = r$, $r \neq e$, L the four-group. Our aim is finding a loop E in which L is embedded and in which $f(x) = r$ has a solution. The loop E used here will be an extension loop [1] of L by $(C_n, +)$, the cyclic group of order n. The construction follows the

Extension Rule. The elements of E are ordered couples (c, a) where $c \in C_n$, $a \in L$. Equality of couples is componentwise. The multiplication in E is defined by $(c_1, a_1)(c_2, a_2) = (c_1 + c_2, a_1a_2 \cdot h(c_1, c_2))$, where $h(c_1, c_2)$ is an element of L depending on c_1 and c_2, assuming the value e except in the case when $c_1 + c_2 = 0$ and $c_1 \neq 0$.

THEOREM 1. *A monic integral loop equation $f(x) = r$ of degree n over a loop L has a solution in an extension loop $E = (C_n, L)$ constructed according to the Extension Rule, with $f(e)h(c, n - c) = r$ whenever $c \neq 0$.*

Proof. If the element b of L is represented in E by $(0, b)$, L is mapped isomorphically into E. Let x be represented in E by $(1, e)$, where 1 is a generator of C_n. All elements of C_n will be written as integers. Then $f(x)$ can be constructed by stages. For every x entering into the successive multiplication one summand 1 appears in the first component. In the second component only the loop elements of $f(x)$ will appear as factors because x has the second component e. The h's

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do not enter the picture until the last step because they depend on the first components, and all multiplications but the last yield $h(c_1, c_2) = e$ in view of $0 \leqq c_1 + c_2 < n$. Thus, at first, the construction of the second component of $f(x)$ in E follows exactly the pattern of the successive multiplication which yielded $f(x)$, with the exception of the factor x. The result is, therefore, the same as though x had been replaced by e, namely $f(e)$. However, the last product, one of whose factors contains by definition at least one x, requires a factor $h(c, n-c)$, $0 < c < n$. Thus the final result is $(n, f(e)h(c, n-c)) = (0, r)$ and consequently $f(e)h(c, n-c) = r$, $c \neq 0$.

THEOREM 2. *An integral loop equation of degree n has in E at least $\varphi(n)$ solutions, φ being Euler's function. For each two of these solutions, x and y, there exists an automorphism of E carrying x into y and leaving L unchanged elementwise.*

Proof. Let again $x = (1, e)$. If k and n are relatively prime, (k, e) is another solution because $nk = 0$ and $h(m, q) = h(km, kq)$ since $m = 0$ or $\neq 0$ according to $km = 0$ or $\neq 0$. There are $\varphi(n)$ distinct k's with the properties $0 < k < n$ and $(k, n) = 1$. This proves the first part of the theorem. Now, $1 \rightarrow k$ is an automorphism of C_n preserving the 0-element and hence also the h's. The loop L is unaffected by these automorphisms, because they act only on the first components.

DEFINITION. An *abelian integral identity* over a loop L is an equation $u(w) = v(w')$, where (i) w and w' are words using the same set of elements of L, but not necessarily in the same order, (ii) $u(w)$ and $v(w')$ are formed from w and w', respectively, by inserting parentheses between the letters of the words so as to make them into uniquely defined expressions if juxtaposition means loop multiplication, (iii) the equality is preserved when the loop elements forming w and w' are replaced by arbitrary elements of L.

In general the validity of abelian integral identities in L, like associativity or the Moufang property, does not carry over into E. However, in the case of degree 2 we are able to obtain the following result.

THEOREM 3. *Let an integral equation of degree 2 over a loop L have the monic form $f(x) = r$. Every abelian integral identity valid in L will hold also in the extension loop E, constructed as in Theorem 1, provided $h(1, 1) = [f(e)]\backslash r$ lies in the center of L.*

Proof. The first component of the elements of E is 1 or 0. Moreover, $h(0, 1) = h(1, 0) = h(0, 0) = 1$; write $h(1, 1) = h$, for short. We have then $h(p, q) = h^{pq}$, where pq is the product of p and q in $GF(2)$, and $h^0 = e$, $h^1 = h$. The addition in $C_2 = \{0, 1\}$ is the addition of $GF(2)$. The loop elements h^0 and h^1 multiply according to the rule $h^p h^q = h^{p+q}$, and, by the hypothesis of the theorem, they lie in the center of L.

Now, the abelian integral identity $u(w) = v(w')$ in E would surely be satisfied for the first components since they behave as elements of C_2, an abelian group. If in the second components the h's are disregarded, the abelian integral identity over E yields an exact replica of the same identity over L. But, as center elements of L, the h's appearing in u and v can indeed be pulled out and shifted to the right of each side. We denote the product of the h's of u by $h^{H(u)}$. In the degenerate case where u consists of one letter only, we define $H(u) = 0$. We have to prove for the second components that $u(w)h^{H(u)} = v(w')^{H(v)}$. Since $u(w) = v(w')$ it will be sufficient to prove $H(u) = H(v)$.

Let the first components of the elements of w be $p_1, \cdots, p_m$. We claim now that $H(u) = \sum_{i\ j=1, i<j}^{m} p_i p_j$, independent of the order of the p's, and that therefore $H(u) = H(v)$. For $m = 2$ we have trivially $H(u) = p_1 p_2$. For $m = 3$, $h^{H(u)} = h^{p_1 p_2} h^{(p_1+p_2)p_3} = h^{p_1 p_2 + p_1 p_3 + p_2 p_3}$. Suppose $H(u) = \sum_{i,j=1,i<j}^{m'} p_i p_j$ has been proved for every word length $m' < m$. If the last multiplication of $u(w)$ is $u'u''$, where u' is a product of $p_1, \cdots, p_k$ and u'' of $p_{k+1}, \cdots, p_m$, then the induction hypothesis yields $H(u') = \sum_{i\ j=1,i<j}^{k} p_i p_j$ and $H(u'') = \sum_{i,j=k+1,i<j}^{m} p_i p_j$. Then

$$\begin{aligned} H(u) &= H(u') + H(u'') + (p_1 + \cdots + p_k)(p_{k+1} + \cdots + p_m) \\ &= \textstyle\sum_{i,j=1,i<j}^{k} p_i p_j + \sum_{i,j=k+1,i<j}^{m} p_i p_j \\ &\quad + (p_1 + \cdots + p_k)(p_{k+1} + \cdots + p_m) = \textstyle\sum_{i,j=1,i<j}^{m} p_i p_j \,. \end{aligned}$$

This completes the proof.

COROLLARY. *The equation $xax = r$ over a group G has a solution in an extension group $E = (C_2, G)$ constructed as in Theorem 1, provided $a^{-1}r$ lies in the center of G. In particular the equation has always a solution in E if G is abelian.*

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A CHARACTERIZATION OF SCALAR TYPE OPERATORS ON REFLEXIVE BANACH SPACES

EARL BERKSON

Introduction. The main purpose of this paper is to characterize scalar operators on reflexive Banach spaces. This is accomplished in 4.2 and 4.4. However, most of the results are not limited to reflexive spaces.

We give a fundamental decomposition theorem for scalar operators in § 2, and show in § 3 that this decomposition is unique.

In what follows, all spaces are over the complex field, all Banach algebras have an identity of norm 1, and an operator will be a bounded linear transformation with range contained in its domain. This understanding will also cover material quoted from other sources.

1. Preliminaries. In this section we reproduce some machinery from [4] and [7] which will be needed in the sequel.

The definitions and results of this paragraph are taken from [4].

DEFINITION. Let X be a vector space. A semi-inner-product on X is a mapping $[\,,]$ of $X \times X$ into the field of complex numbers such that:

(i) $[x + y, z] = [x, z] + [y, z]$ for $x, y, z \in X$.

(ii) $[\lambda x, y] = \lambda[x, y]$ for $x, y \in X$, λ complex.

(iii) $[x, x] > 0$ for $x \neq 0$.

(iv) $|[x, y]|^2 \leqq [x, x][y, y]$.

We then call X a semi-inner-product space (abbreviated s.i.p.s.). If X is a s.i.p.s., then $[x, x]^{1/2}$ is a norm on X. On the other hand, every normed linear space can be made into a s.i.p.s. (in general, in infinitely many ways) so that the semi-innner-product is consistent with the norm—i.e., $[x,x]^{1/2} = \|x\|$, for each $x \in X$. By virtue of the Hahn-Banach theorem this can be accomplished by choosing for each $x \in X$ exactly one bounded linear functional f_x such that $\|f_x\| = \|x\|$ and $f_x(x) = \|x\|^2$, and then setting $[x, y] = f_y(x)$, for arbitrary $x, y \in X$.

DEFINITION. Given a linear transformation T on a s.i.p.s., we denote by $W(T)$ the set, $\{[Tx, x] \mid [x, x] = 1\}$, and call this set the numerical range of T.

An important fact concerning the notion of numerical range is the following:

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1.1 Let X be a Banch space and T an operator on X. Although in principle there may be many different semi-inner-products consistent with the norm of X, nonetheless if the numerical range of T relative to one such semi-inner-product is real, then the numerical range relative to any such semi-inner-product is real. If this is the case, we call T a hermitian operator.
Also important is the result:

1.2 If X is a s.i.p.s., complete with respect to the induced norm on X, and T is a linear transformation on X, bounded with respect to the induced norm, then $|W(T)| \leqq \|T\| \leqq 4|W(T)|$, where $|W(T)|$ denotes the quantity sup $\{|\lambda| \mid \lambda \in W(T)\}$.

In [7], I. Vidav introduces the following notion of hermiticity:

DEFINITION. An element h of a Banach algebra A with identity e will be called hermitian if and only if for α real, $\|e + i\alpha h\| = 1 + o(\alpha)$ as $\alpha \to 0$.
It is shown in [4; § 9] that an operator T on a Banach space X is a hermitian operator if and only if it is hermitian in the sense of Vidav's definition—i.e., if and only if $\|I + i\alpha T\| = 1 + o(\alpha)$ for α real, where I is the identity operator. Thus we have at our disposal two equivalent formulations of the notion of hermiticity for operators on Banach spaces.

The result of [7] very important for our considerations is the following:

1.3 Suppose A is a Banach algebra with identity e. Let H be the set of hermitian elements of A (i.e., $H = \{h \in A \mid \|e + i\alpha h\| = 1 + o(\alpha)$, for α real$\}$). We assume that: (a) every $a \in A$ has a representation $a = u + iv, u, v \in H$; (b) if $h \in H$, then there is a representation $h^2 = u + iv$ such that $u, v \in H$ and $uv = vu$. Then there is an involution on A and a new norm equivalent to the given norm such that in terms of this involution and the new norm A is a C^*-algebra.
It is well known that the Gelfand representation of a commutative C^*-algebra with identity is an isometry onto $C(\mathscr{M})$, the algebra of all continuous complex-valued functions on the maximal ideal space $\mathscr{M}$ (see, for example, [3; § 26A]). Hence we can state:

1.4 Let A be a commutative Banach algebra with identity, and let H be the set of hermitian elements of A. If every $a \in A$ has a representation in the form $a = u + iv$, where $u, v \in H$, then the Gelfand representation of A is a bicontinuous isomorphism of A onto $C(\mathscr{M})$.

2. A fundamental decomposition. Throughout the rest of this paper X will be a fixed Banace space with norm $\| \ \|$, and X^* will be its dual. Throughout this section S will be a fixed scalar operator on X, and E will be its resolution of the identity (see [1] for this terminology). For given $x \in X$, $x^* \in X^*$, we shall denote by var $x^*E(\ )x$ the quantity $\sup \Sigma \,|\, x^*E(\sigma_i)x \,|$, where the supremum is taken over all finite sequences $\{\sigma_i\}$ of disjoint Borel sets in the complex plane p.

It is shown in [1; see proof of Theorem 17] that there is a constant K such that

2.1 $$\operatorname{var} x^*E(\ )x \leqq K \,||\, x \,||\ ||\, x^* \,||, \ x \in X,\ x^* \in X^* .$$

We now show:

2.2 LEMMA. *For each $x \in X$, define $|\, x \,|$ by*

$$|\, x \,| = \sup_{\substack{x^* \in X^* \\ ||x^*||=1}} \operatorname{var} x^*E(\ )x .$$

Then $|\ \ |$ is a norm on X equivalent to $||\ \ ||$.

Proof. It is straightforward to verify that $|\ \ |$ is a seminorm. With K as in 2.1 we have that $|\, x \,| \leqq K \,||\, x \,||$, for $x \in X$. Given $x \in X$, choose $x^* \in X^*$ so that $||\, x^* \,|| = 1$ and $x^*(x) = ||\, x \,||$. Then $||\, x \,|| = |\, x^*(x) \,| = |\, x^*E(p)x \,| \leqq |\, x \,|$. This completes the proof.

2.3 LEMMA. *Relative to the norm $|\ \ |$ defined in 2.2, $E(\sigma)$ is a hermitian operator, for each Borel set σ.*

Proof. We shall also use the symbol $|\ \ |$ to denote the norm of an operator relative to $|\ \ |$. We shall show that if σ is a Borel set, and $E(\sigma) \neq 0$, then

2.4 $$|\, I + i\alpha E(\sigma) \,| = |\, 1 + i\alpha \,|, \text{ for } \alpha \text{ real} .$$

For arbitrary $x \in X$, $x^* \in X^*$, with $||\, x^* \,|| = 1$, real α, and arbitrary finite sequence $\sigma_1, \sigma_2, \cdots, \sigma_n$ of disjoint Borel sets, we have:

$$\sum_{j=1}^{n} |\, x^*E(\sigma_j)(I + i\alpha E(\sigma))x \,|$$
$$= \sum_{j=1}^{n} |\, x^*E(\sigma_j)[E(\sigma')x + (1 + i\alpha)E(\sigma)x] \,|,$$
where σ' is the complement of σ.
$$\leqq \sum_{j=1}^{n} |\, x^*E(\sigma_j)E(\sigma')x \,| + |\, 1 + i\alpha \,| \sum_{j=1}^{n} |\, x^*E(\sigma_j)E(\sigma)x \,|$$
$$\leqq |\, 1 + i\alpha \,| \operatorname{var} x^*E(\ )x,$$ since the sets $\sigma_1 \cap \sigma', \cdots, \sigma_n \cap \sigma', \sigma_1 \cap \sigma, \cdots, \sigma_n \cap \sigma$ are disjoint.
$$\leqq |\, 1 + i\alpha \,|\, |\, x \,| .$$

Hence

$$|(I + i\alpha E(\sigma))x| \leqq |1 + i\alpha||x| .$$

So

$$|I + i\alpha E(\sigma)| \leqq |1 + i\alpha| .$$

On the other hand, if y is in the range of $E(\sigma)$, with $|y| = 1$, then $|(I + i\alpha E(\sigma))y| = |y + i\alpha y| = |1 + i\alpha|$. Thus 2.4 is established. Since $|1 + i\alpha| = 1 + o(\alpha)$, the desired conclusion follows.

2.5 THEOREM. *There are operators R and J such that:*
(1) $S = R + iJ$.
(2) $RJ = JR$.
(3) *Relative to some norm on X equivalent to $\| \;\; \|$, R^mJ^n are hermitian operators for $m, n = 0, 1, 2, \cdots$.*

Proof. We write

2.6 $S = \int Re\lambda dE(\lambda) + i\int Im\lambda dE(\lambda)$, where "$Re$" and "$Im$" denote "real part of" and "imaginary part of," respectively .

We now set $R = \int Re\lambda dE(\lambda)$ and $J = \int Im\lambda dE(\lambda)$. Clearly (1) and (2) hold. For the proof of (3) we use the norm $| \;\; |$, as defined in 2.2. By (2.3), $E(\sigma)$ is a hermitian operator relative to $| \;\; |$, for each Borel set σ. It is clear from the definition in 1.1 that a sum of real multiples of hermitian operators is a hermitian operator, and so is a limit in the uniform operator topology of hermitian operators. The conclusion in (3) is now clear from the fact that for arbitrary positive integers m, n,

$$R^m = \int (Re\lambda)^m dE(\lambda)$$

$$J^n = \int (Im\lambda)^n dE(\lambda)$$

$$R^mJ^n = \int (Re\lambda)^m(Im\lambda)^n dE(\lambda) .$$

REMARK 1. It is not known, in general, if a product of commuting hermitian operators is hermitian, or even if the powers of a hermitian operator are hermitian. Consequently it is not known if part of property (3) of 2.5 is superflous.[1]

REMARK 2. We shall show in § 3 that the decomposition described

[1] See note added in proofreading.

in 2.5 is unique. Thus the representation of S given by 2.6 is characterized by properties (1), (2), and (3) of 2.5. In [2; § 5], Foguel has introduced the representation 2.6, but has characterized it differently, without the notion of hermitian operator. In accordance with his terminology, we call $\int Re\lambda dE(\lambda)$ and $\int Im\lambda dE(\lambda)$ the real and imaginary parts of S, respectively.

3. Uniqueness of the decomposition in 2.5. In this section we show that the decomposition given in the statement of 2.5 is unique. At first glance it might seem that uniqueness is immediate from 1.2; however, given two pairs of operators, each pair satisfying (1)–(3) of 2.5, we do not assume that the norms given in (3) are the same for the two pairs.

Some additional items will be needed. Given an element a of a Banach algebra A, we shall denote by $sp_A(a)$ the spectrum of a in A. We shall denote by $[X]$ the Banach algebra of all operators mapping X into itself. We shall use the fact that if x is a hermitian element of the Banach algebra A, then $sp_A(x)$ is real. This is shown in [7; Lemma 2].

3.1 THEOREM. *Let R and J be any two operators on X satisfying conditions* (2) *and* (3) *of* 2.5, *and let A be the Banach subalgebra of $[X]$ generated by R, J, and I. Further, let $T = R + iJ$, and define the functions f_1 and f_2 on $sp_{[X]}(T)$ by*

$$f_1(\lambda) = Re\lambda, \quad f_2(\lambda) = Im\lambda .$$

Then there is a bicontinuous isomorphism of $C(sp_{[X]}(T))$ onto A such that the image of f_1 is R and the image of f_2 is J.

Proof. We shall assume throughout the proof that X has been renormed with an equivalent norm, $|||\ \ |||$, chosen according to condition (3) of 2.5, and likewise that $[X]$ and A have been renormed with the corresponding operator norm, which we also denote by $|||\ \ |||$. We shall also introduce a semi-inner-product on X (denoted by $[\ ,\ ]$) consistent with $|||\ \ |||$. We first show that A satisfies the hypotheses of 1.4. It is clear that if $Q \in A$, then Q is the limit in the uniform operator topology of a sequence $\{P_n\}$ of polynomials in R and J with complex coefficients. For each n, P_n can be written in the form $P_n = U_n + iV_n$. where U_n and V_n are polynomials in R and J with real coefficients. Thus U_n and V_n belong to A and are hermitian operators on X. For arbitrary positive integers m and n, and for arbitrary $x \in X$ with $|||x||| = 1$,

$$|[(U_m - U_n)x, x] + i[(V_m - V_n)x, x]| = |[(P_m - P_n)x, x]| \leqq |||P_m - P_n||| .$$

Since $[(U_m - U_n)x, x]$ and $[(V_m - V_n)x, x]$ are real,

$$|[(U_m - U_n)x, x]| \leqq |||P_m - P_n||| \text{ and } |[(V_m - V_n)x, x]| \leqq |||P_m - P_n||| .$$

Hence by 1.2,

$$|||U_m - U_n||| \leqq 4|||P_m - P_n||| \text{ and } |||V_m - V_n||| \leqq 4|||P_m - P_n||| .$$

It follows that $\{U_n\}$ and $\{V_n\}$ converge to hermitian operators U and V, respectively, which belong to A. Therefore $Q = U + iV$. Since U and V are hermitian operators lying in A, they are (by § 1, paragraph 3) hermitian elements of the algebra A. Thus 1.4 holds.

To complete the proof we show that there is a one-to-one mapping ψ of $\mathscr{M}$, the maximal ideal space of A, onto $sp_{[X]}(T)$ such that if $\hat{R}$ and $\hat{J}$ denote the Gelfand representatives of R and J, respectively, then $\hat{R}(\psi^{-1}(\lambda)) = Re\lambda$ and $\hat{J}(\psi^{-1}(\lambda)) = Im\lambda$, for each $\lambda \in sp_{[X]}(T)$. To accomplish this, we identify $\mathscr{M}$ with the space of all homomorphisms of A onto the complex numbers. We then define ψ as follows:

3.2 $\qquad \psi(h) = h(R) + ih(J)$, for each homomorphism $h \in \mathscr{M}$.

Since $sp_A(R)$ and $sp_A(J)$ are real, and since R, J, and I generate A, it is clear that ψ is one-to-one. The range of ψ is obviously the range of the Gelfand representative of T, and hence is $sp_A(T)$. Since the Gelfand representation of A is a one-to-one map of A onto $C(\mathscr{M})$, it is clear that the commutative Banach algebra A is semi-simple and completely regular. Hence (see [6; Corollary (3.7.6)]), $sp_A(T) = sp_{[X]}(T)$. The desired conclusions about $\hat{R}$ and $\hat{J}$ are obvious by virtue of 3.2.

3.3 THEOREM. *Let S be a scalar operator on X. The operators R and J of* 2.5 *are uniquely determined by* (1)–(3).

Proof. Let R and J satisfy (1)–(3) of 2.5. Then (in the notation of 3.1) there is a bicontinuous isomorphism ϕ of $C(sp_{[X]}(S))$ onto A such that $\phi(f_1) = R$ and $\phi(f_2) = J$. Let $\mathscr{B}$ be the class of Borel sets in $sp_{[X]}(S)$. By [1; Theorem 18], the adjoint of every operator in A is a scalar type operator of class X, and there is a spectral measure G in X^* of class $(\mathscr{B}, X)$ such that

3.4 $\qquad \phi(f)^* = \int_{sp[X](S)} f(\lambda)dG(\lambda)$, for each $f \in C(sp_{[X]}(S))$.

In particular, $S^* = \int_{sp[X](S)} \lambda dG(\lambda)$. By [1; Lemma 6], the resolution of the identity for S^* (call the resolution of the identity F) is given by

$$F(\sigma) = G(sp_{[X]}(S) \cap \sigma) .$$

Thus, using 3.4, we have

$$R^* = \int_{sp_{[X]}(S)} Re\lambda dF(\lambda) \text{ and } J^* = \int_{sp_{[X]}(S)} Im\lambda dF(\lambda) ,$$

where F is the resolution of the identity for S^*.

So the adjoints of R and J are uniquely determined, and hence so are R and J.

4. Characterization of scalar type operators on reflexive spaces. The first theorem of this section is the converse of 2.5 under the additional hypothesis that X is reflexive. The second theorem contains a summary of preceding results for the special case when X is reflexive.

4.1 THEOREM. *Let X be reflexive, and let T be an operator on X. Then T is a scalar type operator of class X^* if there exist operators R and J satisfying*

(1) $T = R + iJ$.

(2) $RJ = JR$.

(3) *Relative to some norm on X equivalent to $\| \quad \|$, $R^m J^n$ are hermitian operators for $m, n = 0, 1, 2, \cdots$.*

Proof. By 3.1 the Banach subalgebra of $[X]$ generated by R, J, and I is equivalent to $C(sp_{[X]}(T))$. The desired conclusion is now immediate from [1; conclusion (iv) of Theorem 18], which states that if an algebra of operators on a reflexive Banach space Y is equivalent to the algebra of all continuous complex-valued functions on some compact Hausdorff space, then this algebra of operators consists entirely of scalar type operators of class Y^*.

REMARK 3. It is a consequence of 2.5 and 4.1 that a spectral operator on a reflexive Banach space Y is automatically of class Y^*. This is also easy to see directly, since, in the reflexive case, it follows from the Hahn-Banach theorem that a total linear manifold in Y^* is dense in the norm topology of Y^*. Thus for a reflexive space Y, the terms "spectral operator" and "spectral operator of class Y^*" are equivalent.

4.2 THEOREM. *Let X be reflexive, and let T be an operator on X. Then T is a scalar type operator if and only if there exist operators R and J satisfying conditions* (1)–(3) *of* 4.1 *If this is the case, R and J are uniquely determined.*

It is desirable to replace condition (3) occurring above by a condition which involves only the original norm on X rather than some equivalent norm. The author wishes to express his appreciation to G. Lumer for communicating to him the essence of the next theorem, which accomplishes this purpose.

4.3 THEOREM. *Let X be an arbitrary Banach space with norm $\|\ \|$, and let R and J be commuting operators on X. Further, let $\mathscr{A}$ be the set of all polynomials in R and J with real coefficients. In order that there exist a norm on X equivalent to $\|\ \|$ and relative to which R^mJ^n are hermitian operators for $m, n = 0, 1, 2, \cdots$, it is necessary and sufficient that the set $\{e^{iP} \mid P \in \mathscr{A}\}$ be a bounded subset of $[X]$.*

Proof. Suppose that X can be renormed with an equivalent norm, $|||\ |||$, relative to which the operators R^mJ^n are hermitian. Then each $P \in \mathscr{A}$ is clearly a hermitian operator relative to $|||\ |||$, and so by [7; Lemma 1] $|||e^{iP}||| = 1$. Since the renorming is an equivalent one, $\{\|e^{iP}\| \mid P \in \mathscr{A}\}$ is bounded.

Conversely, suppose that the positive number K is an upper bound for $\{\|e^{iP}\| \mid P \in \mathscr{A}\}$. Define $|||\ |||$ on X as follows:

$$|||x||| = \sup_{P \in \mathscr{A}} \|e^{iP}x\| .$$

Clearly $|||x||| \leqq K\|x\|$. Also $\|x\| = \|e^{-iP}e^{iP}x\| \leqq K|||x|||$. Since $|||\ |||$ is obviously a seminorm, we can conclude that it is a norm equivalent to $\|\ \|$. For arbitrary $Q \in \mathscr{A}$ and arbitrary $x \in X$, we have:

$$|||e^{iQ}x||| = \sup_{P \in \mathscr{A}} \|e^{i(P+Q)}x\| .$$

Since $\mathscr{A}$ is obviously a real algebra, it is clear that as P ranges through $\mathscr{A}$, so does $P + Q$. Hence $|||e^{iQ}x||| = |||x|||$, and each operator e^{iQ}, for $Q \in \mathscr{A}$, is an isometry relative to $|||\ |||$. Thus if $Q \in \mathscr{A}$, the operators e^{itQ}, for real t, form a one-parameter group of isometries (relative to $|||\ |||$). Since the families $\{e^{itQ}\}$ and $\{e^{-itQ}\}$, with $t > 0$, are (in particular) semi-groups of contraction operators, we have by [5; Theorem 3.1] that the generators iQ and $-iQ$ are dissipative; hence Q is a hermitian operator relative to $|||\ |||$. In particular, each operator of the form R^mJ^n, where m and n are nonnegative integers, belongs to $\mathscr{A}$ and so is a hermitian operator relative to $|||\ |||$.

Using 4.2 and 4.3, we have:

4.4 THEOREM. *Let X be reflexive, and let T be an operator on X. Then T is a scalar type operator if and only if there exist operators R and J such that:*

(1) $T = R + iJ$.

(2) $RJ = JR$.

(3) *If* $\mathscr{A}$ *denotes the set of all polynomials in* R *and* J *with real coefficients, then* $\{e^{iP} \mid P \in \mathscr{A}\}$ *is a bounded subset of* $[X]$.

If this is the case, R *and* J *are uniquely determined.*

Note Added in Proofreading. G. Lumer has recently shown that the powers of a hermitian operator are not in general all hermitian, even on a reflexive space. This and other matters of interest to readers of this paper will be found in his forth-coming paper, *Spectral operators, hermitian operators, and bounded groups*, to appear in Acta Sci. Math., (Szeged).

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DIVISORIAL VARIETIES

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Introduction. The purpose of the present work is to introduce a new type of algebraic varieties, called Divisorial varieties. The name comes from the fact that the topology of these varieties is determined by their positive divisors. See §3 for a more detailed discussion of the above statement.

In the first two sections we lay the groundwork for our study. The result obtained in Proposition 2.2 is new, and constitutes a natural generalization of a well known result of Serre. (See [3], page 235, and Lemma 2, page 98 of [5]).

Section 3 is devoted to the study of the categorical properties of divisorial varieties. We prove that locally closed subvarieties of divisorial varieties are divisorial, and that products and direct sums of divisorial varieties are divisorial. Furthermore we give a characterization of divisorial varieties which shows how such varieties are a natural generalization of the notion of projective varieties.

We show in §4 that all quasi-projective, and all nonsingular varieties are divisorial. A procedure is also given for constructing a large class of divisorial varieties which are neither quasi-projective nor nonsingular, both reducible and irreducible ones.

In §5 we study the additive group of equivalence classes (under linear equivalence) of locally linearly equivalent to zero divisors of a divisorial variety. We show that such group is generated by the semigroup of those classes which contain some positive members. As a matter of fact the statement of Corollary 5.1 is more general than the one above, but we omit the details here for brevity's sake. The results of §5 are a generalization of the operation of "adding hyper-surface sections," well known to the Italian geometers for projective varieties.

Finally, in §6, we give one instance of a theorem which is known to be true for either quasi-projective or irreducible and nonsingular varieties, and show that it holds for divisorial varieties. The theorem considered, which we refer to as the polynomial theorem of Snapper, is Theorem 9.1 of [6], generalized by Cartier (See [1]) to either quasi-projective or irreducible and nonsingular varieties.

We believe that the notion of divisorial varieties represents a natural extension of the notion of quasi-projective varieties.

Our notation and terminology are essentially those of [3]. The word sheaf always means, unless other-wise specified, algebraic coherent

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sheaf. The symbol $\simeq$ is used to denote all sorts of isomorphisms, and the type has not been specified, unless there is danger of confusion. Whenever the expression $a \otimes a \otimes \cdots \otimes a$, m times, is meaningful, we shall denote it by $a^{(m)}$. When we refer to, say, Theorem 3.2, without any further designation, we mean Theorem 3.2 of the present work, to be found as the second theorem of the third section.

1. We wish to review briefly some of the ideas and theorems concerning the functorial properties of line classes; for a more detailed treatment see [6], §§1 to 5, and [7].

Let X denote an abstract algebraic variety, defined over an algebraically closed groundfield k. Let $\mathcal{O}_X$ denote the sheaf of local rings of X, and $\mathcal{O}_X^0$ the sheaf (not algebraic) of units of $\mathcal{O}_X$. The elements of the (multiplicative) first cohomology group $H^1(X, \mathcal{O}_X^0)$ are called the line classes of X.

Let $f \in H^1(X, \mathcal{O}_X^0)$ and let $\mathcal{U} = (U_i, I)$ be an indexed open covering of X which admits a 1-cocycle b with values in $\mathcal{O}_X^0$ which represents f. We shall briefly say that the system $(\mathcal{U}, b)$ represents f.

If F is an algebraic sheaf over X, there exists a uniquely defined (up to $\mathcal{O}_X$-isomorphisms) algebraic sheaf K, and local isomorphisms u_i: $K \mid U_i \to F \mid U_i$, such that, for every $x \in U_i \cap U_j$ and $a \in F_x$,

$$(u_i u_j^{-1})(a) = [b(i, j)(x)] \cdot a \ .$$

The sheaf K depends only upon F and f, while, of course, the local isomorphisms u_i depend upon the choice of the system $(\mathcal{U}, b)$. We denote the sheaf K by $f(F)$.

In this way f can be looked upon as a functor from the category of (classes of $\mathcal{O}_X$-isomorphic) algebraic sheaves and (classes of equivalent) $\mathcal{O}_X$-homomorphisms into the same category. Such functor is covariant and exact. Furthemore, if F and G are two algebraic sheaves over X, and f and g are two line classes of X, then

$$f(F) \otimes \sigma_X g(G) \simeq fg(F \otimes \sigma_X G) \ ,$$

where fg denotes the product in the group of line classes.

Since F and $f(F)$ are locally isomorphic sheaves, if F is of finite type or coherent so is $f(F)$, and conversely. Furthermore the stalk of $f(\mathcal{O}_X)$ over any point $x \in X$ has a unique maximal submodule, which we shall denote by n_x, corresponding to the unique maximal ideal m_x of $\mathcal{O}_{x,X}$.

2. **Sections of $f(\mathcal{O}_X)$.** We shall keep the same notation as in the previous section. Furthermore, for every sheaf F over X, and any subset U of X, we shall denote by $\Gamma(U, F)$ the set of sections of F over U.

PROPOSITION 2.1. Let X be an abstract algebraic variety, f a line class of X, and $s \in \Gamma[X, f(\mathcal{O}_X)]$. Then the set

$$X_s = \{x \in X \mid s(x) \notin n_x\}$$

is an open subset of X.

Proof. Let $(\mathcal{U}, b)$ be a system representing f, where $\mathcal{U} = (U_i, I)$. Let $u_i: f(\mathcal{O}_X) \mid U_i \to \mathcal{O}_X \mid U_i$ be the local isomorphisms as in §1. If $x \in U_i$ then, by the definition of n_x, $x \in X_s \cap U_i$ if, and only if,

$$(u_i \circ s)(x) \notin m_x$$

or, equivalently, if, and only if, $(u_i \circ s)(x) \in \mathcal{O}_X^\circ, X$. Since $\mathcal{O}_X^\circ$ is open in $\mathcal{O}_X$, and since the u_i is a local homeomorphism, $X_s \cap U_i$ is open in U_i. This proves the proposition.

The following proposition generalizes Proposition 5 of §43 of [3], as well as Lemma 2, page 98 of [5].

PROPOSITION 2.2. Let X be an abstract algebraic variety, f and g two line classes of X, U an open subset of X. Let $s \in \Gamma[X, g(\mathcal{O}_X)]$ and $t \in \Gamma[U, f(\mathcal{O}_X)]$ be given, such that $X_s \subset U$. Then, for a sufficiently high integer n, there exists a section $s^* \in \Gamma[X, fg^n(\mathcal{O}_X)]$, such that $s^* = t \otimes s^{(n)}$ on X_s.

Proof. Let $\mathcal{U} = (U_i, I)$, $\mathcal{W} = (V_\alpha, A)$ be open affine coverings of X which admit 1-cocycles with values in $\mathcal{O}_X^\circ$ representing f and g respectively. We may assume that $\mathcal{W}$ is a refinement of $\mathcal{U}$. Let u_i and v_α be the usual local isomorphisms. Since $X_s \subset U$ we have that $t \in \Gamma[X_s, f(\mathcal{O}_X)]$. Let:

$$t_\alpha = v_\alpha^{-1}(1) \quad ; \quad f_\alpha = v_\alpha \circ t$$
$$s_i = u_j^{-1}(1) \quad ; \quad g_i = u_i \circ s$$

Let $V_\alpha \subset U_i$. Observe that g_i is regular on U_i, and that

$$X_s \cap U_i = \{x \in U_i \mid g_i(x) \notin m_x\} .$$

Since V_α is affine and f_α is defined on $X_s \cap V_\alpha$, we see by Lemma 1 of §55 of [3] that there exists a sufficiently large integer m_α and a section

$$h_\alpha \in \Gamma(V_\alpha, \mathcal{O}_X)$$

such that

$$h_\alpha = f_\alpha \cdot g_i^{m_\alpha}$$

on $V_\alpha \cap X_s$. Since X is compact (we do not include T_2 in the definition of compactness) we may assume all m_α's to be equal, and denote their common value by m.

Let now $s'_\alpha \in \Gamma[V_\alpha, fg^m(\mathcal{O}_X)]$ be defined as follows:

$$s'_\alpha = h_\alpha \cdot (t_\alpha \otimes s_i^{(m)}) .$$

Clearly $s'_\alpha - s'_\beta = 0$ on $V_\alpha \cap V_\beta \cap X_s$. Hence, since g_i is regular on $V_\alpha \cap V_\beta$, we have that the section

$$(s'_\alpha - s'_\beta) \otimes s \in \Gamma[V_\alpha \cap V_\beta, fg^{m+1}(\mathcal{O}_X)]$$

is 0 on $V_\alpha \cap V_\beta$. Therefore the system of sections $s'_\alpha \otimes s$ defines a unique section s^* of $fg^{m+1}(\mathcal{O}_X)$ over X. On $V_\alpha \cap X_s$ we have:

$$\begin{aligned} s^* &= s'_\alpha \otimes s = h_\alpha \cdot t_\alpha \otimes s_i^{(m)} \otimes s \\ &= g_i^m \cdot f_\alpha \cdot t_\alpha \otimes s_i^{(m)} \otimes s = t \otimes s^{(m+1)} \end{aligned}$$

which finishes the proof.

COROLLARY 2.1. *Let X be an abstract algebraic variety, g a line class of X, $s \in \Gamma[X, g(\mathcal{O}_X)]$, h a regular function on X_s. Then, for a sufficiently high integer n, the section $h \cdot s^{(n)}$ can be extended to X.*

Proof. Let $f = 1$, $t = h$, $U = X_s$ in the above proposition.

REMARK. Let X be an irreducible, normal algebraic variety, with constant sheaf (not coherent) of rational functions denoted by E. There exists a group isomorphism between the multiplicative group $H^1(X, \mathcal{O}_X^\circ)$ and the additive group of equivalence classes (under linear equivalence) of locally linearly equivalent to zero divisors of X. If g is a line class of X shall denote by $|g|$ the equivalence class of divisors which corresponds to it. Then there exists an isomorphism between $\Gamma[X, g(E)]$ and $|g|$. Sections of $g(\mathcal{O}_X)$ over X correspond to the positive members of $|g|$. See [6], §5 for the proof of the above statements.

The geometrical meaning of Proposition 2.2 is then the following: if D is a locally linearly equivalent to zero divisor of X, such that the variety of its negative components is contained in the variety of some positive divisor (also locally linearly equivalent to zero), say P, then, for a sufficiently high integer n the divisor $D + nP$ is locally linearly equivalent to zero and positive.

PROPOSITION 2.3. Let X be an abstract algebraic variety, Y a locally closed subvariety of X. Then there exists a homomorphism

$$\varphi_n \colon H^n(X, \mathcal{O}_X^\circ) \longrightarrow H^n(Y, \mathcal{O}_Y^\circ) , \qquad n = 0, 1, \cdots$$

and, for every $f \in H^1(X, \mathcal{O}_X^\circ)$, there exists a homomorphism

$$\varphi_f\colon \Gamma[X, f(\mathcal{O}_X)] \longrightarrow \Gamma[Y, \varphi_1(f)(\mathcal{O}_Y)]$$

such that, for every $s \in \Gamma[X, f(\mathcal{O}_X]$,

$$Y \cap X_s = Y_{\varphi_f(s)} .$$

Proof. There exists a unitary ring epimorphism $\varphi\colon \mathcal{O}_X \mid Y \to \mathcal{O}_Y$, hence a sheaf homomorphism $\varphi'\colon \mathcal{O}_X^\circ \mid Y \to \mathcal{O}_Y^\circ$. This proves the existence of the homomorphisms φ_n.

Let now $f \in H^1(X, \mathcal{O}_X^\circ)$. Let $(\mathscr{U}, b)$ be a system which represents f, where $\mathscr{U} = (U_i, I)$. The system $(\mathscr{U}', b')$, where $\mathscr{U}' = (U_i \cap Y, I)$ and $b'(i, j) = \varphi' \circ b(i, j)$, represents $\varphi_1(f)$. Let

$$u_i\colon f(\mathcal{O}_X) \mid U_i \longrightarrow \mathcal{O}_X \mid U_i$$
$$u_i'\colon \varphi_1(f)(\mathcal{O}_Y) \mid Y \cap U_i \longrightarrow \mathcal{O}_Y \mid Y \cap U_i$$

be the usual local isomorphisms. Let $s \in \Gamma[X, f(\mathcal{O}_X)]$. We define $\varphi_f(s)$ by the formula:

$$\varphi_f(s)(x) = (u_i'^{-1} \circ \varphi \circ u_i \circ s)(x) \qquad x \in Y \cap U_i .$$

We easily verify that $\varphi_f(s)(x)$ does not depend on the index i. We now assert that $\varphi_f(s)$ does not depend on the particular system $(\mathscr{U}, b)$ chosen to represent f. Let therefore $(\mathscr{W}, c)$ be another such system, where $\mathscr{W} = (V_j, J)$. We proceed in steps.

Case 1. $\mathscr{W}$ is a refinement of $\mathscr{U}$, the mapping $t\colon J \to I$ is such that $c = t^*(b)$. From [6], Case 1 of Proposition 2.1 we know that the usual isomorphisms $v_j\colon f(\mathcal{O}_X) \mid V_j \to \mathcal{O}_X \mid V_j$ can be chosen in such a manner that $u_{t(j)} = v_j$ on V_j. The system $(\mathscr{W}', c')$, where $\mathscr{W}' = (Y \cap V_j, J)$ and $c'(j, j') = \varphi' \circ c(j, j')$, clearly represents $\varphi_1(f)$, hence we can furthermore choose the isomorphisms

$$v_j'\colon \varphi_1(f)(\mathcal{O}_Y) \mid Y \cap V_j \to \mathcal{O}_Y \mid Y \cap V_j$$

in such a manner that $u_t'(j) = v_j'$ on $Y \cap V_j$.

Hence, if $x \in Y \cap V_j$ we have

$$(u_{t(j)}'^{-1} \circ \varphi \circ u_{t(j)} \circ s)(x) = (v_j'^{-1} \circ \varphi \circ v_j \circ s)(x)$$

which finishes the proof of Case 1.

Case 2. $\mathscr{U} = \mathscr{W}$, b and c cohomologous. Hence there exists a 0-cochain e of $\mathscr{U}$, with values in $\mathcal{O}_X^\circ$, such that $b^{-1}c$ is the coboundary of e. We can hence choose the isomorphisms v_i in such a manner that, if $x \in U_i$, then $u_i = e(i)(x) \cdot v_i$, on the stalk of $f(\mathcal{O}_X)$ over x. Let

$e' = \varphi_0(e)$. Then it is easily seen that $b'^{-1}c'$ is the coboundary of e', hence, if $x \in Y \cap U_i$, $u'_i = e'(i)(x) \cdot v'_i$, on the stalk of $\varphi_1(f)(\mathscr{O}_Y)$ over x. Hence we have that $v'^{-1}_i = e'(i)(x) \cdot u'^{-1}_i$, and a trivial computation now finishes the proof of Case 2.

Case 3. The systems $(\mathscr{U}, b)$ and $(\mathscr{W}, c)$ are arbitrary. Let $\mathscr{W}'$ be a common refinement of $\mathscr{U}$ and $\mathscr{W}$. Hence there exist two cohomologous 1-cocycles of $\mathscr{W}'$ with values in $\mathscr{O}^\circ_X$, say g and h, such that the systems $(\mathscr{W}', g)$ and $(\mathscr{W}', h)$ represent f, and the pairs (b, g) and (c, h), with their respective coverings, fall under Case 1. Furthermore the pair (g, h) falls under Case 2, and this finishes the proof of Case 3.

The map φ_f is now easily seen to be a homomorphism.

It remains to prove that $Y \cap X_s = Y_{\varphi_f(s)}$. From the definition of $\varphi_f(s)$ we see immediately that, for $x \in Y$, $u_i[s(x)] \notin m_x$ implies $\varphi_f(s)(x)u'^{-1}_i(m'_x)$, where m'_x denotes the unique maximal ideal of $\mathscr{O}_{x,Y}$. Hence $Y \cap X_s$ is contained in $Y_{\varphi_f(s)}$. Conversely, if we have $[u'_i \circ \varphi_f(s)](x) \notin m'_x$, then $(\varphi \circ u_i \circ s)(x) \notin m'_x$, and since $\varphi^{-1}(m'_x) = m_x$, we have $(u_i \circ s)(x) \notin m_x$. Therefore $Y_{\varphi_f(s)} \subset Y \cap X_s$, which completes the proof of the proposition.

3. **Divisorial varieties.** Let X be an abstract algebraic variety, and let G_X denote the collection of open subsets of X. We define

$$B_X = \{U \in G_X \mid U = X_s,\ s \in \Gamma[X, g(\mathscr{O}_X)],\ g \in H^1(X, \mathscr{O}^\circ_X)\}$$

DEFINITION 3.1. An abstract algebraic variety X is called divisorial if B_X constitutes a base for the topology of X.

REMARK. Keeping in mind the remark of the previous section, the geometrical meaning of our definition becomes clear. If Y is irreducible and divisorial, then, for every point $x \in X$ and every closed subset Y of X, not containing x, there exists a positive divisor of X, which is locally linearly equivalent to zero and whose variety contains Y but not x. In other words the topology of X is entirely determined by the positive, locally linearly equivalent to zero divisors. This justifies our terminology.

We now begin the study of the categorical properties of divisorial varieties.

THEOREM 3.1. *Let X be a divisorial algebraic variety, and Y a locally closed subvariety of X. Then Y is a divisorial algebraic variety.*

Proof. Let U' be an open subset of Y and let $x \in U'$. Let U

be an open subset of X such that $U' = Y \cap U$. Since X is divisorial there exist a line class f of X and a section $s \in \Gamma[X, f(\mathcal{O}_X)]$ such that $x \in X_s \subset U$. By Proposition 2.3 the section $\varphi_f(s)$ of the sheaf $\varphi_1(f)(\mathcal{O}_Y)$ over Y is such that

$$Y_{\varphi_f}(s) = Y \cap X_s .$$

Hence $x \in Y_{\varphi_f}(s) \subset U'$, which proves the theorem.

THEOREM 3.2. *The direct sum of divisorial varieties is divisorial.*

Proof. Let X be the direct sum of $X_1, X_2, \cdots, X_n$. It is easily seen that

$$H^1(X, \mathcal{O}_X^\circ) \cong \prod_{i=1}^{n} H^1(X_i, \mathcal{O}_{X_i}^\circ)$$

where the product on the right hand side is direct. Furthermore, if $f_r \in H^1(X_r, \mathcal{O}_{X_r}^\circ)$, and $s_r \in \Gamma[X_r, f_r(\mathcal{O}_{X_r})]$, then the rule

$$s(x) = \begin{cases} s_r(x) & \text{if } x \in X_r \\ 0 & \text{otherwise} \end{cases}$$

defines a section of $(1 \times 1 \times \cdots \times f_r \times \cdots \times 1)(\mathcal{O}_X)$ over X such that $X_s = X_{s_r}$. This proves the theorem.

Before proving that the category of divisorial varieties is a category with product, we need to prove the following very useful characterization of divisorial varieties.

THEOREM 3.3. *Let X be an abstract algebraic variety. A necessary and sufficient condition for X to be divisorial is the following: there exists an open affine covering $\mathcal{U} = (U_i, I)$ of X, line classes $g_1, g_2, \cdots, g_m$ of X, and sections $s_j \in \Gamma[X, g_j(\mathcal{O}_X)]$, $j = 1, 2, \cdots, m$, such that the collection of open sets $\{X_{s_j}, j = 1, 2, \cdots, m\}$ constitutes a covering of X which refines $\mathcal{U}$.*

Proof. The condition is obviously necessary, as it suffices to consider any open affine covering of X, and then use the fact that B_X is a base for the topology of X, and that X is compact.

To prove the sufficiency, let $x \in X$, and let Y be a closed subset of X, not containing x. Let $x \in X_{s_p}$ and $X_{s_p} \subset U_i$. Since U_i is affine, there exists a section h of $\mathcal{O}_X$ over X_{s_p} such that

$$h(x) \notin m_x ; \qquad h(y) \in m_y, \qquad y \in Y \cap X_{s_p} .$$

By Corollary 2.1 there exists a sufficiently high integer n such that the section $h \cdot s_p^{(n)}$ extends to a section $s*$ of $g_p^n(\mathcal{O}_X)$ over X.

Since $s_p(x) \notin n_x$, and $h(x) \notin m_x$, we have

$$s*(x) = h(x) \cdot s_p^{(n)}(x) \notin n'_x$$

where n'_x denotes the unique maximal submodule of $[g_p^n(\mathcal{O}_X]_x$. Furthermore, if $y \in Y \cap X_{s_p}$

$$s*(y) = h(y) \cdot s_p^{(n)}(y) \in n'_y .$$

Finally, n can be chosen high enough so that, if $y \notin X_{s_p}$, then $s*(y) \in n'_y$. Hence $x \in X_{s*} \subset X - Y$, and the proof is finished.

The above proof immediately yields the following corollary.

COROLLARY 3.1. *A necessary and sufficient condition for X to be divisorial is that there exists a finite number of line classes of X, say $g_1, g_2, \cdots, g_m$, such that the collection of open sets $\{X_s\}$, where s ranges among the sections over X of $g_j^n(\mathcal{O}_X)$, $j = 1, 2, \cdots, m$; $n = 1, 2, \cdots$, form a base for the topology of X.*

Proof. The condition is obviously sufficient. If X is divisorial, the proof of the above theorem shows that the line classes given by the criterion in the theorem satisfy the condition stated.

REMARK. Corollary 3.1 shows that the notion of divisorial variety is an extension of the notion of quasi-projective varieties in a natural way. In fact every quasi-projective variety satisfies the condition stated in the Corollary, *with only one line class*, namely the line class p of hyperplane sections, (sections of $p^n(\mathcal{O}_X)$ over X correspond to hypersurface sections) which was introduced by Serre in [3], §54, page 246.

We believe that a slight modification of the condition stated in Corollary 3.1, *with only one line class*, will yield a characterization of quasi-projective varieties.

The above reasoning already shows that every quasi-projective variety is divisorial. We shall give another proof of the same statement in the next section.

Let X, Y be abstract algebraic varieties. There exists a natural monomorphism

$$\mu\colon H^1(X, \mathcal{O}_X^{\circ}) \longrightarrow H^1(X \times Y, \mathcal{O}_{X \times Y}^{\circ})$$

and, for every $g \in H^1(X, \mathcal{O}_X^{\circ})$ a monomorphism

$$\mu_g\colon \Gamma[X, g(\mathcal{O}_X)] \longrightarrow \Gamma[X \times Y, \mu(g)(\mathcal{O}_{X \times Y})]$$

such that

$$(X \times Y)_{\mu_g(s)} = X_s \times Y .$$

The proof of the above statements is entirely straightforward, and we omit it here for brevity's sake. In what follows we will identify $H^1(X, \mathcal{O}_X^\circ)$ and $\Gamma[X, g(\mathcal{O}_X)]$ with their images in $H^1(X \times Y, \mathcal{O}_{X\times Y}^\circ)$ and $\Gamma[X \times Y, \mu(g)(\mathcal{O}_{X\times Y})]$ respectively. Similarly for Y.

THEOREM 3.4. *The product of divisorial varieties is a divisorial variety.*

Proof. Let X, Y be divisorial varieties. We shall use the criterion of Theorem 3.3. Accordingly, let $\mathscr{U} = (U_i, I)$, $g_1, g_2, \cdots, g_m, s_1, s_2, \cdots, s_m$ and $\mathscr{W} = (V_j, J)$, $h_1, h_2, \cdots, h_r, t_1, t_2, \cdots, t_r$ be the affine open coverings, line classes and sections satisfying the condition of Theorem 3.3 for X and Y respectively. Observe that:

$$\begin{aligned}(X \times Y)_{s_p \otimes t_q} &= (X \times Y)_{s_p} \cap (X \times Y)_{t_q} \\ &= (X_{s_p} \times Y) \cap (X \times Y_{t_q}) = X_{s_p} \times Y_{t_q}\end{aligned}$$

for all values of p from 1 to m and of q from 1 to r.

Hence the open affine covering

$$(U_i \times V_j, I \times J)$$

of $X \times Y$, the line classes $g_p h_q$ and the sections $s_p \otimes t_q$, $p = 1, \cdots, m$ and $q = 1, \cdots, r$, satisfy the condition of Theorem 3.3 applied to $X \times Y$. Hence $X \times Y$ is divisorial.

4. Existence of divisorial varieties. As we have already seen in the previous section, all quasi-projective varieties are divisorial. We shall show in the present section that the category of divisorial varieties also includes all nonsingular varieties and lots more.

We call an abstract algebraic variety factorial if the local ring of every one of its points is a unique factorization domain. As Zariski has shown in [9], all nonsingular varieties are factorial.

In what follows, if h is a rational function on an irreducible variety X, we shall denote by (h) the divisor of the function h on X.

THEOREM 4.1. *Every irreducible factorial variety is divisorial.*

Proof. Let X be an irreducible factorial variety, whose function field we shall denote by E. For every irreducible subvariety W of X, we denote by $\mathcal{O}_W$ the local ring of W in E.

Let U be an open subset of X, and let $x \in U$. We proceed in steps.

Case 1. $W = X - U$ is an irreducible subvariety of X. Since

$x \notin W$, it follows that $\mathcal{O}_x \not\subset \mathcal{O}_W$. Let hence $h \in E$ be such that $h \in \mathcal{O}_x$ and $h \notin \mathcal{O}_W$. Let $(h) = D_1 - D_2$, where D_1 and D_2 denote respectively the zeros and poles of the function h. Since $h \in \mathcal{O}_x$, we have $x \notin \mathrm{Var}(D_2)$, where $\mathrm{Var}(D)$ denotes the variety of the divisor D. Furthermore $y \in W$ implies $h \notin \mathcal{O}_y$, hence, since X is normal, $y \in \mathrm{Var}(D_2)$. Therefore $W \subset \mathrm{Var}(D_2)$. Since X is factorial, D_2 is locally linearly equivalent to zero, i.e. there exists an open covering $\mathcal{U} = (U_i, I)$ of X and rational functions $h_i \in E$, such that h_i is regular on U_i and $(h_i) = D_2$ on U_i. Hence, since X is normal, $(h_i/h_j) = 0$ on $U_i \cap U_j$ implies that the system h_i/h_j defines a 1-cocycle of $\mathcal{U}$ with values in $\mathcal{O}_X^\circ$. Let g be the line class of X represented by the system $(\mathcal{U}, h_i/h_j)$, and let u_i: $g(\mathcal{O}_X) \mid U_i \to \mathcal{O}_X \mid U_i$ be the usual isomorphisms. If we define $s(y) = (u_i^{-1} \circ h_i)(y)$, for $y \in U_i$, we clearly obtain a section s of $g(\mathcal{O}_X)$ over X such that

$$X_s \cap U_i = U_i - \mathrm{Var}(D_2) .$$

Hence $x \in X_s = X - \mathrm{Var}(D_2) \subset U$.

Case 2. $W = X - U$ is arbitrary. Let $W_1, W_2, \cdots, W_p$ be the irreducible components of W. From Case 1 we know that there exist line classes $g_1, \cdots, g_p$ of X and sections $s_i \in \Gamma[X, g_i(\mathcal{O}_X)]$ $i = 1, \cdots, p$, such that $x \in X_{s_i} \subset X - W_i$. We easily verify that the section

$$s = s_1 \otimes s_2 \otimes \cdots \otimes s_p \in \Gamma[X, g_1 g_2 \cdots g_p(\mathcal{O}_X)]$$

is such that $X_s = \bigcap_{i=1}^{p} X_{s_i}$, hence $x \in X_s \subset U$. This finishes the proof *of the theorem.*

THEOREM 4.2. *Every factorial variety is divisorial.*

Proof. By definition, every unique factorization domain is an integral domain. Hence every factorial variety is the direct sum of its irreducible components, which, by Theorem 4.1, are all divisorial. Then we apply Theorem 3.2.

THEOREM 4.3. *Every quasi-projective variety is divisorial.*

Proof. Projective space is nonsingular, hence divisorial. Then we apply Theorem 3.1.

THEOREM 4.4. *There exist divisorial varieties which are neither quasi-projective nor nonsingular, of any dimension* > 3.

Proof. There exist nonsingular, nonprojective varieties of any dimension > 2. (See [2]). We use any singular, quasi-projective variety, and apply Theorem 3.4.

REMARK. The above theorems provide us with a large class of divisorial varieties. It is not settled at the moment, though, whether there are divisorial surfaces which are not projective. Such surfaces must necessarily be singular, as it follows from the fact that every nonsingular surface is quasi-projective (See [8]).

For an example of a normal, nonprojective surface see [2], page 492.

5. The group of line classes of a divisorial variety. Let X be an abstract algebraic variety. As in [6], §4, we shall call *regular* any line class g of X such that, for some $s \in \Gamma[X, g(\mathcal{O}_X)]$, $X_s \neq \phi$. Let x be a fixed point of X. A regular line class g is called *free at* x if, for some $s \in \Gamma[X, g(\mathcal{O}_X)]$, $x \in X_s$. The set of line classes which are free at x is easily shown to form a subsemi-group of $H^1(X, \mathcal{O}_X^\circ)$, which we shall denote by L_x.

The following proposition generalizes the well known operation of "adding hypersurface sections." (See [6], Proposition 8.2.).

Let X be a divisorial algebraic variety, and let

$$\mathcal{U} = (U_i, I), g_1, \cdots, g_m, s_1, \cdots, s_m$$

be the open affine covering, line classes and sections satisfying the criterion of Theorem 3.3.

PROPOSITION 5.1. Let X be a divisorial algebraic variety, f a line class of X, x a fixed point of X. Then, for a sufficiently high integer n, and for some integer p between 1 and m, the line class fg_p^n is regular and free at x.

Proof. For a suitable open subset U of X, containing x, we can find a section $t \in \Gamma[U, f(\mathcal{O}_X)]$ such that $t(x) \notin n_x$. By Corollary 3.1 there exist an integer p, with $1 \leqq p \leqq m$, and a sufficiently high integer q such that the sheaf $g_p^q(\mathcal{O}_X)$ has a section s over X with $x \in X_s \subset U$.

Applying Proposition 2.2 to the line classes f and g_p^q, and their respective sections t and s, we see that, for a sufficiently high integer q' the section $t \otimes s^{(q')}$ extends to a section s^* of $fg_p^{qq'}(\mathcal{O}_X)$ over X. We have:

$$s^*(x) = t(x) \otimes s^{(q')}(x) \notin n'_x$$

where n'_x denotes the unique maximal submodule of $[fg_p^{qq'}(\mathcal{O}_X)]_x$. Hence $x \in X_{s^*}$, which finishes the proof of the proposition.

COROLLARY 5.1. *Let X be divisorial, and $x \in X$. The group generated by L_x in $H^1(X, \mathcal{O}_X^\circ)$ is $H^1(X, \mathcal{O}_X^\circ)$.*

Proof. By the above proposition, for any f in $H^1(X, \mathcal{O}_X^\circ)$, $fg_p^n \in L_x$. Clearly $g_p^n \in L_x$.

6. The polynomial theorem of Snapper. Let λ be an additive sheaf function, i.e. a function defined over the category of sheaves, with values in an arbitrary abelian group G, and such that the exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

implies $\lambda(F) = \lambda(F') + \lambda(F'')$. (See [5], §4, page 105, or [1], §3.).

The following theorem is an extension to divisorial varieties of the polynomial theorem proved by Snapper in [6], Theorem 9.1, as well as the more general form given by Cartier in [1], §4.

THEOREM 6.1. *Let λ be an additive sheaf function, and X a divisorial algebraic variety. Then, for every sheaf F over X and every finite set of line classes $f_1, \cdots, f_n$ of X, the expression*

$$\lambda[f_1^{m_1} \cdots f_n^{m_n}(F)]$$

is a polynomial in $m_1, \cdots, m_n$ of degree at most $\dim(\operatorname{Supp} F)$.

Proof. The theorem is an immediate consequence of the following lemma, which generalizes the theorem given in §3 of [1]. The formal algorithm used in §4 of the same paper, identically repeated, proves our theorem. Therefore we limit ourselves to the proof of the lemma.

LEMMA 6.1. *Let X be a divisorial algebraic variety, λ an additive sheaf function, g any line class of X. If $\lambda(F) = 0$ for every sheaf F such that* $\dim(\operatorname{Supp} F) < r$, *then* $\lambda(F) = \lambda[g(F)]$ *for every sheaf F with* $\dim(\operatorname{Supp} F) \leqq r$.

Proof. We proceed in steps.

Case 1. We assume $\dim(\operatorname{Supp} F) < r$. Since F and $g(F)$ are locally isomorphic we have $\dim(\operatorname{Supp} g(F)) < r$, hence

$$\lambda(F) = 0 = \lambda[g(F)]\ .$$

Case 2. We assume $\operatorname{Supp} F \subset S$, where S is an irreducible closed subset of X, and $\dim S \leqq r$. Let $x \in S$. Since X is divisorial, by

Corollary 5.1 we can write $g = f_1/f_2$, where $f_i \in L_x$, $i = 1, 2$. Let therefore $s_i \in \Gamma[X, f_i(\mathcal{O}_X)]$ be such that $x \in X_{s_i}$, $i = 1, 2$. We now define

$$\omega_i \colon F \longrightarrow f_i(F) \qquad i = 1, 2 .$$

as follows:

$$\omega(a) = a \otimes s_i(y) \qquad a \in F_y .$$

Since $x \in X_{s_i}$ we see that $s_i(x)$ generates the stalk of $f_i(\mathcal{O}_X)$ over x, hence ω_i induces an isomorphism on F_x. Therefore Supp (ker ω_i) and Supp (coker ω_i) are proper closed subsets of S, hence

$$\lambda(\ker \omega_i) = \lambda(\operatorname{coker} \omega_i) = 0 .$$

Since λ is additive, the exact sequence

$$0 \longrightarrow \ker \omega_i \longrightarrow F \longrightarrow f_i(F) \longrightarrow \operatorname{coker} \omega_i \longrightarrow 0$$

shows that $\lambda(F) = \lambda[f_i(F)]$. Let $F' = g(F)$. Then F and F' are locally isomorphic, hence $\operatorname{Supp} F = \operatorname{Supp} F'$. Hence, by the above proof applied to F', we obtain:

$$\lambda[g(F)] = \lambda(F') = \lambda[f_2(F')] = \lambda[f_2 g(F)] = \lambda[f_1(F)] = \lambda(F) .$$

Case 3. We only assume $\dim(\operatorname{Supp} F) \leqq r$. Let S_i be the irreducible components of $\operatorname{Supp} F$, and let T be the union of the closed sets $S_i \cap S_j$, for $i \neq j$. We have $\dim S_i \leqq r$, and $\dim T < r$. From [4], page 11, we know that there exist sheaves F_i, G, such that $\operatorname{Supp} F_i \subset S_i$, and $\operatorname{Supp} G \subset T$, and that there exists an exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow \sum F_i \longrightarrow 0_i$$

where the sum at right is direct. Applying Case 2 to each pair (F_i, S_i) we get $\lambda(F_i) = \lambda[g(F_i)]$, and from the exact sequence

$$0 \longrightarrow g(G) \longrightarrow g(F) \longrightarrow g(\sum F_i) \longrightarrow 0$$

we get $\lambda[g(F)] = \lambda[g(\sum F_i)]$. Hence:

$$\begin{aligned} \lambda(F) &= \lambda(\sum F_i) = \sum \lambda(F_i) = \sum \lambda[g(F_i)] \\ &= \lambda[\sum g(F_i)] = \lambda[g(\sum F_i)] = \lambda[g(F)] . \end{aligned}$$

This finishes the proof of the lemma.

Final Remark. We wish to point out the following question, which stems from the above study of divisorial varieties:

If a divisorial variety X has a line class g such that, for any finite set of points $P_1, \cdots, P_n$ of X, there exists an open affine subset

X_s, $s \in \Gamma[X, g(\mathscr{O}_X)]$, containing them, is then X quasi projective?

The above question is more restrictive, in a natural way, than the original one asked by Chevalley, (See [2], footnote to Introduction), and we believe the answer to be in the affirmative.

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STRONGLY REGULAR GRAPHS, PARTIAL GEOMETRIES AND PARTIALLY BALANCED DESIGNS

R. C. BOSE

0. Summary. This paper introduces the concept of a *partial geometry*, which serves to unify and generalize certain theorems on embedding of nets, and uniqueness of association schemes of partially balanced designs, by Bruck, Connor, Shrikhande and others. Certain lemmas and theorems are direct generalizations of those proved by Bruck [5], for the case of nets, which are a special class of partial geometries.

1. Introduction. We use graph theoretic methods for the study of association schemes of partially balanced incomplete block (PBIB) designs. For this purpose it is convenient to switch from graph theoretic language to the language of designs and vice versa as necessary.

As we shall be concerned with finite graphs only, we shall use the word graph in the sense of finite graphs.

A graph G with v vertices is said to be regular if each vertex is joined to n_1 other vertices, and unjoined to n_2 other vertices. Clearly

$$v - 1 = n_1 + n_2 . \tag{1.1}$$

If further any two joined vertices of G, are both joined to exactly p^1_{11} other vertices, and any two unjoined vertices are both joined to exactly p^2_{11} other vertices, then the graph G is defined to be *strongly regular*, with parameters

$$n_1, n_2, p^1_{11}, p^2_{11} . \tag{1.2}$$

The concept of strongly regular graphs is isomorphic with the concept of association schemes of PBIB designs (with two associate classes), which was first introduced by Bose and Shimamoto [4]. Such a scheme they defined as a scheme of relations between v treatements such that

(i) any two objects are either first associates or second associates

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(ii) each treatement has n_i ith associates $(i = 1, 2)$

(iii) If two treatments are ith associates, then the number of treatments common to the jth associates of the first and kth associates of the second is p^i_{jk} and is independent of the pair of treatments with which we start. Also $p^i_{jk} = p^i_{kj}$.

Bose and Clatworthy [1] showed that it is unnecessary to assume the constancy of all the p^i_{jk}'s. If we assume that n_1, n_2, p^1_{11} and p^2_{11} are constant, then the constancy of the $p^1_{21}, p^1_{22}, p^2_{12}, p^2_{21}, p^2_{22}$ follows and $p^1_{12} = p^1_{21}, p^2_{12} = p^2_{21}$.

If we now identify the v treatments of the association scheme with the v vertices of a graph G, and consider two vertices as joined or unjoined according as the corresponding treatments are first or second associates, it is clear that a strongly regular graph G with the parameters (1.2) is isomorphic with an association scheme with the same parameters.

We have introduced for the first time in this paper the concept of a *partial geometry*.

A partial geometry (r, k, t) is a system of undefined *points* and *lines*, and an undefined relation *incidence* satisfying the axioms A1 - A4. To avoid cumbersome expression we may use standard geometric language. Thus a point incident with a line may be said to lie on it and the line may be said to pass through the point. If two lines are incident with the same point, we say that they intersect.

A1. Any two points are incident with not more than one line.

A2. Each point is incident with r lines.

A3. Each line is incident with k points.

A4. If the point P is not incident with the line l, there pass through P exactly t lines $(t \geqq 1)$ intersecting l.

We show that the number of points v and the number of lines b in the partial geometry (r, k, t) are given by

$$v = k[(r-1)(k-1)+t]/t \,, \tag{1.3}$$

$$b = r[(r-1)(k-1)+t]/t \,. \tag{1.4}$$

The graph G of a partial geometry is defined as a graph whose vertices correspond to the points of the geometry, and in which two vertices are joined or unjoined according as the corresponding points are incident or non-incident with a common line. We then prove:

THEOREM. *The graph G of a partial geometry (r, k, t) is strongly regular with parameters.*

$$n_1 = r(k-1), \qquad n_2 = (r-1)(k-1)(k-t)/t \,, \tag{1.5}$$

$$p^1_{11} = (t-1)(r-1)+k-2, \qquad p^2_{11} = rt \,. \tag{1.6}$$

where

$$1 \leqq t \leqq r\ , \qquad 1 \leqq t \leqq k\ . \tag{1.7}$$

Using certain theorems of Bose and Mesner [3] relating to association schemes, we derive:

THEOREM. *A necessary condition for the existence of a partial geometry* (r, k, t) *is that the number*

$$\alpha = \frac{rk(r-1)(k-1)}{t(k+r-t-1)}\ , \tag{1.8}$$

is integral.

A strongly regular graph with parameters (1.5), (1.6) and for which (1.7) is satisfied will be defined to be a *pseudo-geometric* graph with characteristics (r, k, t). Such a graph may or may not be the graph of a partial geometry (r, k, t). It is therefore of interest to study the conditions under which a strongly regular graph, and in particular a pseudo-geometric graph with characteristic (r, k, t) is the graph of a partial geometry (r, k, t).

A subset of vertices of a graph G, any two of which are joined is called a *clique* of G. When G is the graph of a partial geometry (r, k, t) there will exist in G a set Σ of distinct cliques, $K_1, K_2, \cdots$, K_b corresponding to the lines of the geometry satisfying the following axioms

A*1. Any two joined vertices of G are contained in one and only one clique of Σ.

A*2. Each vertex of G is contained in r cliques of Σ.

A*3. Each clique of Σ contains k vertices of G.

A*4. If P is a vertex of G not contained in a clique K_i of Σ there are exactly t vertices in K_i, which are joined to $P (i = 1, 2, \cdots, l)$.

Hence any graph G in which there exists a set Σ of cliques K_1, $K_2, \cdots, K_b$, satisfying axioms A*1 to A*4, is the graph of a partial geometry (r, k, t). In fact G together with the cliques of Σ is isomorphic to a partial geometry (r, k, t) the vertices of G corresponding to the points and the cliques of Σ corresponding to the lines of the geometry. Such a graph will be called *geometrisable* (r, k, t).

One may consider graphs in which there exists a set of cliques $K_1, K_2, \cdots, K_b$ satisfying one or more but not all of the axioms A*1, A*2, A*3, A*4. Thus a previous result due to Bose and Clatworthy [1], is equivalent to the following:

THEOREM. *If in a strongly regular graph G, there exists a set Σ of cliques $K_1, K_2, \cdots, K_b$, satisfying the axioms* A*1, A*2, A*3, *and* *if $k > r$, then the graph is geometrisable (r, k, t), the vertices of G, and the cliques of Σ being the points and lines of the corresponding geometry.*

We further prove

THEOREM. *If in a pseudo-geometric graph with characteristics (r, k, t), there exists a set Σ of cliques $K_1, K_2, \cdots, K_b$, satisfying axioms* A*1 *and* A*2, *and if $k > r$, then G is geometrisable (r, k, t), the vertices of G, and the cliques of Σ being the points and lines of the corresponding geometry.*

There are many interesting examples of partial geometries some of which are given in § 7. In particular the partial geometry (r, k, t) becomes a net of degree r and order k when $t = r - 1$. A pseudo-geometric graph with characteristics $(r, k, r-1)$ may be defined to be a *pseudo-net graph*, of degree r and order k. Bruck [5] has proved a series of lemmas for pseudo-net graphs and in particular has shown that a pseudo-net graph of degree r and order k is geometrisable $(r, k, r-1)$ if

$$k > \tfrac{1}{2}(r-1)(r^3 - r^2 + r + 2)\,. \tag{1.9}$$

The special case $r = 2$, was proved by Shrikhande [19]. In this paper we give the following generalization of Bruck's result.

THEOREM. *A pseudo-geometric graph with characteristics (r, k, t) is geometrisable (r, k, t) if*

$$k > \tfrac{1}{2}[r(r-1) + t(r+1)(r^2 - 2r + 2)]\,. \tag{1.10}$$

We have proved a series of Lemmas which are direct generalizations of the lemmas used by Bruck for his proof. In fact it is surprising how smoothly the technique devised by Bruck for the special case of nets, works in the general case.

In particular the concept of *grand cliques* introduced by Bruck for the case of nets is capable of easy generalization. If G is a pseudo-geometric graph with characteristics (r, k, t), then a *major clique* of G is defined as a clique which contains at least $k - (r-1)^2(t-1)$ points. A major clique which is also maximal is defined as a *grand clique*.

Given a pseudo-geometric graph G with characteristics (r, k, t) the set of grand cliques is unambiguously defined. We may take this set to be the set Σ and enquire under what circumstances, the axioms A*1 - A*4 will be satisfied. We then show that (1.10) is a sufficient condition for this.

A pseudo-geometric graph with characteristics (r, k, t) has the same parmeters as the triangular association scheme if we take $r = t = 2$, $k = n - 1$. Substituting these values in (1.10) we get $n > 8$. Thus for these special values of r, k, t our result is equivalent to the uniqueness of the triangular scheme for $n > 8$, a result first proved by Connor [9]. In fact our result may be interpreted as a generalized uniqueness theorem as explained in § 12.

A net of degree r and order k is defined to have deficiency $d = r + 1 - k$. Bruck [5] showed that a net of order k and deficiency d can be completed to an affine plane by the addition of new lines, if

$$k > \tfrac{1}{2}(d-1)(d^3 - d^2 + d + 2) .$$

We generalize Bruck's result to the following:

THEOREM. *Given a partially balanced incomplete block* (PBIB) *design* $(r, k, \lambda_1, \lambda_2)$, $\lambda_1 > \lambda_2$, *based on an association scheme with parameters*

$$n_1 = (d-1)(k-1)(k-t)/t , \qquad n_2 = d(k-1) ,$$
$$p_{11}^1 = [(d-1)(k-1)(k-t) - d(k-t-1) - t]/t ,$$
$$p_{11}^2 = (d-1)(k-t)(k-t-1)/t ,$$

we can extend the design by adding new blocks, containing the same treatments, in such a way that the extended design is a balanced incomplete block (BIB) *design, with* $r_0 = r + d(\lambda_1 - \lambda_2)$ *replications, block size* k *and in which every pair of treatments occur together in* λ_1 *blocks.*

2. Strongly regular graphs.

A *finite graph* G consists of a finite set of v vertices, and a relation *adjacency* such that any two distinct vertices of G may be either adjacent or non-adjacent. Adjacent vertices may be said to be *joined* and non-adjacent vertices to be *unjoined*. We shall be concerned with finite graphs only, and use the word graph in the sense of finite graphs.

The graph G is said to be regular (of degree n_1) if each vertex of G is joined to exactly n_1 other vertices. In this case each vertex will be unjoined to exactly n_2 other vertices, where

$$n_1 + n_2 = v - 1 . \tag{2.1}$$

A regular graph G will be said to be strongly regular if (i) any two vertices which are joined in G, are both simultaneously joined to exactly p_{11}^1 other vertices (ii) any two pairs of vertices which are unjoined in G, are both simultaneously joined to exactly p_{11}^2 vertices.

A strongly regular graph G thus depends on four parameters

n_1, n_2, p^1_{11} and p^2_{11}, the number of vertices being given by (2.1).

Let two vertices of a strongly regular graph G be called *first associates* if they are joined, and second associates if they are unjoined. If the vertices θ and ϕ of G are ith associates, we shall denote by $p^i_{jk}(\theta, \phi)$ the number of vertices which are jth associates of θ and kth associates of ϕ. From definition the number $p^i_{11}(\theta, \phi)$ is independent of the pair θ, ϕ so long as they are ith associates. Thus

$$p^i_{11}(\theta, \phi) = p^i_{11} \qquad i = 1, 2 .$$

We shall show that a similar situation prevails with respect to all the numbers $p^i_{jk}(\theta, \phi)$, so that we can write

$$p^i_{jk}(\theta, \phi) = p^i_{jk} \qquad i, j, k = 1, 2$$

and that

$$p^i_{jk} = p^i_{kj} . \tag{2.2}$$

This follows from the following theorems proved by Bose and Clatworthy [1], in connection with partially balanced incomplete block (PBIB) designs (if we identify treatments with vertices).

THEOREM 2.1. *Let there exist a relationship of association between every pair among v treatments satisfying the conditions*:

(a) *Any two treatments are either first associates or second associates* (b) *Each treatment has n_1 first and n_2 second associates* (c) *For any pair of treatments which are first associates the number p^1_{11} of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.*

Then, for every pair of first associates among the v treatments the number p^1_{12}, p^1_{21} and p^1_{22} are constants, and $p^1_{12} = p^1_{21}$.

THEOREM 2.2. *Let there exist a relationship of association between every pair among v treatments satisfying the conditions*:

(a) *Any two treatments are either first associates* (b) *Each treatment has n_1 first associates and n_2 second associates* (c) *For any pair of treatments which are second associates, the number p^2_{11} of treatments common to the first associates of the first and the first associates of the second is independent of the pair of treatments with which we start.*

Then, for every pair of second associates among the v treatments the numbers p^2_{12}, p^2_{21} and p^2_{22} are constants, and $p^2_{12} = p^2_{21}$.

It appears from the proof of Bose and Clatworthy that

$$p^1_{12} = n_1 - p^1_{11} - 1 = p^1_{21} , \quad p^1_{22} = n_2 - n_1 + p^1_{11} + 1 , \tag{2.3}$$

$$p^2_{12} = n_1 - p^2_{11} = p^2_{21} , \quad p^2_{22} = n_2 - n_1 + p^2_{11} - 1 , \tag{2.4}$$

Actually the relations (2.3), (2.4) were obtained much earlier by Bose and Nair [3], but in their formulation they started with the constancy of all the numbers p^i_{jk} $(i, j, k = 1, 2)$. It is also easy to prove as shown by Bose and Nair [3], that

$$n_1 p^1_{12} = n_2 p^2_{11} , \quad n_1 p^1_{22} = n_2 p^2_{12} . \tag{2.5}$$

The concept of a strongly regular graph is isomorphic with the concept of an association scheme with two associate classes, as introduced by Bose and Shimamoto [4], in connection with the theory of partially balanced designs, if treatments are identified with vertices, a pair of first associates with a pair of joined vertices, and a pair of second associates with a pair of unjoined vertices. Thus strongly regular graphs first arose in connection with the theory of partially balanced designs.

The numbers v, n_1, n_2 p^i_{jk} are called the parameters of the regular graph G. They are connected by the relations (2.1)-(2.5), and only four are linearly independent. These may be conveniently chosen as n_1, n_2, p^1_{11} and p^2_{11}.

3. **Partial geometries.** Consider two undefined classes of objects called *points* and *lines*, together with a relation *incidence*, such that a point and a line, may or may not be incident. Then the points and lines are said to form a partial geometry (r, k, t) provided that the following axioms are satisfied:

A1. A pair of distinct points is not incident with more than one line.

A2. Each point is incident with exactly r lines.

A3. Each line is incident with exactly k points.

A4. Given a point P not incident with a line l, there are exactly t lines $(t \geqq 1)$ which are incident with P, and also incident with some point incident with l.

If there were two distinct lines l and m each incident with two distinct points P_1 and P_2, then A1 would be contradicted. Hence we have,

A′1. A pair of distinct lines is not incident with more than one point.

For convenience we will use the ordinary geometric language. Thus if a point is incident with a line, we shall say that the point lies on the line or is contained in the line, and that the line passes through the point. If two points are incident with the line, then we speak of the line as joining the two points. By A1 there cannot be more than one line joining two points. Thus either two points are unjoined or joined by a unique line. If two lines are incident with a common point, they are said to intersect, and the common point is said to be their point of intersection. By A'1 two lines cannot intersect in more than one point. Hence two lines are either non-intersecting or intersect in a unique point.

THEOREM 3.1. *Given a partial geometry (r, k, t) there exists a dual partial geometry (k, r, t) obtained by calling points of the first, the lines of the second; and the lines of the first the points of the second.*

This follows by noting the duality A1 and A'1, the duality A2 and A3 and the self dual nature of A4. In fact A4 can be rephrased as

A'4. Given a line l not incident with a point P there are exactly t points which are incident with l and also incident with some line incident with P.

In terms of the alternative geometric language introduced we may write A4 and A'4 as

A4. Through any point P not lying on a line l there pass exactly t lines intersecting l.

A'4. On any line l not passing through a point P, there lie exactly t points, joined to P.

The equivalence of A4 and A'4 is now obvious.

4. Graph of a partial geometry. The graph G of a partial geometry (r, k, t) is defined as follows: The vertices of G are the points of the partial geometry. Two vertices of G are joined (adjacent) if the corresponding points of the geometry are joined (incident with the same line). Two vertices of G are unjoined (non-adjacent) if the corresponding points of the partial geometry are unjoined (i.e. there exists no line incident with both the points).

THEOREM 4.1. *The graph G of partial geometry (r, k, t) is strongly regular with parameters*

(4.1) $$n_1 = r(k-1)\ , \qquad n_2 = (r-1)(k-1)(k-t)/t\ ,$$

(4.2) $$p_{11}^1 = (t-1)(r-1) + k - 2\ , \qquad p_{11}^2 = rt\ ,$$

(4.3) $$1 \leqq t \leqq r\ , \qquad 1 \leqq t \leqq k\ .$$

Let there be v points and b lines in the partial geometry. Since the points of the geometry have been identified with the vertices of the graph G, we can call two points of the geometry first associates if they are joined by a line, and second associates if they are not joined by a line. Now through any point P of the geometry there pass r lines, each of which contains $k-1$ other points besides P. Hence P has exactly $r(k-1)$ first associates. Hence

$$n_1 = r(k-1) . \tag{4.4}$$

This shows that G is a regular graph. Consider the $b-r$ lines not passing through P. From A4 each of these lines contains exactly t first associates of P. Any particular first associate Q of P, lies on $r-1$ such lines, since one of the r lines passing through Q joins it to P. Hence the number of first associates is

$$n_1 = t(b-r)/(r-1) . \tag{4.5}$$

Comparing (4.4) and (4.5) we have

$$b = r[(r-1)(k-1)+t]/t . \tag{4.6}$$

Again each of the $b-r$ lines not passing through P contains exactly $k-t$ second associates of P. Any particular second associate R of P lies on r such lines. Hence the number of second associates of P is

$$n_2 = (k-t)(b-r)/r .$$

Substituting for b from (4.6) we have

$$n_2 = (r-1)(k-1)(k-t)/t . \tag{4.7}$$

Consider any two points P and Q which are first associates. They are joined by a line l. We shall count the number of points which are first associates to each of P and Q. The $k-2$ points on l other than P and Q are first associates of both P and Q. Now there pass $(r-1)$ lines through P, other than l. By A4 each of these contains $t-1$ first associates of Q other than P. Thus these $(t-1)(r-1)$ points are first associates of both P and Q. It is easy to see that there are no other first associates of both P and Q. Hence

$$p_{11}^1 = (t-1)(r-1)+k-2 . \tag{4.8}$$

Consider two points P and R which are second associates. There is no line joining them. Each of the r lines passing through P contains t first associates of R. Hence

$$p_{11}^2 = rt . \tag{4.9}$$

We have now verified the formulae (4.1) and (4.2), showing that G is a strongly regular graph. The values of the other parameters p^i_{jk} follow from the identities (2.3) and (2.4). The inequality (4.3) clearly follows from axiom A4.

COROLLARY. *The number of points v and the number of lines b in a partial geometry (r, k, t) is given by*

$$v = k[(r-1)(k-1)+t]/t\ , \tag{4.10}$$

$$b = r[(r-1)(k-1)+t]/t\ . \tag{4.11}$$

In view of the isomorphism of association schemes with two associate classes, and strongly regular graphs, the definition of partially balanced incomplete block (PBIB) designs given by Bose and Shimamoto [4], may now be rephrased as follows:

Given a strongly regular graph G with parameters $n_1, n_2, p^1_{11}, p^2_{11}$, we may identify its v vertices with v treatments. Then a PBIB design is an arrangement of the v treatments into b sets (called blocks) such that.

(α) Each treatment is contained in exactly r blocks

(β) Each block contains k distinct treatments

(γ) Any two treatments which are first associates (joined in G) occur together in exactly λ_1 blocks. Any two treatments which are second associates (unjoined in G) occur together in λ_2 blooks

The design may be called a PBIB design $(r, k, \lambda_1, \lambda_2)$ based on the strongly regular graph G.

Given a partial geometry (r, k, t), with graph G, it is clearly a PBIB design $(r. k, 1, 0)$ based on G, and this PBIB design is a connected design. This follows because two first associates always occur together in a block, and if two treatments θ_0 and θ_2 are second associates, we can find a treament θ_1 in $p^2_{11} = rt > 0$ ways, such that θ_0 and θ_1 are first associates, and θ_1 and θ_2 are first associates. The *incidence matrix* of a partial geometry may be defined as the matrix $N = (n_{ij})$ where $n_{ij} = 1$ if the ith point is incident with the jth line and 0 otherwise. Then N is also the incidence matrix of the corresponding PBIB design. Now Connor and Clatworthy [11] and Bose and Mesner [2], have shown that for a connected PBIB design, with two associate classes, NN' has only three distinct characteristic roots, whose multiplicites are 1, α and β where

$$\alpha, \beta = \frac{n_1+n_2}{2} \mp \frac{(n_1-n_2)+\gamma(n_1+n_2)}{2\sqrt{\Delta}}\ , \tag{4.12}$$

and

$$\gamma = p^2_{12} - p^1_{12}\ , \qquad \Delta = \gamma^2 + 2(p^1_{12} + p^2_{12}) + 1\ .$$

Now these multiplicities are necessarily integral. Using the formulae (4.1), (4.2), (2.3) and (2.4) we find that

$$\alpha = \frac{rk(r-1)(k-1)}{t(k+r-t-1)} \,. \tag{4.13}$$

Hence we have the theorem

THEOREM (4.2). *A necessary condition for the existence of a partial geometry* (r, k, t) *is that the number*

$$\alpha = \frac{rk(r-1)(k-1)}{t(k+r-t-1)}$$

is a positive integer.

5. Partially balanced designs, which are partial geometries. We have already shown that a partial geometry (r, k, t) is isomorphic to a PBIB design $(r, k, 1, 0)$ based on the graph (association scheme) of the geometry. However a PBIB design based on a strongly regular graph need not necessarily be a partial geometry. It would therefore be of interest to find sufficient conditions under which a PBIB design $(r, k, 1, 0)$ based on a strongly regular graph is isomorphic to a partial geometry.

Now Bose and Clatworthy [1] have shown that if there exists a PBIB design $(r, k, 1, 0)$ based on a strongly regular graph G for which $r < k$, then the parameters of G are given by the formulae (4.1), (4.2) i.e. are the same as the parameters of the graph of some partial geometry (r, k, t). We shall show that the design is indeed a partial geometry and thus establish the following theorem:

THEOREM (5.1). *If there exists a* PBIB *design* $(r, k, 1, 0)$ *based on a strongly regular graph* G, *then if* $r < k$, *the design must be a partial geometry* (r, k, t) *for some* $t \leqq r$. *The parameters of* G *are given by the formulae* (4.1), (4.2).

The parameters of G are given by (4.1), (4.2) in view of the result of Bose and Clatworthy already cited. Also the axioms A1, A2, A3 for a partial geometry are implicit in the definition of a PBIB design $(r, k, 1, 0)$. It therefore only remains to prove axiom A4 which amounts to saying that each block of the design contains exactly t treatments which are first associates of a given treatment not contained in the block.

Let K be the set of k treatments contained in a particular block, and let $\bar{K}$ be the set of the remaining $v - k$ treatments. Let $g(x)$ denote the number of treatments in $\bar{K}$ which have exactly x first associates in K. Then

(5.1) $$\sum_{x=0}^{k} g(x) = v - k = k(k-1)(r-1)/t \ .$$

Let us count the number of pairs (P, Q), where P is a treatment in K, Q is a treatment in $\bar{K}$, and P and Q are first associates. Now each treatment in K has $k-1$ first associates in K, and consequently $n_1 - k + 1$ first associates in $\bar{K}$. Hence the required number is $k(n_1 - k + 1)$. Again there are $g(x)$ treatments in $\bar{K}$, which have exactly x treatments in K. These treatments contribute $xg(x)$ to our count. Hence

(5.2) $$\sum_{x=0}^{k} xg(x) = k(n_1 - k + 1) = k(r-1)(k-1) \ .$$

Again let us count the number of triplets (P_1P_2, Q) where P_1P_2 is an ordered pair of distinct treaments in K, and Q is a treatment in $\bar{K}$ which is a first associate of both P_1 and P_2. Since P_1 and P_2 have $k-2$ common first associates in K, they have $p_{11}^1 - k + 2$ common first associates in $\bar{K}$. Hence the required number of triplets like (P_1P_2, Q) is $k(k-1)(p_{11}^1 - k + 2)$. Now each of the $g(x)$ treatments in $\bar{K}$, which have x first associates in K contribute $x(x-1)g(x)$ to our count. Hence we have the equation

(5.3) $$\begin{aligned}\sum_{x=0}^{k} x(x-1)g(x) &= k(k-1)(p_{11}^1 - k + 2) \\ &\doteq k(k-1)(t-1)(r-1) \ .\end{aligned}$$

Using (5.1), (5.2) and (5.3) a simple calculation shows that

$$\bar{x} = \Sigma xg(x)/\Sigma g(x) = t \ ,$$

i.e. the average value of x (the number of first associates in K of any treatment of $\bar{K}$ is t. Also

$$\sum_{x=0}^{k} g(x)(x-t)^2 = 0 \ ,$$

which shows that x must always have the value t. This proves our theorem.

6. Geometrisable and pseudo-geometric graphs. A strongly regular graph G which has parmeters (4.1) and (4.2) and for which the inequality (4.3) is satisfied, is defined to be a *pseudo-geometric* graph with characteristics (r, k, t). Thus a pseudo-geometric graph with characteristics (r, k, t) has the same parameters as the graph of a partial geometry (r, k, t). However a graph may be pseudo-geometric without being the graph of a partial geometry.

A subset of vertices of a graph G, any two of which are joined is called a *clique* of G. When G is the graph of a partial geometry there will exist in G a set Σ of distinct cliques $K_1, K_2, \cdots, K_b$, corresponding to the lines of the geometry satisfying the following axioms;

A*1. Any two joined vertices of G are contained in one and only one clique of Σ.

A*2. Each vertex of G is contained in r cliques of Σ.

A*3. Each clique of Σ contains k vertices of G.

A*4. If P is a vertex of G not contained in a clique K_i of Σ, there are exactly t vertices in K_i which are joined to P ($i = 1, 2, \cdots, b$).

Hence any graph G in which there exists a set Σ of cliques $K_1, K_2, \cdots, K_b$, satisfying axioms A*1 – A*4 is the graph of a partial geometry (r, k, t). In fact G together with the cliques of Σ is isomorphic to a partial geometry (r, k, t), the vertices of G corresponding to the points, and cliques of Σ to the lines of the geometry. Such a graph will be said to be *geometrisable* (r, k, t).

One may consider graphs in which there exist a set Σ of cliques $K_1, K_2, \cdots, K_b$ satisfying one or more but not all of the axioms A*1, A*2, A*3, A*4, and investigate under what additional conditions the graph will be geometrisable. Thus theorem (5.1) may be rephrased as

THEOREM (6.1). *If there esists a set Σ of cliques $K_1, K_2, \cdots, K_b$ in a strongly regular graph G, satisfying axioms* A*1, A*2, A*3 *and if $k > r$, then G is geometrisable (r, k, t).*

THEOREM (6.2). *Let G be a pseudo-geometric graph with characteristics (r, k, t). If it is possible to find in G a set Σ of cliques $K_1, K_2, \cdots, K_b$, satisfying axioms* A*1 *and* A*2, *and if $k > r$, then G is geometrisable (r, k, t).*

We shall prove that each of the cliques $K_1, K_2, \cdots, K_b$ contains exactly k vertices, and that if Q is any vertex not contained in any clique K_i $(1 \leqq i \leqq b)$, then there are exactly t vertices in K_i which are first associates of (joined to) Q. This will show that if the vertices of G are taken as points, and the cliques $K_1, K_2, \cdots, K_b$ as lines, then we have a partial geometry (r, k, t).

Let P be any vertex. Without loss of generality we can take $K_1, K_2, \cdots, K_r$ to contain P. Now the sets $K_1 - P, K_2 - P, \cdots, K_r - P$ are disjoint and must contain between them all the $r(k - 1)$ first associates of P. From this it follows that the average number of vertices in the r cliques of the set $K_1, K_2, \cdots, K_r$ is k. Hence there exists a clique containing at least k vertices. Let K_j be such a clique. Let us take a subset K of K_j, such that K contains exactly k vertices. Let $\bar{K}$ be the of vertices not contained in K. Let $g(x)$ be the number of vertices in $\bar{K}$ which have exactly x first associates in K. Then it follows exactly as in the proof of Theorem (5.1) that

$$\sum_{x=0}^{k} g(x) = k(k-1)(r-1)/t ,$$

and

$$\sum_{x=0}^{k} xg(x) = k(r-1)(k-1) .$$

Hence the average value of x is t. Also as before

$$\sum_{x=0}^{k} (x-t)^2 g(x) = 0 ,$$

which is only possible if x is constant and equal to t. Hence every point of $\bar{K}$ has exactly t first associates in K.

If K_j contains any vertex Q other than those already contained in K, then Q belongs to $\bar{K}$, and therefore has exactly t first associates in K. But each point of K is a first associate of Q hence $t = k$, which contradicts $1 \leqq t \leqq r < k$. This shows that none of the cliques K_j containing P, contains more than k vertices. Thus each contains exactly k vertices. Since each clique contains at least one vertex, each of the clique $K_1, K_2, \cdots, K_b$ contains exactly k vertices. Also if Q is a point not contained in K_i $(1 \leqq i \leqq b)$, then there are exactly t vertices in K_i which are first associates of Q_i. This completes the proof of the theorem.

N. B. Compare Theorems (6.1) and (6.2).

7. Examples of partial geometries. (a) A net (r, k) of degree r and order k is a system of undefined points and lines together with an incidence relation subject to the following axioms (i) There is at least one point (ii) The lines of the net can be partitioned into r disjoint, nonempty, "parallel classes" such that each point of the net is incident with exactly one line of each class and given two lines belonging to distinct classes there is exactly one point of the net which is incident with both lines.

For convenience we can use phrases such as "point is on a line" istead of speaking of incidence. Then it can be readily proved (see for example Bruck [5]) that

(1) Each line of the net contains exactly k distinct points where $k \geqq 1$.

(2) Each point of the net lies on exactly r distinct lines where $r \geqq 1$.

(3) The net has exactly rk distinct lines. These lines fall into r parallel classes of k lines each. Distinct lines of the same parallel class have no common points. Two lines of different classes have one common point.

(4) The net has exactly k^2 distinct points.

We shall show that *a net* (r, k) *of degree* r *and order* k *is a partial geometry* $(r, k, r-1)$. The properties (1) and (2) above show that axioms A3 and A2 of a partial geometry hold.

Two lines cannot intersect in more than one point, for they either belong to the same parallel class and have no common point, or different parallel classes in which they have one common point. Form this follows the fact two distinct points cannot be incident with more then one line. Hence axiom A1 for a partial geometry holds.

Again given a point P not incident with the line l. There are exactly r lines through P, one belonging to each parallel class. One of these is parallel to l (i.e. belongs to the same parallel class as l), and ones not intersect l. The other $r-1$ lines through P each intersects l in one point, these points being all distinct. Hence axiom A4 for a partial geometry holds with $t = r - 1$. This completes the proof of our statement.

It follows from Theorem 4.1, that the parameters of the graph G_N of a net (r, k) are given by

$$n_1 = r(k-1)\ , \qquad n_2 = (k-1)(k-r+1)\ , \tag{7.1}$$

$$p_{11}^1 = (r-2)(r-1) + (k-2)\ , \qquad p_{11}^2 = r(r-1)\ . \tag{7.2}$$

If a strongly regular graph has parameters (7.1), (7.2) we shall call it a pseudo-net graph with characteristic (r, k). A pseudo-net graph with characteristics (r, k) is pseudo-geometric with characteristics $(r, k, r-1)$.

Bruck [5], defines the deficiency d of a net (r, k) by

$$d = k - r + 1\ . \tag{7.3}$$

The interpretation of the deficiency d is that if it were possible to add d more parallel classes, each consisting of k lines, so that the extended net now has $k+1$ classes of parallels, the net would become an affine plane, in which any two points are joined by a unique line.

If we take the k^2 points of the net as treatments and the rk lines as blocks, we obtain what is known as the *lattice* design. This is a PBIB design $(r, k, 1, 0)$ based on the strongly regular graph G_N with parameters given by (7.1), (7.2). Lattice designs were introduced by Yates [24]. The association scheme corresponding to G_N is the L_r scheme defined by Bose and Shimanoto [4].

It is well known that a latice design with r replications and block size k is equivalent to a system of $r-2$ mutually orthogonal Latin squares of order k.

If $r-2$ mutually orthogonal Latin square of order k are given, we can superpose them. Then each cell contains $r-2$ symbols belonging in order to the different Latin squares. The k^2 cells are now identified with k^2 treatments. Treatments belonging to the same row give one set of k blocks. Treatments bolonging to the same column give another set of k blooks. Treatments (cells) which contain the same symbol of the ith Latin square give a set of blocks for each value of i $(i = 1, 2, \cdots, r-2)$. We thus get r sets of blocks. The treatments and blocks so obtained constitute a lattice design.

Conversely given a Lattice design with r replications and block size k, we can construct a set of $r-2$ mutually orthogonal Latin squares of order k.

If $r-2$ mutually orthogonal Latin squares of order k are given, we can superpose them. Then each cell contains $r-2$ symbols belonging in order to the different Latin squares. The k^2 cells are now identified with k^2 treatments. Treatments belonging to the same row give one set of k blocks. Treatments belonging to the same column give another set of k blocks. Treatments (cells) which contain the same symbol of the ith Latin square give a set of blocks for each value of i $(i = 1, 2, \cdots, r-2)$. We thus get r sets of blocks. The treatments and blocks so obtained constitute a lattice design.

Conversely given a Lattice design with r replications and block size k, we can construct a set of $r-2$ mutually orthogonal Latin squares of order k.

(b) Take an $n \times n$ squares and write down the numbers $1, 2, \cdots, n(n-1)/2$ in the cells above the main diagonal. Fill up the cells below the main diagonal symmetrically. The case $n = 5$ is exemplified below.

*	1	2	3	4
1	*	5	6	7
2	5	*	8	9
3	6	8	*	10
4	7	9	10	*

Fig. 1

The cells containing the same number are identified with the same point. Thus there are two different cells representing the same point, there being $v = n(n-1)/2$ points all together. Let the n rows constitute lines. Thus there are n lines. It is clear that axioms A1, A2, A3 of a partial geometry are satisfied with $r = 2$, $k = n - 1$. It is easy to see that any two lines intersect in one point. Thus $t = 2$, and we have a partial geometry $(2, n-1, 2)$.

If two points which lie on a line are called first associates, and two points which do not lie on any line are called second associates, we have the triangular association scheme first defined by Bose and Shimamoto [4], and extensively studied by Connor [9], Shrikhande [21], Hoffman [13, 14] and Chang [6, 7]. The parameters of the association scheme or the corresponding strongly regular graph are

$$n_1 = 2(n-2)\ , \qquad n_2 = (n-2)(n-3)/2\ , \tag{7.4}$$

$$p_{11}^1 = n - 2\ , \qquad p_{11}^2 = 4\ . \tag{7.5}$$

If a strongly regular graph has the parameters (7.4), (7.5) we shall call it a pseudo-triangular graph with characteristic n. A pseudo-triangular graph with characteristic n is pseudo-geometric with characteristics $(2, n-1, 2)$.

(c) A balanced incomplete block design BIB is an arrangement of a set of v_0 objects or treatments in b_0 sets or blocks, such that (i) each block contains k_0 distinct treatments (ii) each treatment is contained in r_0 blocks (iii) each pair of distinct treatments is contained in λ_0 blocks. This design has sometimes been called a (v_0, k_0, λ_0) configuration. The dual of a design is defined as a new design whose treatments and blocks are in (1, 1) correspondence with the blocks and treatments of the original design, and incidence is preserved (where a block and a treatment are incident if the treatment is contained in the block, and non-incident otherwise), Shrikhande [23] has shown that the dual of a BIB design with $\lambda_0 = 1$ is a PBIB design. Now a BIB design with $\lambda_0 = 1$ is clearly a partial geometry (r_0, k_0, k_0). Hence the dual design is the dual partial geometry (k_0, r_0, k_0). If we set $k_0 = r$ and $r_0 = k$, so as to make r the replication number and k the block size in the dual design, then the dual design is the partial geometry (r, k, r). Any two blocks (lines) of this design intersect in a unique treatment (point). Hence the association scheme of this design has been called the SLB (singly linked block) scheme by Bose and Shimamoto [4]. The parameters of the corresponding strongly regular graph can be written down directly from Theorem (4.1). We have

$$n_1 = r(k-1)\ , \qquad n_2 = (k-r)(r-1)(k-1)/r\ , \tag{7.6}$$

$$p_{11}^1 = (r-1)^2 + k - 2\ , \qquad p_{11}^2 = r^2\ . \tag{7.7}$$

If a strongly regular graph has the parameters (7.6), (7.7) we shall call it a pseudo-SLB graph with characteristics (r, k). A pseudo-SLB graph with characteristics (r, k) is a pseudo-geometric graph with characteristics (r, k, r).

(d) To conclude we shall give a rather less obvious example of a partial geometry.

Consider an elliptic non-degenerate quadric Q_5 in the finite projective space $PG(5, p^n)$. This quadric is ruled by straight lines, called generators, but contains no plane. As shown by Primrose [16] and Ray-Chaudhuri [17], there are $(s^3 + 1)(s + 1)$ points and $(s^3 + 1)(s^2 + 1)$ generators in Q_5, each generator contains $s + 1$ points, and through each point there pass $s^2 + 1$ generators, where $s = p^n$.

If P is a point on Q_5 not contained in a generator l, then the polar 4-space of P intersects l in a single point P^*, and PP^* is a generator of Q_5. It can be readily verified by using theorems proved by Ray-Chaudhuri that PP^* is the only generator through P, which intersects l. This shows that if we consider the points and generators of Q_5 as points and lines, they constitute a partial geometry $(s^2 + 1, s + 1, 1)$. The parameters of the graph of this partial geometry can be easily written down using Theorem (4.1). They are

$$n_1 = s(s^2 + 1) , \qquad n_2 = s^4 + 1 ,$$
$$p_{11}^1 = s - 1 , \qquad p_{11}^2 = s^2 + 1 .$$

The partial geometry $(s^2 + 1, s + 1, 1)$ is of course a PBIB design. This was obtained by Ray-Chaudhuri [18], the special case $s = 2$ was given earlier by Bose and Clatworthy [1]. An interesting point in the present formulation is that to verify that the configuration of points and generators on Q_5 is a PBIB design we have only to check the constancy of r, k and t instead of the constancy of $r, k, n_1, n_2, p_{11}^1, p_{11}^2$ as was done by Bose and Clatworthy [1] and by Ray-Chaudhuri [18]. The dual partial geometry $(s + 1, s^2 + 1, 1)$ is also of interest.

In the same way one can show that the configuration of points and generators on a non-degenerate quadric Q_4 in $PG(4, p^n)$ is a partial geometry $(s + 1, s + 1, 1)$ where $s = p^n$. The corresponding design was first obtained by Clatworthy [8].

8. Lemmas on claws in pseudo-geometric graphs. We have shown in Theorem (5.1), that if we can base a PBIB design $(r, k, 1, 0)$ on a strongly regular graph G, then G is pseudo-geometric and the design is a partial geometry. We can ask the converse question: If the graph G is pseudo-geometric (r, k, t) can we base a PBIB design $(r, k, 1, 0)$ on it? In graph theoretic language this question may be put as: If the graph G is pseudo-geometric (r, k, t) can we find a set of cliques $K_1, K_2, \cdots, K_b$ satisfying the axioms A*1 - A*4 of § 6.

In the rest of the paper we shall frequently use the following functions:

$$\gamma(r, t) = 1 + (r-1)^2(t-1) , \tag{8.1}$$

$$q(r, t) = 1 + (r-1)(2r-1)(t-1) , \tag{8.2}$$

$$\rho(r, t) = rt + (r-1)(t-1)(2r-1) , \tag{8.3}$$

$$p(r, t) = \tfrac{1}{2}[r(r-1) + t(r+1)(r^2-2r+2)] . \tag{8.4}$$

We note that in view of the inequality $1 \leqq t \leqq r$

$$p(r, t) \geqq \rho(r, t) \geqq q(r, t) \geqq \gamma(r, t) . \tag{8.5}$$

The concept of a *claw* was suggested by Alan Hoffman in conversation with R. H. Bruck and the author. By a claw $[P, S]$ of a pseudo-geometric graph G is meant an ordered pair consisting of a vertex P, the *vertex* of the claw, and a nonempty set S of vertices distinct from P such that every vertex in S is joined to P in G but no two vertices in S are joined in G.

The number of elements in a finite set S will be denoted by $|S|$. The order of the claw $[P, S]$ is defined as $s = |S|$.

In Lemmas 8.1 – 8.3, G denotes a pseudo-geometric graph (r, k, t).

LEMMA 8.1. *If $k > \gamma(r, t) = 1 + (t-1)(r-1)^2$ then for any s, $1 \leqq s \leqq r$, each vertex P of G is the vertex of a claw $[P, S]$ of order s. We can choose S to include any vertex A joined to P in G.*

Let P be a vertex of G. Suppose there exists a claw $[P, S]$ of order s. Let T be the set of all vertices other than those belonging to $[P, S]$ and which are joined to P (are first associates of P). Let $f(x)$ be the number of vertices Q in T, such that Q is joined to exactly x vertices in S ($f(x)$ is the number of vertices in T each of which has $f(x)$ first associates in S). Then we have

$$\sum_{x=0}^{s} f(x) = n_1 - s = r(k-1) - s , \tag{8.6}$$

since the left hand side of (8.6) counts all first associates of P, which are not in S. Now let us count pairs (A, Q) where A is in S and Q is in T and is a first associate of both A and P. Since A and P have exactly p_{11}^1 common first associates (none of which can belong to S by the definition of a claw) we have sp_{11}^1 pairs like (A, Q).

Again there are $f(x)$ vertices in T each of which has exactly x first associates in S. They contribute $xf(x)$ to our count, Hence

$$\sum_{x=0}^{s} xf(x) = sp_{11}^1 = s\{(t-1)(r-1) + (k-2)\} . \tag{8.7}$$

Hence

$$(8.8) \qquad f(0) - \sum_{x=1}^{s} (x-1)f(x) = (r-s)(k-1) - s(t-1)(r-1) .$$

Hence if $s < r$ and $k > \gamma(r, t) = 1 + (r-1)^2(t-1)$, then $f(0)$ is positive, i.e. there is at least one vertex A_{s+1} in T which is not joined to $A_1, A_2, \cdots, A_s$. We can therefore add A_{s+1} to S and get a claw of order $s+1$. In this way we can go on extending a claw till we get a claw of any required order not exceeding r. We can start this process with any claw $[P, A]$ of order 1. This proves the lemma.

LEMMA (8.2). *If $k > \gamma(r, t) = 1 + (r-1)^2(t-1)$, and if $[P, S]$ is a claw of order $r-1$, then there exist at least $k - (r, t)$ distinct vertices Q of G such that $[P, S \cup Q]$ is a claw of order r.*

If $[P, S]$ is a claw of order $r-1$, then from (8.8)

$$f(0) \geqq k - \gamma(r, t) .$$

Hence there exist at least $k - \gamma(r, t)$ vertices Q, each of which taken together with S give a claw $[P, S^*]$ of order r where $S^* = S \cup Q$.

LEMMA (8.3). *If $k > p(r, t) = \frac{1}{2}[r(r-1) + t(r+1)(r^2-2r+2)]$ then there exists in G no claw of order $r+1$.*

Let $[P, S]$ be a claw of order s in G. Let the set T be as in Lemma (8.1). We shall count the number of triplets (A_1A_2, Q) where A_1, A_2 is an ordered pair of distinct vertices in S, and Q is a vertex in T which is a first associate of both A_1 and A_2. Since A_1 and A_2 are second associates they have exactly $p_{11}^2 - 1$ common first associates other than P. Some of these may not lie in T. Hence an upper bound for the required number of triplets is $s(s-1)\ p_{11}^2$. However the $f(x)$ vertices in T, each of which has exactly x first associates in S, contribute $x(x-1)f(x)$ to our count. Hence

$$(8.9) \qquad \sum_{x=0}^{s} x(x-1)f(x) \leqq s(s-1)(p_{11}^2 - 1) = s(s-1)(rt-1) .$$

If possible let $s = r+1$. Then adding (8.8) to (8.9) multiplied by a half, and noting that on the left hand side of (8.9) the term $x = 0$ contributes nothing we get

$$\begin{aligned} f(0) &+ \tfrac{1}{2} \sum_{x=1}^{s} (x-1)(x-2)f(x) \\ &\leqq -k + \tfrac{1}{2}[r(r-1) + t(r+1)(r^2-2r+2)] . \end{aligned}$$

Hence if $k > p(r, t)$, there cannot exist a claw of order $r + 1$, as the left hand side is essentially positive.

9. Lemmas on cliques in pseudo-geometric graphs. In the Lemmas (9.1) to (9.6), G denotes a pseudo-geometric graph (r, k, t). The definition of a major clique generalizes Bruck's definition, and the concept of a grand clique is taken over from Bruck [5].

A *major clique* K of G is a clique such that

$$|K| \geqq k + 1 - \gamma(r, t) = k - (r-1)^2(t-1) .$$

A *grand clique* is a major clique which is also a maximal clique.

If K and L are distinct maximal cliques, then $K \cup L$ cannot be a clique. Since K and L are distinct, there must be a vertex P in one of them (say K), not belonging to the other (L). Now if $K \cup L$ is a clique, then P is joined to every vertex of L. Thus $P \cup L$ is a clique which contradicts the fact that L is maximal. Since grand cliques are maximal *the union of two grand cliques cannot be a clique.*

If we take the set of grand cliques as the set Σ of § 6, we may enquire under what conditions the axioms A*1, A*2, A*3 and A*4 are satisfied. The lemmas which follow are directed to this purpose.

LEMMA (9.1). *If $k > (r, t)$ and if G has no claw of order $r+1$, then for every pair of distinct joined vertices P and Q in G, there exists at least one major clique containing both P and Q.*

From Lemma (8.1) we can find a claw $[P, S]$ of order r such that $Q \in S$. Let $A_1, A_2, \cdots, A_{r-1}$ be other vertices of S. Let Ω be the set of vertices R which when adjoined to S, give a claw P, S^* of order r, $S^* = S \cup R$. From Lemma (8.2), the number of points in $\Omega = |\Omega| \geqq k - \gamma(r, t)$.

The vertices in Ω are all joined to one another. If any two were not joined they could be added to $A_1, A_2, \cdots, A_{r-1}$ to give a claw of order $r + 1$. Thus P and the vertices in Ω are all mutually joined. Hence $P \cup \Omega = K$ is a clique of order $\geqq k + 1 - \gamma(r, t)$, i.e. a major clique.

COROLLARY 1. *In Lemma* (9.1) *the hypothesis may be replaced by $k > p(r, t)$.*

This follows from Lemma (8.3) by noting that $p(r, t) \geqq \gamma(r, t)$ for $1 \leqq t \leqq r$.

COROLLARY 2. *When the conditions of Lemma* (9.1) *or Corollary* 1 *are satisfied, P and Q are contained in at least one grand clique.*

We can extend the major clique K by adding new vertices till it becomes maximal, and therefore a grand clique.

LEMMA (9.2). *If K and L are cliques of G and $K \cup L$ is not a clique then $|K \cap L| \leqq rt$.*

Since $K \cup L$ is not a clique, there must exist in $K \cup L$ a pair of vertices P_1, P_2 not joined to one another such that $P_1 \in K$, $P_2 \in L$. Any vertex belonging to $|K \cap L|$ must be joined to both P_1 and P_2. Hence the number of vertices in $|K \cap L|$ cannot exceed p^2_{11}. Thus

$$|K \cap L| \leqq rt .$$

LEMMA (9.3). *If K and L are cliques of G and $K \cap L$ contains at least two vertices A and B, then*

$$|K \cup L| \leqq k + (r-1)(t-1) .$$

Every vertex in $K \cup L$, other than A and B, is a first associate of both A and B. Hence

$$|K \cup L| \leqq p^1_{11} + 2 = k + (r-1)(t-1) .$$

LEMMA (9.4). *If K and L are cliques of G such that* (i) *$K \cup L$ is not a clique* (ii) *$K \cap L$ contains at least two vertices, then*

$$|K| + |L| \leqq k + rt + (r-1)(t-1) .$$

Since conditions of Lemmas (9.2) and (9.3) are satisfied

$$\begin{aligned} |K| + |L| &= |K \cap L| + |K \cup L| \\ &\leqq rt + k + (r-1)(t-1) . \end{aligned}$$

LEMMA (9.5). *If $k > \rho(r, t) = rt + (r-1)(t-1)(2r-1)$ and if G has no claw of order $r+1$, then two distinct vertices of G which are joined, are contained in one and only one grand clique.*

Suppose there are two distinct grand cliques K and L both containing P and Q. Then $K \cup L$ cannot be a clique. Also $K \cap L$ has at least two vertices P and Q. Hence from Lemma (9.4),

$$|K| + |L| \leqq rt + k + (r-1)(t-1) .$$

Since K and L are grand cliques, they are both major. Hence

$$\begin{gathered} 2[k - (r-1)^2(t-1)] \leqq |K| + |L| \leqq rt + k + (r-1)(t-1) \\ k \leqq rt + (r-1)(t-1)(2r-1) = \rho(t) , \end{gathered}$$

which is a contradiction.

COROLLARY. *In Lemma* (9.5) *the hypothesis may be replaced by* $k > p(r, t)$.

LEMMA (9.6) *If* (i) $k > q(r, t) = 1 + (r-1)(2r-1)(t-1)$, (ii) *two distinct vertices of* G *are contained in utmost one grand clique of* G, (iii) *there exists no claw of order* $r+1$ *in* G, *then each point of* G *is contained in exactly* r *grand cliques.*

From Lemma (9.1), Corollary 2 any two vertices are contained in at least one grand clique. It follows from assumption (ii), that any two vertices of G are contained in one and only one grand clique. Again from Lemma (8.1), there exists a claw $[P, S]$ of order r, where $S = \{A_1, A_2, \cdots, A_r\}$. As in Lemma (8.1), let T be the set of first associates of P, other than $A_1, A_2, \cdots, A_r$. We define H_j as the set consisting of P, A_j and all $Q \in T$, such that Q is a first associate of A_j but not of A_i when $i \neq j$. Let $f(x)$ be as in Lemma (8.1). Now $f(0) = 0$, since there are no claws of order $r+1$. Hence from (8.6) and (8.7) we have

$$\sum_{x=1}^{r} f(x) = n_1 - r = r(k-2) ,$$

$$\sum_{x=1}^{r} x f(x) = r p_{11}^{1} = r(k-2) + r(r-1)(t-1) ,$$

$$\sum_{x=2}^{r} (x-1) f(x) = r(r-1)(t-1) .$$

Now

$$\sum_{2}^{r} f(x) \leqq \sum_{2}^{r} (x-1) f(x) \leqq (r-1) \sum_{2}^{r} f(x) .$$

$$r(t-1) \leqq \sum_{2}^{r} f(x) \leqq r(r-1)(t-1) .$$

$$r(t-1) \leqq r(k-2) - f(1) \leqq r(r-1)(t-1) .$$

Hence we have

$$r(k-t-1) \geqq f(1) \geqq r\{(k-2) - (r-1)(t-1)\} . \tag{9.1}$$

Any two vertices in H_j are joined together otherwise there would be a claw of order $r+1$. Thus H_j is a clique.

If we put $H_j^* = H_j - (A_j \cup P)$, then H_j^* consists of exactly those vertices of T which are joined to A_j but to no other vertex of S. Hence $H_1^*, H_2^*, \cdots, H_r^*$ are disjoint sets and the total number of vertices in these sets is $f(1)$, which satisfies (9.1).

Now there is a unique grand clique K_j containing A_j and $P (j = 1, 2, \cdots, r)$. The number of vertices in K_j cannot be less than the number of vertices in H_j. If possible let $|K_j| < |H_j|$. Since K_j is a grand clique it follows that H_j is a major clique and is contained in some grand clique K'_j. Since A_j and P are contained in K_j and K'_j, they must coincide. Hence K_j contains H_j which contradicts $|K_j| < |H_j|$.

Now consider the r grand cliques $K_1, K_2, \cdots, K_r$. Then $K_1 - P$, $K_2 - P, \cdots, K_r - P$ are disjoint. For if $K_i - P$ and $K_j - P$ $(i \neq j)$ have a common point Q, then K_i and K_j would coincide and would contain both A_i and A_j which is impossible since A_i is not joined to A_j. Now

$$\begin{aligned} &|K_1 - P| + |K_2 - P| + \cdots + |K_r - P| \\ &\quad \geqq |H_1 - P| + |H_2 - P| + \cdots + |H_r - P| \\ &\quad = r + |H_1^*| + |H_2^*| + \cdots + |H_r^*| \\ &\quad = r + f(1) \\ &\quad \geqq r\{(k-1) - (r-1)(t-1)\} . \end{aligned}$$

If possible suppose there is another grand clique $K_r + 1$ containing P. The vertices in $K_{r+1} - P$ must be disjoint from the vertices in $K_1 - P, \cdots, K_r - P$. Also $K_{r+1} - P$ must have at least $(k-1) - (r-1)^2(t-1)$ vertices. If we remember that the number of first associates of P is $r(k-1)$ we have

$$r\{(k-1) - (r-1)(t-1)\} + (k-1) - (r-1)^2(t-1) \leqq r(k-1) .$$
$$k \leqq 1 + (r-1)(2r-1)(t-1) .$$

Hence $k \leqq q(r, t)$, which gives a contradiction. This finally proves our lemma.

COROLLARY 1. *The hypothesis of the Lemma may be replaced by* (i) $k > \rho(r, t)$ *and* (ii) *there exists no claw of order* $r + 1$ *in* G.

This follows from Lemma (9.5), and the inequality $\rho(r, t) \geqq q(r, t)$.

COROLLARY 2. *The hypothesis of the lemma may be replaced by* $k > p(r, t)$.

This follows from Corollary 1, Lemma (8.3), and the inequality $p(r, t) \geqq \rho(r, t)$.

THEOREM (9.1). *Let* G *be a pseudo-geometric graph* (r, k, t). *If*

(i) $k > q(r, t)$,

(ii) *two distinct vertices of G are contained in utmost one grand clique,*

(iii) *there exists no claw of order $r+1$ in G; then G is geometrisable (r, k, t).*

THEOREM (9.2). *Let G be a pseudo-geometric graph (r, k, t). If*
(i) $k > \rho(r, t)$,

(ii) *there exists no claw of order $r+1$ in G; then G is geometrisable (r, k, t).*

THEOREM (9.3). *Let G be a pseudo-geometric graph (r, k, t). If $k > p(r, t)$, then G is geometrisable (r, k, t).*

If we take the set of grand cliques of G, as the set of § 6, then Lemmas (9.5) and (9.6), together with their corollaries show that the axioms A*1 and A*2 are satisfied, under the conditions of any of the Theorems (9.1), (9.2), (9.3). The result now follows from Theorem (6.2).

10. The uniqueness of the triangular association scheme for $n > 8$. Consider a pseudo-triangular graph with characteristic n, which is a pseudo-geometric graph with characteristics $(2, n-1, 2)$ and with parameter (7.4), (7.5). In this case $r = 2$, $t = 2$ and the function $p(r, t)$ given by (8.4) is equal to 7. Hence from Theorem (9.3) the graph is geometrisable $(2, n-1, 2)$ if $n - 1 > 7$, i.e. $n > 8$.

Now $v = n(n-1)/2$ and $r = 2$. Thus each point occurs in exactly two lines. Given any point P not contained in a line b, the two lines m_1 and m_2 containing P, must both intersect b, since $r = t = 2$. Hence any two lines intersect in a unique point. If we designate the lines by the numbers $1, 2, \cdots, n$; then we may make a (1,1) correspondence between points and the unordered number of pairs (i, j), $i \neq j$, $i, j = 1, 2, \cdots, n$, where the point corresponding to (i, j) is the intersection of the lines i and j. If we now take an $n \times n$ square and write down in the cells (i, j) and (j, i) the treatment corresponding to the unordered pair (i, j), then clearly the points occurring in the same row (or same column) are those occurring in the same line (see Fig. 1 for the case $n = 5$). Thus the association relations between the vertices of the graph will be exhibited in the form known as the triangular scheme for $n > 8$. This result was first obtained by Connor [9]. Of course when we use design of experiments language the vertices of the graph or points are treatments.

Shrikhande [21] has proved the uniqueness of the triangular scheme for $n = 5, 6$ and Hoffman [13] and Chang [6] have proved the same for $n = 7$. Both Hoffman [14] and Chang [7] have shown that for $n = 8$, the parameters (7.4), (7.5) do not completely determine the scheme. There are three other possible schemes with the same parameters besides the triangular. This may be expressed as follows: *There are four non-isomorphic strongly regular graphs with parameters*

$$n_1 = 12, \qquad n_2 = 15, \qquad p^1_{11} = 6, p^2_{11} = 4$$

only one of which is geometrisable (2, 7, 2).

Consider a BIB design

$$(9.1) \qquad v^* = \tfrac{1}{2}(n-1)(n-2), \qquad b^* = \tfrac{1}{2}n(n-1), \qquad r^* = n, \qquad k^* = n-2, \qquad \lambda^* = 2$$

Hall and Connor [12] have shown that if this design exists then it can be embedded in a symmetric BIB design

$$(9.2) \qquad v_0 = b_0 = \tfrac{1}{2}n(n-1) + 1, \qquad r_0 = k_0 = n, \qquad \lambda_0 = 2\ .$$

Their proof does not cover the case $n = 8$, for which Connor [10] separately showed that the design (9.1) does not exist.

Shrikhande [22], has proved the Hall-Connor theorem for the case $n \neq 8$ by using the uniqueness of the triangular scheme for $n \neq 8$. It is interesting to observe that $n = 8$, the case not covered in Hall and Connor's entirely different proof, is exactly the case when the parameters (7.4), (7.5) do not uniquely characterize the scheme as triangular.

11. Theorems of Shrikhande, Bruck and Mesner on the uniqueness of the L_r scheme. Consider G_N as a pseudo-net graph with characteristic (r, k) or the corresponding association scheme with parameters (7.1), (7.2). Since $t = r - 1$ in this case, G_N is geometrisable $(r, k, r-1)$ if

$$(11.1) \qquad k > p(r, r-1) = \tfrac{1}{2}(r-1)(r^3 - r^2 + r + 2)\ .$$

In particular if $r = 2$, this reduces to $k > 4$.

In the case $r = 2$, the geometry consists of two sets of parallel lines. Each parallel class contains k lines, and each line contains k points. Lines of the same class do not intersect. Lines of different classes intersect in a point. Thus each point is uniquely determined as the intersection of one line from each class. We can number the lines of each class $1, 2, \cdots, k$; and we can number the points or vertices of the graph $1, 2, \cdots, k^2$. We now take a $k \times k$ square and identify the ith line of the first class with the ith row, the jth line of the second class with the jth column, and the cell (i, j) with the point which is intersection of the line i of the first class, with the line j of the second class, then the association relations between the vertices of the graph are exhibited as an L_2 scheme. This proves the uniqueness of the L_2 scheme for $n > 4$ a result obtained by Shrikhande [19] and by Mesner [15].

In the general case the geometry consists of a set of k^2 points (the vertices of G_N) and r classes of parallel lines, each class containing k lines. Lines of the same class do not intersect. Lines of different classes intersect in one point. Let the parallel classes be designated by $(R), (C), (U_1), \cdots, (U_{r-2})$. To the k lines within each class we assign the symbols $1, 2, \cdots, k$. Each point is uniquely given by the intersection of a line of (R), with a line of (C). Hence as in the case $r = 2$, the lines of (R) may be identified with the rows, and the lines of (C) with the columns of a $k \times k$ square. Then the intersection of the line of (R) and the line of (C) is identified with the cell (i, j). If in each cell (i, j) we put the number of the line of (U_α) which passes through the point corresponding to the cell, we get Latin squares L_α $(\alpha = 1, 2, \cdots, r - 2)$ and the Latin squares $L_1, L_2, \cdots, L_{r-2}$ are mutually orthogonal. Two points (cells) are first associates if they lie in the same row, same column or correspond to the same letter of the same Latin square. Thus the association relations between the points or vertices of G_N can be exhibited by the L_r scheme defined by Bose and Shimamoto [4]. Thus the L_r scheme with parameters (7.1) - (7.2) is unique (up to type) if (11.1) holds. It is necessary to add the words up to type, since there may be many non-isomorphic sets of $r - 2$ mutually orthogonal Latin squares. This result is implicit in Bruck's paper [5]. A slightly weaker result was proved by Mesner [15].

12. The SLB scheme and the general uniqueness theorem. Let us consider the SLB scheme or the pseudo-SLB graph with characteristics (r, k) for which the parameters are given by (7.6), (7.7). Then Theorem (9.3) states that the graph is geometrisable (r, k, r) if

$$k > \tfrac{1}{2} r(r^3 - r^2 + r + 1) . \tag{12.1}$$

In the language of designs this would mean that if there is an association scheme with parameters (7.6), (7.7) then if (12.1) holds the association relations can be exhibited by the dual of a BIB design (with $r_0 = k$ and $k_0 = r$, $\lambda_0 = 1$) so that the first associates are exactly those which occur together in a block of this dual and the second associates are those which do not occur together in a block of this dual. Thus (7.6), (7.7) determine the structure of the association scheme up to type. It is necessary to add the words up to type since there will exist in general non-isomorphic BIB designs with $(r_0 = k, k_0 = r, \lambda_0 = 1)$ and their duals will automatically be non-isomorphic. When (12.1) does not hold we cannot say that their will exist a dual of a BIB design i.e. a partial geometry (r, k, r), whose structure will exhibit the association relations.

In general then we can say that *if we have a pseudo-geometric association scheme with parameters* (4.1), (4.2), *then if*

$$k > p(r, t) = \tfrac{1}{2}[r(r-1) + t(r+1)(r^2 - 2r + 2)] . \tag{12.2}$$

the association structure can be exhibited by means of a partial geometry (r, k, t), the first associates being those treatments which correspond to points on a line of the geometry. Such schemes may called geometric schemes. Thus when (12.2) is true, the association scheme will be determined up to type, for there will exist non-isomorphic partial geometries with the same parameters r, k, t. This may be regarded as a generalized uniqueness theorem. When (12.2) is not true very little is known except for schemes which have the same parameters as the triangular scheme or the L_2 scheme $(r = 2, k, t = 1)$. These two cases have fully investigated.

13. A general embedding theorem.

THEOREM (13.1). *Given a PBIB design* $(r, k, \lambda_1, \lambda_2)$, $\lambda_1 > \lambda_2$ *based on a strongly regular graph* G *(association scheme) with parameters*

$$n_1 = (d-1)(k-1)(k-t)/t , \qquad n_2 = d(k-1) , \tag{13.1}$$

$$p_{11}^1 = [(d-1)(k-1)(k-t) - d(k-t-1) - t]/t , \tag{13.2}$$

$$p_{11}^2 = (d-1)(k-t)(k-t-1)/t .$$

We can extend the design by adding new blocks, containing the same treatments, in such a way that the extended design is a balanced incomplete block (BIB) design with $r_0 = r + d(\lambda_1 - \lambda_2)$ replications, block size k and in which every pair of treatments occur together in λ_1 blocks, provided that

$$k > p(d, t) = \tfrac{1}{2}[d(d-1) + t(d+1)(d^2 - 2d + 2)] . \tag{13.3}$$

Let G^* be the complementary of G, i.e. G^* is the graph with the same vertices as G, but with the relation of adjacency reversed, i.e. just those vertices in G^* are joined which were unjoined in G. This means that first associates become second associates and vice versa. The parameters of G^* are obtained from G by interchanging the subscripts and superscripts 1 and 2. Hence for G^*

$$n_1^* = d(k-1) , \qquad n_2^* = (d-1)(k-1)(k-t)/t , \tag{13.3}$$

$$(13.4)\qquad p_{22}^{2*} = [(d-1)(k-1)(k-t) - d(k-t-1) - t]/t\,, \qquad p_{22}^{1*} = (d-1)(k-t)(k-t-1)/t$$

Using the identities (2.3), (2.4), we find that

$$p_{11}^{1*} = (t-1)(d-1) + k - 2\,, \qquad p_{11}^{2*} = dt\,.$$

Hence G^* is pseudo-geometric with characteristics (d, k, t).

In view of (13.3) it follows from theorem (9.3) that G^* is geometrisable (d, k, t). From (4.1) the geometry has $d[(d-1)(k-1)+t]/t$ blocks, and every pair of treatments which were second associates in the original PBIB design occur once in the new blocks. If we add these new blocks repeated $\lambda_1 - \lambda_2$ times to the original blocks then each pair occurs λ_1 times and each treatment occurs $r + d(\lambda_1 - \lambda_2)$ times. This proves our theorem.

We shall now derive from this the embedding theorem on orthogonal Latin squares due to Shrikhande and Bruck.

In Theorem (13.1) take $t = d - 1$, then G^* is a pseudo-geometric graph $(d, k, d-1)$ i.e. a pseudo-net graph. If

$$k > p(d, d-1) = \tfrac{1}{2}(d-1)(d^3 - d^2 + d + 2)\,,$$

it is geometrisable.

Also let us take $r = k + 1 - d, \lambda_1 = 1, \lambda_2 = 0$. Then the PBIB design becomes the design $(k+1-d, k, 1, 0)$ based on the strongly regular graph with parameters (7.1), (7.2). This is easy to check by substituting $d = k + 1 - r$ in (13.1), (13.2) and noting that they reduce to (7.1), (7.2). Hence the PBIB design is a net of degree $k + 1 - d$, or a lattice design with $r = k + 1 - d$ and block size k. Hence the extended design is a BIB design with $r + d$ i.e. $k + 1$ replications in which every pair of treatments occurs in one block. This is an affine plane of order k. Hence we have

THEOREM (13.2A). *A lattice design (or a net) with $r = k + 1 - d$ and block size k can be completed to an affine plane*

$$v_0 = k^2, \quad b_0 = k(k+1), \quad r_0 = k + 1, \quad k_0 = k, \quad \lambda_0 = 1\,,$$

by adding kd new blocks, if $k > \frac{1}{2}(d-1)(d^3 - d^2 + d + 2)$.

Now we have already noticed the equivalence of a lattice design with r replications and block size k, with a set of $r - 2$ mutually orthogonal Latin squares of order k. Since an affine plane of order k can be regarded as a lattice with $k + 1$ replications, Theorem (13.2A) may alternatively be stated as

THEOREM (13.2B). *If there exist $k - 1 - d$ mutually orthogonal*

Latin squares of order k, it is possible to get a complete set of $k-1$ mutually orthogonal Latin squares, by adding d new suitably chosen squares, provided that $k > \frac{1}{2}(d-1)(d^3-d^2+d+2)$.

The case $d = 2$, was first obtained by Shrikhande [20] and the general case was obtained by Bruck [5].

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FINITE NETS, II. UNIQUENESS AND IMBEDDING

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1. Introduction. In discussing the present paper we have a choice of three languages: (a) the language of orthogonal latin squares; (b) the language of incomplete block designs, as used in connection with design of experiments; and (c) the geometric language of nets. As far as proofs are concerned, either (b) or (c) affords a useful symmetry which is missing in (a); it is merely a matter of taste that we choose (c). Here let us begin with (a).

Let C be a collection of t mutually orthogonal latin squares of side n. We assume $n > 1$, $t \geqq 1$. The inequality $t \leqq n - 1$ necessarily holds; if $t = n - 1$, C is said to be *complete*. As is well known, a complete set of orthogonal latin squares of side n determines and is determined by an affine plane of order n. We define the *degree*, k, and *deficiency*, d, of C by

$$k = t + 2\,, \qquad d = n - 1 - t\,, \tag{1.1}$$

so that

$$k + d = n + 1\,. \tag{1.2}$$

Here k is, in language (b), the number of *constraints:* one constraint for the rows of the squares, one for the columns, and one for each of the t squares. On the other hand, if C can be enlarged to a complete set, C', of $n - 1$ mutually orthogonal latin squares, then d is the number of squares in C' which are not in C; or the number of constraints missing in C. In language (c) we may describe C as a net N of order n, degree k, deficiency d. For an example of such a net N, we may begin with an affine plane π of order n—with its n^2 points and $n + 1$ parallel classes of lines, n lines per class—and retain the points but delete some d parallel classes.

Before discussing the results of the paper, it will be convenient to define two polynomials $p(x)$, $q(x)$:

$$p(x) = \tfrac{1}{2}x^4 + x^3 + x^2 + \tfrac{3}{2}x\,, \tag{1.3}$$

$$q(x) = 2x^3 - x^2 - x + 1\,, \tag{1.4}$$

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and to note the following table:

d	$(d-1)^2$	$q(d-1)$	$2(d-1)^3$	$p(d-1)$	$d^4/2$
1	0	1	0	0	1/2
2	1	1	2	4	8
3	4	11	16	23	40 1/2
4	9	43	54	81	128
5	16	109	128	214	312 1/2
6	25	221	250	470	648

We note from (1.1) that the side n and deficiency d of the collection C (of mutually orthogonal latin squares) satisfy the inequality $n \geqq d + 2$. Assuming that $d \geqq 1$, we are interested in conditions under which C can be completed; that is, can be enlarged to a complete set, C', of mutually orthogonal latin squares of side n. Our first result is:

(A) *If $n > (d-1)^2$, and if C can be completed at all, then it can be completed uniquely, aside from trivialities.*

This follows from Theorem 3.1. However, examples show that the condition $n > (d-1)^2$ does not ensure completion. On the other hand:

(B) *If $n > p(d-1)$, C can always be completed.*

This follows from Theorem 4.3. The result (B) is known to be best possible for $d = 1$ (folk-lore) and for $d = 2$ (Shrikhande [9]). Whether (B) is best possible for $d > 2$ is unknown to the author. Before mentioning further results, intermediate between (A) and (B), which take into account the structure of C, it seems worthwhile to give a simple consequence of (B).

Bose and Shrikhande defined $m(n)$ to be the maximum number of mutually orthogonal latin squares of side n. As a result of the work of Bose, Shrikhande and Parker (see, for example, [1]), Chowla, Erdos and Straus [5] were able to prove that

$$m(n) > \tfrac{1}{3}n^{1/91} \tag{1.5}$$

for all sufficiently large n (the lower bound on n being unknown.) In view of (B) we may state a dichotomy:

Either (I) $m(n) = n - 1$

or (II) $n \leqq p(n - 2 - m(n)) < \frac{1}{2}[n - 1 - m(n)]^4$.

As an easily stated consequence:

(1.6) *If* $m(n) < n - 1$, *then* $m(n) < n - 1 - (2n)^{1/4}$.

We note that (I) holds precisely when there exists an affine or projective plane of order n. Thus (I) holds for infinitely many n, for example, for every prime-power. However, by the Bruck-Ryser Theorem (Bruck and Ryser [3]), (II) also holds for infinitely many n. We may add that, just as Chowla et al. state that their methods would not allow (1.5) to be improved to

$$(1.7) \qquad m(n) > n^{1/2}$$

(although (1.7) is not known to be false for large n), so it seems likely that the present methods would not allow (1.6) to be improved to

$$(1.8) \qquad \textit{If } m(n) < n-1 \text{ , } \textit{then } m(n) \leqq n-2-n^{1/2} \text{ .}$$

Note that (1.8) would result from (II) if we could replace $p(x)$ by x^2. —A more reasonable possibility is that $p(x)$ could be replaced by $q(x)$, by dint of a more penetrating discussion of maximal incomplete sets of orthogonal latin squares. This would give an exponent 1/3, instead of 1/2, in (1.8).—But even if (1.7), (1.8) could both be proved, they would still leave a great gap in our knowledge of $m(n)$.

The refinements of (A), (B) are conveniently stated in terms of graphs. From the collection C—or, equivalently, from the corresponding net N of order n, degree k, deficiency d—we define a graph G_1 with n^2 vertices, whose edges are the unordered pairs of distinct points lying on a common line of the net N. If G_2 is the complementary graph of G_1 then G_2 has (at least superficially) the type of structure that one would associate with the graph of a net of order n, degree d, deficiency k. (Note that the roles of k and d have been interchanged.) We abstract from this superficial structure a definition of what we call a *pseudo net-graph* of order n, degree d, deficiency k. Our first observation is that, to enlarge C to a complete set C', or, equivalently, to imbed the net N in an affine plane of order n, we must introduce a suitable collection of lines into the complimentary graph G_2 in such a way as to turn G_2 into the graph of a net of order n, degree d, deficiency k. We actually prove our results for pseudo net-graphs. Thus (B) is obtained as a consequence of:

(B′) *If* $n > p(d-1)$, *every pseudo net-graph of order* n, *degree* d *is the graph of a uniquely defined net of order* n, *degree* d.

The corresponding theorem for $d = 2$ was proved by Shrikhande [10], who also refers to unpublished results of Dale Mesner for $d \geqq 2$. In the language (b) used by Shrikhande, (B′) could be restated as:

(B″) *If* $n > p(d-1)$, *and if the parameters of the second kind* *for a partially balanced incomplete block design with n^2 treatments with two associate classes are given by*

$$n_1 = d(n-1)\ , \quad p_{11}^1 = n - 2 + (d-1)(d-2)\ , \quad p_{11}^2 = d(d-1)\ ,$$

then the design has L_d association scheme.

Now we require the notion of a *claw*. If G is a pseudo net-graph, a claw P, S of G is a pair consisting of a vertex P and a nonempty set S of vertices distinct from P such that P is joined in G to every vertex in S but no two vertices in S are joined in G. The *order* of the claw is the cardinal number, $|S|$, of S. If G is the graph of a net of degree d then, obviously, G has no claws of order $d+1$. We may state a partial converse (see Theorem 4.2):

(C) *If $n > 2(d-1)^3$, and if G is a pseudo net-graph of order n, degree d which possesses no claws of order $d+1$, then G is the graph of a uniquely defined net of order n, degree d.*

This result is also given by Shrikhande [10] for $d = 2$. We may remark here that the inequality in (C) could probably be sharpened to

$$n > 2(d-1)^3 - (d-1)^2\ .$$

This could be done if the right-hand side of formula (4.7) in Lemma 4.2 could be replaced by $d-1$, as seems likely.

To state our final result in this direction we need the notion of a *grand clique*. A clique (of a pseudo-net graph G of order n, degree d) is a set of vertices every two of which are joined in G. And a grand clique is a maximal clique containing at least

$$n - (d-1)^2(d-2)$$

vertices. Our result is (Theorem 4.1):

(D) *Assume $n > q(d-1)$, and let G be a pseudo net-graph of order n, degree d such that (i) no two distinct grand cliques of G have more than one common vertex and (ii) G has no claws of order $d+1$. Then G is the graph of a uniquely defined net of order n, degree d.*

We may remark that, for $d = 1$ or 2, condition (i) of (D) may be dropped. Indeed, in these cases, grand cliques have exactly n vertices, and this simplifies matters considerably. On the other hand, for $d > 2$, (i) is needed to help us prove that grand cliques have exactly n vertices and are in fact the lines of a net.

These are perhaps the main results of the paper. However, other items are also worthy of note. In § 2 we find it worthwhile to formalize the familiar process of "enumeratinng in two ways." We feel that this process would repay formal study, just as the formal study of equality has led to a rich theory of equivalence relations. In § 5 (originally conceived as a section designed to end all study of incidence matrices, but now recast) we uncover a one-to-one correspondence, apparently unknown until the present, between sets of $k-2$ mutually orthogonal latin squares of side n and sets of k mutually orthogonal matrices of order n^2. (Theorem 5.1). The suitable modification for pseudo net-graphs is given in Theorem 5.2. We also show in § 5 that a conjecture concerning adjacency matrices of finite graphs (originally advanced by Harary and disproved by Bose) is hopelessly beyond repair.

The paper [4], of like title to the present one, was compressed at the suggestion of the editors. A good deal of material—some of which appears in almost unrecognizable form in §§ 3, 5 of the present paper—was omitted, including all examples. There are some grounds for our belief that the result was to hamper theory of latin squares. As a case in point, a counterexample contained in the original version of [4], and known to the author in 1949, served in 1961 to halt an extensive high-speed machine program on latin squares. With this in mind, we have tried in the concluding section (§ 6) to include a reasonable selection of remarks and examples.

In conclusion, the author would like to express his appreciation to The RAND Corporation of Santa Monica and to all the participants of the 1961 Summer Symposium on Combinatorial Mathematics of Project RAND. The present paper has been largely molded in discussions with Alan Hoffman, R. C. Bose and E. T. Parker. Hoffman is certainly the father of Lemma 4.4 (though he is not responsible for (4.19)), and Hoffman and Bose must share some guilt in connection with the birth of Theorem 5.1—which they, however, have never seen.

2. Counting in two ways. During the course of this paper we shall have many occasions to use the familiar process of "counting in two ways." In order to ensure brevity without loss of clarity, it seems worthwhile to state the process as a formal lemma. Here, for any set S, $|S|$ denotes the cardinal number of S.

LEMMA 2.1. *Let A, B be nonempty sets, ρ be a finite subset of the direct product set $A \times B$. For each a in A, let $a\rho$ denote the subset of B consisting of all b in B such that (a, b) is in ρ; and, for each b in B, let ρb be the subset of A consisting of all a in A such* *that* (a, b) *is in* ρ. *Then*

$$\sum_{a \in A} |a\rho| = \sum_{b \in B} |\rho b| . \tag{2.1}$$

Proof. For each a in A, the set $(a, a\rho)$, consisting of all pairs (a, b) with b in $a\rho$, contains precisely $|a\rho|$ elements of ρ. Also, the sets $(a, a\rho)$, as a ranges over A, partition ρ—provided we ignore the empty sets which may turn up. Hence the left-hand side of (2.1) is equal to $|\rho|$. Similarly for the right-hand side of (2.1). This completes the proof.

It goes without saying that the value of (2.1) in any particular case depends upon skill in choosing the sets A and B (these may often be complex sets constructed from others more immediately at hand) and the relation (or finite subset) ρ. I would conjecture that all proofs by enumeration may be reduced to a sequence of applications of the apparently innocuous Lemma 2.1. Be that as it may, there were several instances at the 1961 Combinatorial Symposium of Project RAND in which Lemma 2.1 provided a simpler alternative to proofs involving matrix calculations.

3. Nets. We begin with a positive integer (or, more generally, with any cardinal number) k such that

$$k \geqq 3 . \tag{3.1}$$

A k-net, N, is a system of undefined points and lines, together with an incidence relation, subject to the following axioms: (i) N has at least one point. (ii) The lines of N are partitioned into k disjoint, nonempty, "parallel classes" such that (a) each point of N is incident with exactly one line of each class; (b) to two lines belonging to distinct classes there corresponds exactly one point of N which is incident with both lines. For convenience, we shall use phrases such as "point is on line" instead of speaking of incidence.

The axioms, coupled with (3.1), ensure the existence of two distinct lines L, L' of N and a parallel class K containing neither of L, L'. Since each point of L lies on a unique line of class K, and since each line of class K meets L in a unique point, there is a one-to-one correspondence between the points of L and the lines of K. Similarly, there is a one-to-one correspondence between the points of L' and the lines of K. Furthermore, each point of N lies on exactly one line of K. Hence, if some line of N contains exactly n distinct points, the following statements are true:

(I) Each line of N contains exactly n distinct points, where $n \geqq 1$.

(II) Each point of N lies on exactly k distinct lines, where $k \geqq 1$.

(III) N has exactly kn distinct lines. These fall into k parallel classes of n lines each. Distinct lines of the same parallel class have no common points. Two lines of different classes have exactly one common point.

(IV) N has exactly n^2 distinct points.

A system N satisfying (I)-(IV) we shall call a net of *order* n, *degree* k. If (3.1) fails—in particular, if $k = 1$ or 2—we shall call the net *degenerate*. And if $n = 1$ we shall call the net *trivial*. In the sequel we study finite nontrivial nets (n and k finite) but we cannot entirely avoid degenerate nets.

For each finite nontrivial net N of order n, degree k we introduce integers d, n_i and p^i_{jk} as follows:

$$k + d = n + 1 \tag{3.2}$$

$$n_1 = k(n-1)\,, \qquad n_2 = d(n-1)\,, \tag{3.3}$$

$$\begin{aligned}
p^1_{11} &= n - 2 + (k-1)(k-2)\,,\\
p^1_{12} = p^1_{21} &= (k-1)d\,,\\
p^1_{22} &= d(d-1)\,,\\
p^2_{22} &= n - 2 + (d-1)(d-2)\,,\\
p^2_{21} = p^2_{12} &= (d-1)k\,,\\
p^2_{11} &= k(k-1)\,.
\end{aligned} \tag{3.4}$$

In (3.3), (3.4) we are using the notation of R. C. Bose [2]. We call the integer d the *deficiency* of N. It is to be observed that interchange of k and d preserves (3.2) and has the effect in (3.3), (3.4) of interchanging the subscripts and superscripts 1, 2.

Before making clear the significance of the above definitions, it will be convenient to introduce further notation. If P, Q are two distinct points of N we say that P, Q are *joined in* N if there exists a line PQ of N (necessarily unique) which contains both P and Q; if the line PQ does not exist, we say that P, Q are *not joined* in N. By a *partial transversal*, S, of N we mean a nonempty set, S, of points of N such that every two distinct points in S are not joined in N. By a *transversal* of N we mean a partial transversal with exactly n distinct points (where n is the order of N). We are now ready for an important elementary lemma.

LEMMA 3.1. *Let N be a nontrivial finite net of order n, degree k, deficiency d.*

(i) *If S is a partial transversal of N, then $|S| \leqq n$.*

(ii) *If P is a point of N, then, of the $n^2 - 1$ points of N distinct from P, n_1 are joined to P in N and n_2 are not joined to P in N.*

(iii) *If P is a point of N and if L is a line of N not containing P, then P is joined to $k - 1$ points of L and P is not joined to d points of L.*

(iii′) *If P is a point of N and if T is a transversal of N not containing P, then P is not joined to $d - 1$ points of T and P is joined to k points of T.*

(iv) *If P, Q are distinct points joined in N, then, of the remaining $n^2 - 2$ points, p^1_{11} are joined to both of P, Q; p^1_{12} are joined to P and not joined to Q; p^1_{21} are not joined to P and joined to Q; p^1_{22} are not joined to P and not joined to Q.*

(iv′) *If P, Q are distinct points not joined in N, then, of the remaining $n^2 - 2$ points, p^2_{22} are not joined to P and not joined to Q; p^2_{21} are not joined to P and joined to Q; p^2_{12} are joined to P and not joined to Q; p^2_{11} are joined to P and joined to Q.*

REMARKS. (1) The statement of Lemma 3.1 is intended to emphasize a duality of importance for the sequel. Item (i) merely points out that transversals are maximal partial transversals. (However, not every maximal partial transversal is a transversal.) We note that if "joined" and "not joined" are interchanged, then (ii) remains true provided n_1, n_2 are interchanged; (iii) and (iii′) are interchanged provided "line L" and "transversal T" are interchanged, as well as k and d; and (iv), (iv′) are interchanged provided the subscripts and superscripts 1, 2 are interchanged.

(2) In view of (ii)—since $n_2 = d(n - 1)$—or (iii) we see that *the deficiency, d, of a finite net N, is a nonnegative integer.*

(3) In view of (ii) we see that *a finite net, N, of order n, deficiency zero is precisely an affine plane of order n.* Thus the deficiency measures the extent to which a net fails to be an affine plane—namely, it lacks d classes of parallel lines.

(4) In view of (iii′) we see that *if $d = 0$, then N has no transversals.* (Indeed, if $d = 0$, each partial transversal of N has exactly one point, since every two distinct points are joined in N—by (ii), (3.3).)

Proof (i). Let $s = |S|$ and let K be any parallel class of lines of N. Each of the s points of S lies on a unique line of K. Two distinct points of S are not joined in N and hence lie on distinct lines of K. Therefore $s \leqq |K| = n$. This proves (i).

(ii) We note from (3.3), (3.2) that $n_1 + n_2 = n^2 - 1$. Each of the k lines through P contains $n - 1$ points in addition to P. The $n_1 = k(n - 1)$ points so obtained are distinct and are all the points joined to P. This proves (ii).

(iii) We note from (3.2) that $(k - 1) + d = n$. One of the k lines through P is parallel to L. The rest meet L in $k - 1$ distinct points. Moreover, L has exactly n distinct points. This proves (iii).

(iii′) The n distinct points of T lie one each on the n distinct lines of each parallel class (cf. the proof of (i)). Hence the k lines through P meet T in k distinct points. Since $k + (d - 1) = n$, there remain $d - 1$ points of T not joined to P. This proves (iii′).

(iv) Here P, Q lie on a line PQ of N. There are $n - 2$ points of PQ which are joined to both P and Q. Each of the $k - 1$ lines through P, other than PQ, is met by the $k - 1$ lines through Q, other than PQ, in $k - 2$ distinct points (there being a case of parallelism). This gives a total of

$$n - 2 + (k - 1)(k - 2) = p^1_{11}$$

distinct points joined to both P and Q. Since

$$p^1_{11} + p^1_{12} = n_1 - 1$$

by (3.4), (3.2), (3.3), and since P is joined (by (ii)) to exactly $n_1 - 1$ points distinct from itself and Q, then there are exactly p^1_{12} distinct points joined to P but not to Q. (And, of course, there are p^1_{21} distinct points joined to Q but not to P.) Since

$$p^1_{21} + p^1_{22} = n_2$$

by (3.4, (3.2), (3.3), and since P is not joined (by (ii)) to exactly n_2 distinct points, then there are exactly p^1_{22} distinct points joined to neither P nor Q. This proves (iv).

(iv′) Here P, Q are not joined in N. Since each of the k lines through P is met by the k lines through Q in exactly $k - 1$ points (one case of parallelism) and since none of these intersection points is P or Q, there are exactly

$$k(k - 1) = p^2_{11}$$

distinct points joined to both P and Q. Since

$$p^2_{11} + p^2_{12} = n_1 ,$$

there are exactly p^2_{12} points joined to P but not Q. Since

$$p^2_{21} + p^2_{22} = n_2 - 1 ,$$

and since P is not joined to exactly $n_2 - 1$ points in addition to Q, the proof of (iv′) and of Lemma 3.1 is now complete.

It will be convenient at this point to make a brief review of some well known facts about nets. Let $n \geqq 2$ be any given positive integer, and let us construct a square of side n, containing n^2 cells. We regard the cells as points of a net N. If we define two distinct cells to be joined in N if and only if they lie in the same row, then N is a (degenerate) net of order n, degree 1, with the n rows of cells as the n lines of its single parallel class. If we allow both rows and columns of cells as lines, we have a (degenerate) net of order n, degree 2. If we now mark the cells with the numbers 1 through n in such a way as to form a latin square and allow, in addition to the row-lines and column-lines, lines consisting of n cells marked with the same number, we get a net of order n, degree 3. Similarly, for any integer k in the range $3 \leqq k \leqq n+1$, a set of $k-2$ mutually orthogonal latin squares of side n may be used to define a (non-degenerate) net of order n, degree k. Conversely, any net of order n, degree k ($k \geqq 1$) can be obtained in the manner indicated, usually in many ways.

To imbed a net N of order n, degree k (where $k < n+1$) in a net N_1 of order n, degree $k+1$ which has the same points as N and has k of its line classes identical with those of N is equivalent to finding a single new "parallel class." This must consist of n distinct transversals of N, no two with a point in common. To imbed N in an affine plane N_2 of order n (with the same points as N and with k of its line classes identical with those of N) is equivalent to finding $d = n + 1 - k$ new "parallel classes," consisting of d sets of n parallel transversals, such that two distinct transversals belonging to the same set have no common point and two belonging to different sets have exactly one common point. It is easy to see that each of the n^2 points should lie in exactly d of the transversals. Indeed, to imbed net N of order n, degree k, deficiency $d > 0$ in an affine plane is equivalent to defining a *complementary* net, N', of order n, degree d, deficiency k, whose points are identical with those of N and whose lines are a suitably selected set of transversals of N.

Several problems arise. A given net may have no complementary net or several complementary nets. How can we ensure existence or uniqueness of a complementary net? A given net may have several classes of parallel transversals, or no complete parallel class of transversals, or no transversals at all. How can we ensure existence of a suitable collection of transversals?

One case in which transversals are embarrassingly common is worth mentioning. A net of order 10, degree 3, is essentially a latin square of side 10. Here the deficiency is $d = 8$. To imbed such a net in an affine plane of order 10 we would need a suitable collection of $dn = 80$ transversals, 8 through each point. No such collection has ever been found. However, E. T. Parker, in a machine search for an orthogonal mate to suitably selected latin squares, usually finds an average of about 120 transversals per cell—or about 15 times as many transverals per point of the net as we would want. As we shall see, the situation changes when the order n is somewhat "larger" compared with the deficiency d.

LEMMA 3.2. *Let N be a finite nontrivial net of order n, degree k, deficiency $d > 0$. Let T be a transversal of N and let S be a partial transversal of N not contained in T but containing at least two points of T. Then*

$$|S \cap T| \leqq d - 1 , \tag{3.5}$$

$$|S| \leqq (d - 1)^2 . \tag{3.6}$$

COROLLARY. *If N is a finite nontrivial net of order n, deficiency $d > 0$, and if $n > (d - 1)^2$, then two distinct transversals of N can have at most one common point.*

Proof. By hypothesis, S contains at least one point R which is not in T. By Lemma 1 (iii′), there are precisely $d - 1$ points of T not joined to R. Among these $d - 1$ points must be the points of $S \cap T$, since R is joined to no other point of S. Hence we have (3.5). Again, by hypothesis, $S \cap T$ contains at least two distinct points P, Q. By Lemma 1 (iv′), there are precisely p_{22}^2 points joined to neither P nor Q, and the points of $S \cup T - \{P, Q\}$ must be among these p_{22}^2 points. Hence

$$|S \cup T| \leqq p_{22}^2 + 2 = n + (d - 1)(d - 2) . \tag{3.7}$$

By this and (3.5), we have

$$\begin{aligned} |S| + |T| &= |S \cap T| + |S \cup T| \\ &\leqq (d - 1) + n + (d - 1)(d - 2) = n + (d - 1)^2 . \end{aligned}$$

However, $|T| = n$, since T is a transversal. Therefore we have (3.6).

If we assume that S is also a transversal, (3.6) yields

$$n \leqq (d - 1)^2 . \tag{3.8}$$

At this point we note that if S, T are any two distinct transversals, then S must have a point not in T. Hence, if we further assume that S, T have at least two common points, we get (3.8). Thus, by denying (3.8), we get the Corollary. This completes the proof.

Two remarks are in order. First, if (3.8) holds, then (3.6) is trivial in view of Lemma 1 (i). Secondly, the Corollary to Lemma 3.2 is "best possible" of its kind. One class of examples may be obtained as follows: Let m be any integer (for example, any prime power) for which there exists an affine plane π of order $n = m^2$ which possesses an affine subplane π_1 of order m. We form a net N of order $n = m^2$, degree $k = m^2 - m$, deficiency $d = m + 1$ whose points are the points of π and whose lines are the k parallel classes of π containing no lines of π_1. The net is degenerate if $m = 2$, and nondegenerate otherwise. Among the transversals of N are the $m^2 + m$ lines of π_1 (that is, the lines of π containing at least two and hence exactly m points of π_1) and each two of these intersect in at most one point. But there is another transversal, namely the set consisting of the $n = m^2$ points of π_1, and this has exactly m points in common with each of the lines of π_1. In this class of examples we have $n = (d-1)^2$. In addition, when equality holds in (3.8), transversals seem to behave as the above discussion indicates. Indeed:

LEMMA 3.3. *Let N be a finite net of order $n = m^2$, degree $k = m^2 - m$, deficiency $d = m + 1$. Assume $m > 2$, so that N is nontrivial and nondegenerate.*

(i) *If S, T are distinct transversals with more than one common point, then they have exactly $d - 1 = m$ common points. Moreover (a) each point of $S - T$ is joined to each point of $T - S$ and (b) if P, Q are any two distinct points of the intersection $S \cap T$, then every point not in the union $S \cup T$ is joined to at least one of P, Q.*

(ii) *If S, T, U are three distinct transversals such that S has m points in common with each of T, U, then T, U have at most one common point.*

Proof. For (i), we use the proof of Lemma 3.2, assuming that S, T are distinct transversals with at least two common points. Then (3.5), (3.7) become

$$(3.9) \qquad |S \cap T| \leqq m\,, \qquad |S \cup T| \leqq 2 + p_{22}^2 = m^2 + m(m-1)\,,$$

and we get

$$2n = |S| + |T| = |S \cup T| + |S \cap T| \leqq 2m^2 = 2n\,.$$

Hence we must have equality in (3.9). Thus $|S \cap T| = m$, and, moreover, (b) holds. Again, if R is any point in $S - T$, then R is not joined to exactly $d - 1 = m$ points of T, and these points must be the points of $S \cap T$. Consequently, R must be joined to every point of $T - S$. This proves (a) and completes the proof of (i).

To prove (ii), we begin by assuming that $S \cap T \cap U$ has at least two distinct points P, Q. Then, by (i) (b), since $S \cap T$ has m points and since no point of U is joined to P or Q, we must conclude that U is in $S \cup T$. Since, by (i) (a), every point of $S - T$ is joined to every point of $T - S$, we see that U cannot contain both a point of $S - T$ and a point of $T - S$. Therefore either $U \subset S$ or $U \subset T$. But then, since $|U| = |S| = |T|$, either $U = S$ or $U = T$, in contradiction to hypothesis. Consequently,

$$|S \cap T \cap U| \leqq 1 . \tag{3.10}$$

By hypothesis, $|S \cap T| = m = |S \cap U|$. By (3.10), $S \cap T, S \cap U$ have at most one common point. These two facts, taken together, tell us that U has at least one point of $S - T$ (indeed, at least $m - 1$ such points). Therefore U, having a point of $S - T$, can have no point of $T - S$. This means that

$$T \cap U \subset S \cap T \cap U . \tag{3.10a}$$

And (13.10a), (3.10) complete the proof of Lemma 3.3.

There are many other examples indicating that the Corollary to Lemma 3.2 is best possible. One comes from the nets of order $n = 6$, degree $k = 3$, deficiency $d = 4$. Here we have $n = 6 < 9 = (d - 1)^2$. Such nets are given by latin squares of order 6. There are 17 types, and at least one has two distinct transversals with 3 common points. (See Fisher and Yates [6].)

As far as construction is concerned, the nets satisfying (3.8) are the most important at present. Nevertheless, there is a great deal to be learned about the remaining nets, and we shall be concerned here with inequalities at least as strong as

$$n > (d - 1)^2 . \tag{3.11}$$

The most obvious consequences of (3.11*) are summed up in the following theorem.

THEOREM 3.1. *Let N be a finite nontrivial net of order n, degree k, deficiency d, satisfying* (3.11*). *Let N^* be the system whose points are the points of N and whose lines are the lines of N together with the transversals of N, and whose incidence relation is the natural one.*

(i) *If t is the total number of distinct transversals of N, then*

$$t \leqq dn . \tag{3.12}$$

(ii) *A necessary and sufficient condition that N be imbeddable in an affine plane of order n is that equality hold in* (3.12).

(iii) *If N is imbeddable in an affine plane N_1 of order n, then*

N_1 is isomorphic to N^*. In summary, N^* is the only candidate for an affine plane of order n containing N.

Proof. By the Corollary to Lemma 3.2, two distinct transversals of N have at most one common point. Moreover, two distinct lines of N have at most one common point, and a line and a transversal of N have exactly one common point. Consequently, *two distinct lines of N^* have at most one common point.*

For each point P of N (and N^*), let $t(P)$ be the number of distinct transversals of N containing P. Thus the number of distinct lines of N^* containing P is exactly

$$k + t(P) .$$

Two such lines have only the point P in common. Therefore the number of points, distinct from P, to which P can be joined in N^* (not N!) is

$$[k + t(P)](n - 1) \leqq n^2 - 1 .$$

Since $n > 1$, we deduce that

$$k + t(P) \leqq n + 1 = k + d$$

and hence that

$$t(P) \leqq d \tag{3.13}$$

for every point P in N. Moreover, for any fixed P, equality holds in (3.13) precisely when P can be joined (in N^*) to every other point. By summing (3.13) over the n^2 points P of N, and remembering that every transversal has exactly n points, we see that (3.12) holds, with equality precisely when every two distinct points are joined in N^*. In particular, (i) is true.

If N is imbeddable in an affine plane N_1 of order n, then (when N is considered as a subsystem of N_1) every line of N_1 is either a line of N or a transversal of N. Hence every line of N_1 is a line of N^*. Since every two distinct points are joined in N_1, we must conclude that equality holds in (3.12).

Now suppose, conversely, that equality holds in (3.12). Then, also, equality holds in (3.13) for every point P, and every two distinct points are joined in N^*. We consider a transversal T and a line L of N and note the T, L have a unique common point, Q. Let P be any point of L distinct from Q. Then P is not in T. Hence, by Lemma 1 (iii′), there are axactly $d - 1$ distinct points of T not joined in N to P. Each of these is joined to P by a unique line of N^*, giving, in all, $d - 1$ distinct transversals of N which contain P and intersect T. Since $t(P) = d$, there remains a unique transversal which contains P and if parallel to T. As P varies over the $n - 1$ points of L distinct from Q, we get in this way $n - 1$ distinct transversals parallel to T. No two of these transversals intersect, for a common point R would lie on two distinct transversals parallel to T. Consequently, when we include T, we get a set of n distinct, mutually parallel transversals. These must contain all the points of N, namely n points on each of n transversals. It should now be clear that the $t = dn$ transversals of N form d distinct parallel classes of lines of N^*, distinct from the kn lines of N. Therefore N^* is a net of order n, degree $k + d = n + 1$, deficiency 0. That is, N^* is affine plane.

Putting the last two paragraphs together, we see that (ii) and (iii) are true. This completes the proof of Theorem 3.1.

It would be wrong to assume that the N^* of Theorem 3.1 is always an affine plane. If N is the net of order $n = 4$, degree $k = 3$, deficiency $d = 2$ given by the cyclic group of order 4 then (3.11) holds but N has no transversals. If N is the net of order $n = 5$, degree $k = 3$, deficiency $d = 3$ given by any loop of order 5 other than the cyclic group (there are only two nets of order 5, degree 3) then (3.11*) holds but N has exactly 3 transversals; one point lies on all three, 12 points lie each on one, and 12 points lie on none. Moreover (see Norton [8]) there exists a net of order $n = 7$, degree $k = 5$, deficiency $d = 3$ with too few transversals to be imbedded in a net of degree 6, deficiency 2. Precise necessary and sufficient conditions, in the presence of (3.11*), that N^* be an affine plane still await exploration.

In the section which follows we show, in particular, that a suitable strengthening of the inequality (3.11*) suffices to ensure that N^* is an affine plane.

4. Net-graphs and pseudo net-graphs. From a net N of order n, degree k, deficiency d we form a *net-graph* G_1 *of order* n, *degree* k, *deficiency* d (namely, the *graph of* N) as follows: G_1 has n^2 vertices, namely the n^2 points of N. Two distinct points P, Q of N form an (unordered) edge $\{P, Q\}$ of G_1 if and only if P, Q are joined in N (that is, lie on a common line of N.) Since the edges of G_1 are unordered, G_1 is a *symmetric* graph. Since each vertex of G_1 lies on exactly $n_1 = k(n - 1)$ edges of G_1, the graph is *regular*. But G_1 has still more regularity, given in Lemma 3.1 (iv) and (iv′) in terms of the *constants of connection* p^i_{jk}.

For any symmetric graph G, the *complementary* graph G' is a symmetric graph with the same vertices as G, such that, if P, Q are distinct vertices of G, then $\{P, Q\}$ is on edge of G' precisely when $\{P, Q\}$ is not an edge of G.

In particular, if G_1 is as in the first paragraph, and if G_2 is the complement of G_1, then G_2 is an example of what we shall call a *pseudo* net-graph of order n, degree d, deficiency k. And the question as to whether N can be imbedded (in at least one way) in an affine plane of order n is (as essentially noted in § 3) equivalent to the question as to whether G_2 is the net-graph of at least one net N' of order n, degree d, deficiency k, namely a net complementary to N. Moreover, by Theorem 3.1, if $n > (d-1)^2$, and if G_2 is a net-graph, then the corresponding net is uniquely defined by G_2.

By a *pseudo net-graph* G of order n, degree d, deficiency k, where n, d, k are nonnegative integers related by

$$d + k = n + 1 , \tag{4.1}$$

we mean a symmetric graph with n^2 vertices such that

(i) each vertex of G is joined (by an edge of G) to exactly

$$n_1 = d(n-1)$$

other vertices of G;

(ii) two distinct vertices P, Q of G which are joined in G are together joined to exactly

$$p_{11}^1 = n - 2 + (d-1)(d-2)$$

other vertices of G;

(iii) two distinct vertices P, Q which are not joined in G are together joined to exactly

$$p_{11}^2 = d(d-1)$$

other vertices in G.

It will be noted that we have interchanged k and d and the indices 1 and 2 in formulas (3.2), (3.3), (3.4). This is merely a matter of convenience in view of the application to imbedding of nets. We shall have little need to refer to the deficiency, k, of G. However, to avoid trivialities, we shall assume throughout that

$$n \geqq d \geqq 1 . \tag{4.2}$$

By a *clique* of graph G we mean a subgraph of G every two of whose vertices are joined in the subgraph. That is, a clique is a complete subgraph of G. We are interested in introducing certain cliques as lines. Specifically, if G is a pseudo net-graph of order n, we define a *line* of G to be a clique with exactly n vertices. When G is the complementary graph, G_2, of a net N, the cliques of G are the partial transversals of N, and the lines of G are the transversals of N. In this case, by Lemma 3.1 (i), no clique of G has more than n elements. The same fact is true for pseudo net-graphs, but requires a different proof.

LEMMA 4.1. *Let G be a pseudo net-graph of order n, degree d, and let L be a line of G. Then*

(i) *each vertex of G which is not in L is joined in G to exactly $d-1$ distinct vertices of L; and*

(ii) *L is a maximal clique of G.*

COROLLARY. *No clique of G has more than n elements.*

Proof. Let L' be the set consisting of the n^2-n vertices of G which are not in L. For each integer x in the range $0 \leqq x \leqq n$, let $g(x)$ denote the number of vertices in L' which are joined in G to exactly x distinct vertices in L. We shall first make use of the formulas

$$\sum g(x) = n^2 - n \,, \tag{4.3}$$

$$\sum xg(x) = (d-1)(n^2-n) \,, \tag{4.4}$$

$$\sum x^2 g(x) = (d-1)^2(n^2-n) \,, \tag{4.5}$$

where the sum in each case is over the range of x, and then establish them later. From these formulas we deduce that

$$\begin{aligned}&\sum (d-1-x)^2 g(x)\\ &\quad = (n^2-n)[(d-1)^2\cdot 1 - 2(d-1)\cdot(d-1) + 1\cdot(d-1)^2] = 0 \,,\end{aligned}$$

and thence that $g(x)=0$ for $x \neq d-1$. At this point, (4.3) yields $g(d-1) = n^2-n$. And now (i) follows. From (i) and the fact that n exceeds $d-1$, we see that for every vertex P in L', the set $L \cup \{P\}$ is not a clique, since P is joined to only $d-1$ vertices, and therefore is not joined to all vertices, in L. This means that L is a maximal clique.—The Corollary should be obvious.

We prove the formulas by appeal to Lemma 2.1. In each case, the set B of that lemma is L'. For (4.3), A is any one-element set, and ρ is $A \times B$. For (4.4), A is L and ρ is the set of all pairs (a, b) with a in A, b in B such that $\{a, b\}$ is an edge. The left side of (4.4) is a double sum; $xg(x)$ counts all $|\rho b|$ with b joined to exactly x edges, and $\sum xg(x)$ gives the complete sum. For the right-hand side, we note that there are n choices of a in A. Each a lies on $d(n-1)$ edges, including $n-1$ edges joining it to points of $A = L$. Hence

$$\sum |a\rho| = n\cdot[d(n-1) - (n-1)] = (d-1)(n^2-n) \,.$$

To get (4.5), we take A to be the set of $n(n-1)$ ordered pairs of distinct vertices of L, and ρ to be the subset of $A \times B$ consisting of all pairs (a, b) such that b is joined by an edge to both of the vertices making up a. Then $x(x-1)g(x)$ is the sum of $|\rho b|$ over all b which are joined to exactly x vertices of L, and

$$\sum x(x-1)g(x) = \sum |\rho b| .$$

On the other hand, for each element a of A,

$$|a\rho| = p_{11}^1 - (n-2) = (d-1)(d-2) .$$

Thus

$$\sum x(x-1)g(x) = (n^2 - n)(d-1)(d-2) ,$$

whence, by addition of (4.4), we get (4.5). This completes the proof of Lemma 4.1. It seems worth remarking that, although Lemma 4.1 and its proof both seem pretty obvious, the proof was still lacking for several weeks after everything which follows in this section had been established subject to the conjecture that no clique had more than n vertices.

In the proofs which follow, we first establish the existence of certain cliques called grand cliques, and eventually prove, on the basis of Lemma 4.1, that these are lines. We make two definitions, relative to a pseudo net-graph of order n, degree k:

A *major* clique, K, is a clique such that

$$|K| \geqq n - (d-1)^2(d-2) . \tag{4.6}$$

A *grand* clique is a major clique which is also a maximal clique. We note from Lemma 4.1 that, if $d = 1$ or 2, major cliques and grand cliques are the same as lines. There is a lemma for graphs completely analogous to Lemma 3.2 (with lines and cliques replacing transversals and partial transversals) but here we need something weaker:

LEMMA 4.2. *Let G be a pseudo net-graph of order n, degree d, and let K, L be two distinct cliques of G.*

(i) *If $K \cup L$ is not a clique, then*

$$|K \cap L| \leqq d(d-1) . \tag{4.7}$$

(ii) *If $K \cap L$ has at least two vertices, then*

$$|K \cup L| \leqq n + (d-1)(d-2) . \tag{4.8}$$

(iii) *If (4.7), (4.8) hold, then*

$$|K| + |L| \leqq n + 2(d-1)^2 . \tag{4.9}$$

COROLLARY. *If G is a pseudo net-graph of order n, degree d,* *and if* $n > 2(d-1)^3$, *then two distinct grand cliques of* G *can have at most one common vertex.*

REMARK. Analogous results hold for partial transversals in a net of order n, deficiency d.

Proof. (i) If $K \cup L$ is not a clique, there must exist a vertex P in $K - L$ and a vertex Q in $L - K$ such that P, Q are not joined in G. Then P, Q are together joined to exactly

$$p^2_{11} = d(d-1)$$

other vertices, and these must include $K \cap L$. This proves (i).

(ii) If $K \cap L$ contains two distinct vertices R, S, then R, S are are joined in G and hence are together joined to exactly

$$p^1_{11} = n - 2 + (d-1)(d-2)$$

other vertices. Among these must be included $K \cup L - \{R, S\}$. This proves (ii); and (iii) follows immediately.

Now suppose that K, L are two distinct maximal cliques with at least two common vertices. Then (ii) holds. Moreover, $K \cup L$ cannot be a clique, so (i) holds. Therefore we have (iii). If K, L are also both major cliques, (4.9) yields

$$2[n - (d-1)^2(d-2)] \leqq n + 2(d-1)^2$$

and hence

$$n \leqq 2(d-1)^3 \, . \tag{4.10}$$

Consequently, two distinct grand cliques cannot have two common vertices unless (4.10) holds. This proves the Corollary.

To establish the existence of major and grand cliques, we need the concept of a claw—a concept suggested in conversation by Alan Hoffman. By a claw, P, S, of a pseudo net-graph, G is meant an ordered pair consisting of a vertex P, the *vertex* of the claw, and a nonempty set S of vertices distinct from P such that every vertex in S is joined to P in G but no two vertices in S are joined in G. By the *order* of the claw P, S we mean the number, $|S|$, of vertices in S.

When G is the complementary graph of a net N of deficiency d, it is easy to see that a claw P, S of order d exists for every vertex P. Indeed, let L be any line of N not containing P, and let S consist of the d distinct points of L not joined to P in N; then every two points of S are joined in N. Hence, in G, P, S is a claw of order d with vertex P.

We need several lemmas concerning claws, and it is convenient to begin with a fairly general preliminary lemma.

LEMMA 4.3. *Let G be a pseudo net-graph of order n, degree d, and let P, S be a claw of G of order $|S| = s$. Let T be the set of all vertices of G other than P and those in S. For each x in the range $0 \leqq x \leqq s$, let $f(x)$ be the number of vertices in T which are joined to P and, in addition, are joined to exactly x vertices in S. Then*

$$\sum_0^s f(x) = d(n-1) - s \,, \tag{4.11}$$

$$f(0) - \sum_2^s (x-1)f(x) = (d-s)(n-1) - s(d-1)(d-2) \,, \tag{4.12}$$

$$\begin{aligned} 2f(0) &+ \sum_3^s (x-1)(x-2)f(x) \\ &= \alpha_s + 2(d-s)(n-1) - 2s(d-1)(d-2) \,, \end{aligned} \tag{4.13}$$

where α_s is an integer such that

$$0 \leqq \alpha_s \leqq s(s-1)(d^2-d-1) \,, \tag{4.14}$$

and the upper bound is attained in (4.14) *precisely when every vertex of T which is joined to at least two distinct vertices of S is also joined to P.*

REMARK. If $s \leqq 2$, the summation on the left side of (4.13) should be omitted. Similarly, if $s = 1$, the summation on the left side of (4.12) should be omitted.

Proof. The left-hand side of (4.11) is the number of vertices of T which are joined to P. As for the right-hand side of (4.11), P is joined in G to exactly $d(n-1)$ distinct vertices; of these vertices, s are in S and the rest are in T. This proves (4.11).

Next we prove

$$\sum_1^s xf(x) = s[n-2+(d-1)(d-2)] \,, \tag{4.15}$$

by applying Lemma 2.1. We take A to be the set of all vertices in T which are joined to P, B to be S, and ρ to be the subset of $A \times B$ consisting of all $(a, b), a \in A, b \in B$, such that $\{a, b\}$ is an edge of G. For any $x \geqq 1$, $xf(x)$ is the sum of $|a\rho|$ as a ranges over the vertices in A which are joined to exactly x vertices in $B = S$; hence the left-hand side of (4.15) is $|\rho| = \sum |a\rho|$. For any b in $B = S$, since P and b are joined, there are exactly p_{11}^1 vertices in G joined to both P and b; and these are in A. Hence

$$|\rho b| = p_{11}^1 = n - 2 + (d-1)(d-2)$$

and therefore, since $|B| = |S| = s$, $\sum |\rho b|$ is the right-hand side of (4.15).

This proves (4.15). To get (4.12), we subtract (4.15) from (4.11). Next we prove

$$\sum_{2}^{s} x(x-1)f(x) = \alpha_s \tag{4.16}$$

where α_s satisfies (4.14). To do this we first define, for every ordered pair U, V of distinct vertices in S, $f_1(U, V)$ to be the number of vertices in T which are joined to U, V and also to P, and $f_0(U, V)$ to be the number of vertices in T which are joined to U, V but not to P. For each such pair U, V, there are exactly p_{11}^2 vertices in G joined to both of U, V; one of these vertices is P and the rest are in T. Hence

$$f_1(U, V) + f_0(U, V) = p_{11}^2 - 1 = d^2 - d - 1 .$$

We define

$$\alpha_s = \sum f_1(U, V)$$

where the sum is over the $s(s-1)$ ordered pairs of vertices U, V in S, and observe that

$$\alpha_s + \sum f_0(U, V) = s(s-1)(d^2 - d - 1) .$$

Since the second sum is a nonnegative integer, we see that the integer α_s satisfies (4.14) and attains its upper bound under the conditions stated in the lemma. To prove (4.16) we use Lemma 2.1 with A as before and with B defined to be the set of all ordered pairs U, V of distinct vertices in S. Also, ρ is the subset of $A \times B$ consisting of all triples (a, U, V) with a joined to both of U, V. From the definition of α_s, $|\rho| = \sum |\rho b| = \alpha_s$. For each $x \geqq 2$, $x(x-1)f(x)$ is the sum of $|a\rho|$ over all a in A which are joined to exactly x elements of S. Thus we have (4.16). To obtain (4.13), we multiply (4.12) by 2 and add the result to (4.16). This completes the proof of Lemma 4.3.

The author is indebted to Allan Hoffman for suggesting the importance of the non-existence of claws of order $d+1$, and for sketching a non-existence proof for n large compared with d. In the next lemma we give precise details in terms of the polynomial $p(x)$ defined by

$$p(x) = \tfrac{1}{2}x^4 + x^3 + x^2 + \tfrac{3}{2}x . \tag{4.17}$$

It will be convenient to note that

$$(4.18) \quad 2[p(d-1)-1] = (d+1)d(d^2-d-1) - 2(d+1)(d-1)(d-2) .$$

LEMMA 4.4. *If G is a pseudo net-graph of order n, degree d, and if*

$$(4.19) \qquad n > p(d-1) ,$$

then G has no claws of order $d+1$.

Proof. Assume, by way of obtaining a contradiction, that G has a claw P, S of order $d+1$. Then we may quote Lemma 4.3 with $s = d+1$. The left hand side of (4.13) is a nonnegative integer. Hence, certainly, if we replace α_{d+1} by its upper bound in (4.14), the right hand side of (4.13) must be nonnegative. This gives

$$(d+1)d(d^2-d-1) - 2(n-1) - 2(d+1)(d-1)(d-2) \geqq 0$$

and hence, by (4.18),

$$2(n-1) \leqq 2[p(d-1)-1] ,$$

in contradiction to (4.19). This proves Lemma 4.4.

The next three lemmas may conveniently be stated and proved together:

LEMMA 4.5. *Let G be a pseudo net-graph of order n, degree d such that*

$$(4.20) \qquad n - 1 > (d-1)^2(d-2) .$$

Then to every pair P, Q of distinct vertices joined in G there corresponds at least one claw P, S of order d such that S contains Q.

LEMMA 4.6. *Let G be a pseudo net-graph of order n, degree d such that* (4.20) *holds and G has no claws of order $d+1$. Then every edge of G is contained in at least one grand clique of G.*

LEMMA 4.7. *Let G be a pseudo net-graph of order n, degree d subject to the following three conditions:* (i) *G has no claws of order $d+1$;* (ii) *two distinct grand cliques of G have at most one common point and* (iii) *$n > q(d-1)$ where*

$$(4.21) \qquad q(x) = 2x^3 - x^2 - x + 1 .$$

Then every vertex of G lies in exactly d distinct grand cliques, and every grand clique of G is a line of G.

Proof of Lemma 4.5. We begin by noting that $P, \{Q\}$ is a claw of order one. If $d = 1$, the proof of Lemma 4.5 is complete. Therefore we consider the case $d > 1$ and assume inductively that there exists a claw P, S of order s such that S contains Q and $1 \leqq s \leqq d - 1$. Since $s \leqq d - 1$, the right-hand side of (4.12) (see Lemma 4.3) is at least

$$n - 1 - (d - 1)^2(d - 2) .$$

Since the sum on the left-hand side of (4.12) is non-negative, we deduce that

$$f(0) \geqq n - 1 - (d - 1)^2(d - 2) > 0 , \tag{4.22}$$

the last inequality following from (4.20). If R is any one of the $f(0)$ vertices in T which are joined to P but to no vertex in S, then P, $S \cup \{R\}$ is a claw of order $s + 1$. Therefore, by mathematical induction, we have the conclusion of Lemma 4.5.

Proof of Lemma 4.6. Let $\{P, Q\}$ be any edge of G. By Lemma 4.5, there exists at least one claw P, S' of order d such that S' contains Q. We write $S' = \{Q\} \cup S$ where S does not contain Q. Then (in the notation of Lemma 4.3) let H be the set of all elements of T which are joined to P but to no element of S. Clearly H contains Q. Moreover, $|H| = f(0)$, and $f(0)$ satisfies (4.22). Hence if $K = \{P\} \cup H$,

$$|K| \geqq n - (d - 1)^2(d - 2) . \tag{4.23}$$

We claim that K is a clique. Indeed, every element of H is joined to P. Therefore, if K contains two distinct vertices A, B not joined in G, then $P, S \cup \{A, B\}$ is a claw of order $d + 1$, contrary to hypothesis. In view of (4.23), the clique K is major. Therefore, if K' is any maximal clique containing K, then K' is a grand clique containing the edge $\{P, Q\}$. This completes the proof of Lemma 4.6.

Proof of Lemma 4.7. We first note that

$$q(d - 1) - 1 = (d - 1)^2(d - 2) + d(d - 1)(d - 2) . \tag{4.24}$$

Hence the inequality $n > q(d - 1)$ implies the inequality (4.20). If P is any vertex of G, there exists, by Lemma 4.5, at least one claw P, S of order d with vertex P. We denote the d vertices in S by $A_1, A_2, \cdots, A_d$. For each i in the range $1 \leqq i \leqq d$, we denote by H_i the set of vertices, distinct from P and the A_j for $j \neq i$; which are joined to P but to no vertex A_j for $j \neq i$. As the proof of Lemma 4.6 shows, $P \cup H_i$ is, for each i, a major clique containing P and A_i. We denote by K_i a grand clique containing $P \cup H_i$. Since, for $i \neq j$, H_i and H_j have no common elements, it follows from our uniqueness hypothesis (ii) that the only common element of K_i and K_j is P. We wish to show that the d grand cliques $K_1, K_2, \cdots, K_d$ are the only grand cliques containing P.

We begin by recalling that P, S is a claw of order d and that (in the notation of Lemma 4.3) the set

$$H = H_1 \cup H_2 \cup \cdots \cup H_d$$

consists of S and of all vertices in T which are joined to exactly one of the vertices $A_1, \cdots, A_d$ of S and are also joined to P. That is (when we take $s = d$ in Lemma 4.3),

$$|H| = f(1) + d .$$

Moreover, since G has no cliques of order $d + 1$, $f(0) = 0$. Thus (4.11), (4.12), with $s = d$, can be rewritten as

$$|H| + \sum_2^d f(x) = d(n-1) , \tag{4.25}$$

$$\sum_2^d (x-1)f(x) = d(d-1)(d-2) . \tag{4.26}$$

If $d = 1$, the summation disappears in (4.25) and the inequalities

$$d[n-1-(d-1)(d-2)] \leqq |H| \leqq d[n-1-(d-2)] \tag{4.27}$$

hold trivially. If $d > 1$, (4.26) yields

$$\sum_2^d f(x) \leqq d(d-1)(d-2) \leqq (d-1)\sum_2^d f(x) ,$$

whence

$$-d(d-1)(d-2) \leqq -\sum_2^d f(x) \leqq -d(d-2) .$$

The latter inequalities, combined with (4.25), yield (4.27).

Now let us suppose that P is contained in at least one grand clique K distinct from $K_1, K_2, \cdots K_d$. Then each of the $d(n-1)$ vertices (distinct from P) which are joined to P, is contained in at most one of the $d+1$ grand cliques. Moreover, $K_1, \cdots, K_d$ together contain at least $|H|$ of these vertices, and K, being a grand clique, contains at least

$$n - 1 - (d-1)^2(d-2)$$

more. Therefore, by (4.27),

$$d[n-1-(d-1)(d-2)] + n - 1 - (d-1)^2(d-2) \leqq d(n-1)$$

and hence (see (4.24))

$$n - 1 \leqq (d-1)^2(d-2) + d(d-1)(d-2) = q(d-1) - 1 \, .$$

This yields $n \leqq q(d-1)$, in contradiction to our hypothesis.

At this stage we have proved that each vertex P lies in exactly d distinct grand cliques $K_1, \cdots, K_d$. If Q were a vertex joined to P but in none of $K_1, \cdots, K_d$, then, by Lemma 4.6, there would be a grand clique K, distinct from $K_1, \cdots, K_d$, containing P and Q. Hence each of the $d(n-1)$ vertices joined to P lies in one (and only one, by uniqueness) of $K_1, \cdots, K_d$. By Lemma 4.1, no maximal clique can have more than n elements. Consequently, each of $K_1, \cdots, K_d$ must contain exactly n vertices. That is, each K_i is a line of G.

If K is any grand clique of G, we fix attention on a vertex P contained in K and use the fact, just proved, that every grand clique containing P is a line. Hence K is a line of G. This completes the proof of Lemma 4.7.

We did not need the upper bound in (4.27) for the proof of Lemma 4.7. This upper bound shows, however, that, for $d > 2$, the major cliques $\{P\} \cup H_i$ constructed in the proof are not all lines—else the upper bound would have to be at least $d(n-1)$.

Now we shall state and prove three theorems—three, because the varying hypotheses apply to different classes of graphs. We also state (4.2), which we have tacitly assumed up until this point.

THEOREM 4.1. *Let G be a pseudo net-graph of order n, degree d, with $n \geqq d \geqq 1$, which is subject to the following conditions:* (i) *G has no claws of order $d+1$;* (ii) *two distinct grand cliques of G have at most one common point;* (iii) *$n > q(d-1)$, where the polynomial q is given by* (4.21). *Then G is the graph of one and only one net of order n, degree d.*

COROLLARY. *Assume, in addition to the hypotheses of Theorem 4.1, that G is the complementary graph of a nontrivial net N of order n, deficiency d. Then N can be imbedded uniquely in an affine plane π of order n, and G is graph of the net complementary to N in π.*

Proof. We may apply Lemma 4.7. Since every grand clique of G is a line, we see that each vertex of G lies on exactly d distinct lines of G. Let P be a vertex of G and let L be a line of G not containing P. By Lemma 4.1 (i), P is joined to exactly $d-1$ distinct vertices in L. By Lemma 4.6 (which we may apply in view of (4.24)) these $d-1$ vertices lie one each on $d-1$ lines through P. Thus there is one and only one line, L', through P which is parallel to L

(has no vertex in common with L.) If we choose any line M which meets L, we see that through each vertex in M but not L there passes a unique line parallel to L. By uniqueness, no two such parallels can intersect. Hence L determines a parallel class L consisting of n lines, including L itself, each two of which are parallel. It is now clear that the vertices of G (considered as points) and the lines of G constitute a net of order n, degree d. Since two distinct vertices of G are joined in G if and only if they lie on a common line of G, we see that G is the graph of the net. The Corollary is immediate, in view of the discussion in § 3. This completes the proof of Theorem 4.1 and Corollary.

THEOREM 4.2. *Let G be a pseudo net-graph of order n, degree d such that* (i) *G has no claws of order $d+1$ and* (ii) $n > 2(d-1)^3 \geqq 0$ *(and, in case $d = 1$, also $n > 1$.) Then G is the graph of one and only one net of order n, degree d.*

COROLLARY. *Assume, in addition to the hypotheses of Theorem 4.2, that G is the complementary graph of a net N of order n, deficiency d. Then N can be imbedded uniquely in an affine plane π of order n, and G is the graph of the net complementary to N in π.*

Proof. We need merely show that the hypotheses of Theorem 4.1 are verified. Hypothesis (i) of Theorem 4.2 is identical with hypothesis (i) of Theorem 4.1. In view of the Corollary of Lemma 4.2, hypothesis (ii) of Theorem 4.2 implies hypothesis (ii) of Theorem 4.1. Since

$$2(d-1)^3 - q(d-1) = (d-1)^2 + (d-1) - 1 > 0 \qquad \text{for } d > 1$$

and

$$q(0) = 1 \ ,$$

hypothesis (ii) of Theorem 4.2 also implies hypothesis (iii) of Theorem 4.1. This completes the proof.

THEOREM 4.3. *Let G be a pseudo net-graph of order n, degree d such that $n > p(d-1)$, where p is the polynomial given by* (4.17), *and either $d = 1, n > 1$ or $d > 1$. Then G is the graph of one and only one net of order n, degree d.*

COROLLARY. *Assume, in addition to the hypotheses of Theorem 4.3, that G is the complementary graph of a net N of order n, deficiency d. Then N can be imbedded uniquely in an affine plane*

π of order n, and G is the graph of the net complementary to N in π.

Proof. $p(0) = q(0) = 1$ and

$$p(d-1) - 2(d-1)^3 = \tfrac{1}{2}\{(d-1)^3(d-3) + 2(d-1)^2 + 3(d-1)\} > 0$$

if $d > 1$. Thus we may apply Lemma 4.4 to get the hypotheses of Theorem 4.2.

5. Incidence matrices. We are going to show that certain sets of k mutually orthogonal symmetric matrices of order n^2 are closely akin to nets of order n, degree k—and thus to sets of $k-2$ mutually orthogonal latin squares of side n. Surprising as it may seem, in view of the coincidence of the adjective "orthogonal" in "orthogonal matrices" and "orthogonal latin squares," we have it on the authority of R. C. Bose that, when orthogonal latin squares are used in the analysis of statistics, no orthogonal matrices arise such as the ones here defined. Thus the correspondence seems to be new.

It will be convenient to have a name for the matrices we study, and we adopt the adjective "germaine" as a pseudonym for "akin." By a *germaine* matrix, F, of order n^2 we mean a matrix F of n^2 rows and columns such that (i) F is symmetric; (ii) every entry on the main diagonal of F is $n-1$; (iii) every other entry of F is either $n-1$ or -1; (iv) $F^2 = n^2F$.

If F is a germaine matrix of order n^2, then, by (iv), the matrix $E = n^{-2}F$ is idempotent and, by (ii), E has trace $n-1$. Since (over a field of characteristic zero) the trace of an idempotent matrix is equal to its rank, we see that E and F have rank $n-1$. When $n = 2$, there are germaine matrices of order 4 which we want to avoid, e.g., the matrix with every entry equal to 1. This is an exception to the general rule (which will be clear in a moment) that germaine matrices have zero row-sums.

In order to avoid complications of notation, we begin with two lemmas concerning one and two germaine matrices respectively.

LEMMA 5.1. *Let $n \geqq 2$ be an integer. To each enumeration $1, 2, \cdots, n^2$ of the n^2 points of a net N of order n, degree 1 there corresponds a germaine matrix F of order n^2 defined as follows: F has $n-1$ down the main diagonal; for $i \neq j$, F has $n-1$ or -1 in position (i, j) according as the points i, j of N lie or do not lie on a line of N. Moreover, F has zero row sums. Conversely, if F is a germaine matrix of order n^2 (and if, in case $n = 2$, F has zero row sums) then F arises from a net N of order n, degree 1 in the manner indicated.*

Proof. If N is given, and if F is defined from N as described, it is a straightforward matter to verify that F is germaine and has zero row-sums. Conversely, let

$$F = (f_{ij})$$

be a given germaine matrix of order n^2 (with zero row-sums in case $n = 2$). Since F is symmetric, condition (iv) for a germaine matrix may be written as

$$n^2 f_{ij} = \sum_k f_{ik} f_{jk} \qquad (i, j = 1, 2, \cdots, n^2) \tag{5.1}$$

where k ranges from 1 to n^2. Let G be a graph whose n^2 vertices are the integers $1, 2, \cdots, n^2$. For $i \neq j$, let vertices i, j form an edge if and only if $f_{ij} = n - 1$. Since $n - 1 \neq -1$, the edges of F are well-defined, by (ii), (iii). Moreover, by (i), G is symmetric.

Our first task is to show that G is a pseudo net-graph of order n, degree $d = 1$. Thus, in the sense of § 4, we must show that

$$n_1 = n - 1\,, \qquad p_{11}^1 = n - 2\,, \qquad p_{11}^2 = 0\,. \tag{5.2}$$

Consider some fixed vertex i of G and suppose that i is joined to x and not joined to $n^2 - 1 - x$ of the remaining $n^2 - 1$ vertices. Taking $j = i$ in (5.1), and using the properties of F, we get

$$n^2(n-1) = (n-1)^2(1+x) + 1\cdot(n^2 - 1 - x)\,,$$

whence

$$n^2(n-2) = n(n-2)(1+x)\,.$$

If $n > 2$, we get $1 + x = n$, $x = n - 1$. In any case, the sum of the ith row of F is

$$(n-1)(1+x) - (n^2 - 1 - x) = n(x - n + 1)\,,$$

and this is zero precisely when $x = n - 1$. Therefore we have $n_1 = x = n - 1$, whence

$$n_1 = n - 1\,, \qquad n_2 = n^2 - n\,, \tag{5.3}$$

where, of the $n^2 - 1$ vertices distinct from i, n_1 are joined and n_2 are not joined to i. Next consider two distinct vertices i, j, joined in G. Of the $n^2 - 2$ vertices distinct from i, j let $p_{11}^1 = y$ be joined to both i and j. Then (since i, j are joined) $p_{12}^1 = n_1 - 1 - y = n - 2 - y$ are joined to i but not j, and $p_{21}^1 = n - 2 - y$ are joined to j but not i, and $p_{22}^1 = n_2 - p_{21}^1 = n^2 - 2n + 2 + y$ are joined to neither i nor j. Using (5.1) for the given i, j, we get

$$n^2(n-1) = 2(n-1)^2 + (n-1)^2 p_{11}^1 - (n-1)(p_{12}^1 + p_{21}^1) + p_{22}^1$$
$$= n^2 + n^2 y ,$$

whence $y = n - 2$. Thus

$$p_{11}^1 = n - 2 , \quad p_{12}^1 = p_{21}^1 = 0 , \quad p_{22}^1 = n(n-1) . \tag{5.4}$$

Finally, consider two distinct vertices i, j not joined in G. This time we may set

$$p_{11}^2 = z , \quad p_{12}^2 = p_{12}^2 = n_1 - z = n - 1 - z ,$$
$$p_{22}^2 = n_2 - 1 - p_{21}^2 = n^2 - 2n + z .$$

From (5.1) we get

$$n^2(-1) = 2(n-1)(-1) + (n-1)^2 p_{11}^2 - (n-1)(p_{12}^2 + p_{21}^2) + p_{22}^2$$
$$= -n^2 + n^2 z .$$

Therefore $z = 0$ and

$$p_{11}^2 = 0 , \quad p_{12}^2 = p_{21}^2 = n - 1 , \quad p_{22}^2 = n - 2 + (n-1)(n-2) . \tag{5.5}$$

This completes the proof that G is a pseudo net-graph of order n, degree $d = 1$, deficiency $k = n$.

Since $p(0) = 0$ and since $d = 1, n > 1$, we may conclude from Theorem 4.3 that G is the graph of a net N of order n, degree $d = 1$. This completes the proof of Lemma 5.1.

We recall that two matrices A, B are *orthogonal* provided $AB = BA = 0 =$ the zero matrix.

LEMMA 5.2. *Let $n \geqq 2$ be an integer. To each enumeration $1, 2, \cdots, n^2$ of the n^2 points of a net N of order n, degree 2, and to each enumeration 1, 2 of the two line-classes of N, there corresponds an ordered pair F_1, F_2 of orthogonal germaine matrices of order n^2, such that, for $\alpha = 1, 2$, F_α corresponds in the sense of Lemma 5.1 to the net N_α of order n, degree 1 with the same points as N and with the lines of class α as its lines. Conversely, if F_1, F_2 is an ordered pair of orthogonal germaine matrices, (with zero row sums, in case $n = 2$) then F_1, F_2 arises from a net N of order n, degree 2 in the manner indicated.*

Proof. If N is given, and if F_1, F_2 are defined from N as described, then F_1, F_2 are germaine by Lemma 5.1, and, by a straightforward computation, F_1, F_2 are orthogonal. Now we assume, conversely, that F_1, F_2 are orthogonal and germaine. By Lemma 5.1, for $\alpha = 1, 2$, F_α defines a net N_α of order n, degree 1 on the points $1, 2, \cdots, n^2$. We let N be the system with the same points as N_1 and

N_2 and with the lines of N_1 as its first parallel class and the lines of N_2 as its second parallel class. Then each line of N contains exactly n distinct points, and each point of N lies on exactly one line of each class. To prove that N is a net of order n, degree 2, we need only show that two lines of distinct classes have one and only one common point.

If we write $F_1 = (a_{ij})$, $F_2 = (b_{ij})$, then, since F_1, F_2 are symmetric, orthogonality is expressed by the condition

$$\sum_k a_{ik} b_{jk} = 0 \qquad (i, j = 1, 2, \cdots, n^2) . \tag{5.6}$$

First we consider a point i and suppose that, of the $n^2 - 1$ other points, x_{11} are joined to i both by a line of class 1 and a line of class 2, x_{10} are joined to i by a line of class 1 but not by a line of class 2, and similarly for x_{01}, x_{00}. Setting $x_{11} = x$, we see that

$$x_{10} = x_{01} = n - 1 - x \ , \quad x_{00} = n^2 - 2n + 1 + x \ .$$

From (5.6) with $i = j$, we get (since F_1, F_2 are germaine)

$$\begin{aligned} 0 &= (n-1)^2(1 + x_{11}) - (n-1)(x_{10} + x_{01}) + x_{00} \\ &= n^2 x \ . \end{aligned}$$

Hence $x = 0$. That is, if two lines of distinct classes have a common point i, then they have no other common point.

There remains the possibility that there are two lines of different classes with no common point. Suppose that, for some $i \neq j$, the line of class 1 through i and the line of class 2 through j have no common point. Then, from (5.6), for the given i, j, we get

$$\begin{aligned} 0 &= n(n-1)(-1) + n(n-1)(-1) + [(n^2 - 2n)(-1)^2] \\ &= -n^2 \ , \end{aligned}$$

a contradiction. This completes the proof of Lemma 5.2.

Now we are ready for the main theorem.

THEOREM 5.1. *Let n, k be integers, with $n \geqq 2, 1 \leqq k \leqq n+1$. To each enumeration $1, 2, \cdots, n^2$ of the n^2 points of a net N of order n, degree k, and to each enumeration $1, 2, \cdots, k$ of the k line-classes of N, there corresponds an ordered set*

$$F_1, F_2, \cdots, F_k \tag{5.7}$$

of mutually orthogonal germaine matrices of order n^2 such that, for $\alpha = 1, 2, \cdots, k$, F_α corresponds in the sense of Lemma 5.1 to the net N_α of order n, degree 1 with the same points as N and with the lines of class α as its lines. Conversely, if (5.7) *is an ordered set* *of k mutually orthogonal germaine matrices of order n^2 (each with zero row-sums, in case $n = 2$) then* (5.7) *arises from a net N of order n, degree k in the manner indicated.*

Proof. In view of Lemmas 5.1, 5.2, we need only treat the case $k \geqq 3$. In this case, for $\alpha < \beta$, if N is given, let $N_{\alpha\beta}$ be the net of order n, degree 2 with the same points as N and with the line-classes α, β as its two line-classes. By Lemma 5.2, $N_{\alpha\beta}$ determines the ordered pair F_α, F_β of orthogonal germaine matrices of order n^2. Conversely, the pair determines $N_{\alpha\beta}$. It should now be clear that the set (5.7) determines a net N of order n, degree k. Indeed, the only point which could be at issue is whether two lines of distinct classes in N have a unique common point, and this follows from the fact that each $N_{\alpha\beta}$ is a net of degree 2. The proof of Theorem 5.1 is now complete.

For a (non-trivial) net N of order n, degree k, deficiency d we also define matrices F_0, F^*, F_∞ in addition to (5.7). First we define

$$F^* = \sum_{i=1}^{k} F_i \, . \tag{5.8}$$

In addition, F_0 (usually called J or S) is the matrix of order n^2 with every entry equal to 1, and, finally, F_∞ is defined by the equation

$$n^2 I = F_0 + F^* + F_\infty \, , \tag{5.9}$$

where I is the identity matrix of order n^2. We shall give a direct description of F^* and F_∞: The matrix F^* has $k(n-1)$ down the main diagonal and, for $i \neq j$, has $d - 1$ or $-k$ in place (i, j) according as the points i, j of N are joined in N (by a line of any class) or not joined in N. By contrast, the matrix F_∞ has $d(n-1)$ down the main diagonal and, for $i \neq j$, has $-d$ or $k - 1$ in place (i, j) according as the points i, j are joined or not joined in N. Clearly it is reasonable to associate F^* with the graph G_1 of N and F_∞ with the complementary graph G_2. We note that if N is an affine plane, so that $d = 0$ and every two points of N are joined, then F_∞ is the zero matrix, and F^* has $n^2 - 1$ down the main diagonal, -1 off the main diagonal.

If the graph G of the theorem which follows is the complementary graph of the above net N, then the matrix F of the theorem is F_∞ (except that the words "joined" and "not joined" have been interchanged):

THEOREM 5.2. *Let n, d, k be positive integers with*

$$d + k = n + 1 \, . \tag{5.10}$$

To each ordering $1, 2, \cdots, n^2$ *of the* n^2 *vertices of a pseudo net-graph* G *of order* n, *degree* d, *deficiency* k *there corresponds a matrix* F *of order* n^2 *with the following properties:* (i) F *is symmetric;* (ii) *every entry in the main diagonal of* F *is* $d(n-1)$; (iii) *every other entry of* F *is either* $k-1$ *or* $-d$. (iv) $F^2 = n^2F$. *Specifically, we define* F *by insisting on* (ii) *and, for distinct vertices* i, j, *by putting* $k-1$ *or* $-d$ *in place* (i, j) *of* F *according as* i, j *are joined or not joined in* G. *The matrix* F, *so defined, has zero row sums. Conversely, if* F *is a matrix of order* n^2 *with properties* (i)—(iv) (*and if* F *has zero row-sums in case* $n = 2d$), *then* F *arises from a pseudo net-graph* G *of order* n, *degree* d, *deficiency* k *in the manner indicated.*

Sketch of proof. We note that, when $d = 1$, Theorem 5.2 is essentially Lemma 5.1 (stated for a graph instead of a net). The direct part of the proof is straightforward and the converse part can be stated so as to reduce to the main part of the proof of Lemma 5.1 when $d = 1$. The only difference is that we do not claim—and, for n small compared with d, we cannot claim—that F determines a net. This should suffice for the proof of Theorem 5.2.

It should be observed that if the edges of the graph G of Theorem 5.2 can be partitioned into two sets so that G can be regarded as made up of two graphs G_1, G_2 on the same vertices, where G_α is a pseudo net-graph of order n, degree d_α (and $d = d_1 + d_2$) then the matrix F of Theorem 5.2 can be decomposed ($F = F_1 + F_2$) into the sum of a pair of orthogonal matrices F_1, F_2, where F_α is defined for G_α in the manner of Theorem 5.2. Precisely when G is a net-graph, F can be decomposed into a sum of d mutually orthogonal germaine matrices.

Returning again to a net N of order n, degree k, deficiency d, and to the matrices exhibited in (5.8), (5.9), we wish to discuss briefly *point-point* incidence matrices for N. First we define

$$E_i = n^{-2}F_i\,(0 \leqq i \leqq k)\,, \quad E^* = n^{-2}F^*\,, \quad E_\infty = n^{-2}F_\infty \tag{5.11}$$

and observe that these E's are mutually orthogonal idempotent matrices. Moreover

$$E^* = \sum_{i=1}^{k} E_i\,, \tag{5.12}$$

$$I = E_0 + E^* + E_\infty\,, \tag{5.13}$$

$$\text{rank } E_0 = 1\,, \qquad \text{rank } E_i = n - 1 \ (1 \leqq i \leqq k)\,, \tag{4.14}$$

$$\text{rank } E^* = k(n-1)\,, \qquad \text{rank } E_\infty = d(n-1)\,. \tag{5.15}$$

Next let

$$x, y_1, \cdots, y_k, z$$

be $k + 2$ rational numbers and define the *point-point incidence* matrix

$$A(x, y_1, \cdots, y_k, z) = (a(i, j)) \tag{5.16}$$

of order n^2 as follows: $a(i, i) = x$ for all i; if $i \neq j$, and if the points i, j of N lie on a line of class α, $a(i, j) = y_\alpha$; if $i \neq j$, and if the points i, j are not joined in N, $a(i, j) = z$. We may express the F's and E's in terms of $A(x, y_1, \cdots, y_k, z)$ for suitable choices of x, the y's and z. Specifically, we get F_0 by taking x, the y's and z all equal to 1. We get $F_\alpha (1 \leqq \alpha \leqq k)$ by taking $x = y_\alpha = n - 1, z = -1$ and $y_\beta = -1$ for $\beta \neq \alpha$. And we get F_∞ by taking $x = d(n - 1), y_\alpha = -d \,(1 \leqq \alpha \leqq k)$ and $z = k - 1$. Conversely, we may easily verify that

$$A(x, y_1, \cdots, y_k, z) = XE_0 + \sum_{i=1}^{k} Y_i E_i + ZE_\infty \tag{5.17}$$

where

$$\begin{aligned} X &= x + (n - 1)y^* + d(n - 1)y \,, \\ Y_i &= x + ny_i - y^* - dz \,, \qquad (i \leqq i \leqq k) \\ Z &= x - y^* + (k - 1)z \,, \end{aligned} \tag{5.18}$$

$$y^* = \sum_{i=1}^{k} y_i \,. \tag{5.19}$$

Since the E's are mutually orthogonal idempotents of known ranks, we see at once that *the characteristic roots of $A(x_1, y_1, \cdots, y_k, z)$ are: X of multiplicity 1; (for $1 \leqq i \leqq k$) Y_i of multiplicity $n - 1$; and Z of multiplicity $d(n - 1)$.*—For certain choices of x, the y's and z, some of these roots coincide; then their multiplicities must be added.

The results of the preceding paragraph may be used to show that a conjecture originally advanced by Harary and later disproved by Bose (along the present lines), is quite impossible to repair. We recall that, for any finite symmetric graph G with s vertices $1, 2, \cdots, s$, the *adjacency matrix* of G is a matrix of order s with 0 down the main diagonal, and with 1 or 0 in non-diagonal position (i, j) according as the vertices i, j are joined or not joined in G. Harary's conjecture was that (to within an isomorphism) a finite symmetric graph was determined by the characteristic roots of its adjacency matrix, taken with their multiplicities. However, if G_1, G_2 are the graph and the complementary graph of the net N of the preceding paragraph, the adjacency matrix of G_1 comes from (5.16) by taking $x = z = 0$ and all y's equal to 1, and the adjacency matrix of G_2 comes by taking $x = 0, z = 1$ and all y's equal to 0. In either case, the characteristic roots and their multiplicities depend only upon n, k and d. Let us concentrate on G_1 and note, from Theorem 4.3, that, if $n > p(k-1)$, G_1 uniquely determines N. It follows that, for $n > p(k-1)$, there are precisely as many graphs G_1 as there are nets N. However, even in the special case $k = 3$, corresponding to a single latin square, the number of nets of order n, degree k increases astronomically with n. (Cf. Hall [7].)

Finally, we wish to mention *line-point* incidence matrices, again for the same net N. With each line L of the net N we associate a row-vector of n^2 columns, having 1 or 0 in column i according as point i lies or does not lie on L. With the αth line class of N we associate a matrix M_α of n rows, n^2 columns, the rows of M_α being those for the lines of class α, in any order. Finally, we define M to be the matrix of kn rows, n^2 columns, given by

$$M = \begin{pmatrix} M_1 \\ M_2 \\ \vdots \\ M_k \end{pmatrix} \tag{5.20}$$

Then M is the *line-point* incidence matrix of N. We merely wish to note that, where A^T denotes the transpose of matrix A,

$$M_\alpha^T M_\alpha = nE_0 + nE_\alpha\ , \qquad 1 \leqq \alpha \leqq k\ , \tag{5.21}$$

$$M^T M = knE_0 + nE^*\ . \tag{5.22}$$

As a consequence, the germaine matrices of Theorem 5.1 bear a simple relationship to the matrices $M_\alpha^T M_\alpha$.

6. Remarks and examples. In [4] we assigned to each non-trivial non-degenerate net N of order n, degree k (and deficiency d) a numerical invariant $\phi(N)$ with properties like Euler's totient. In particular, a necessary condition for the existence for a transversal to N is that $\phi(N) = 1$. (Consequently, by Theorem 4.3, $\phi(N) = 1$ if $n > p(d-1)$.)

It was remarked in [4] that the necessary condition $\phi(N) = 1$ was not sufficient for the existence of a transversal, but this statement seems to have been missed. Accordingly, we remark here that if N is the net of the latin square

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 5 & 1 & 6 & 2 & 4 \\ 4 & 6 & 2 & 1 & 3 & 5 \\ 5 & 4 & 6 & 2 & 1 & 3 \\ 6 & 1 & 5 & 3 & 4 & 2 \end{matrix} \tag{6.1}$$

then $\phi(N) = 1$ but N has no transversals.—This example recently disproved a conjecture and stopped a computer program.

If G is the complementary graph of the net of (6.1), then G has order $n = 6$, degree $d = 4$, deficiency $k = 3$, and G has no lines whatever. For other examples of this type, we need the *MacNeish number*, $M(n)$, of the positive integer n: If n is a prime-power, $M(n) = n$. If n is a product of prime-powers involving distinct primes, then $M(n)$ is the least of these prime powers. MacNeish showed that, for every positive integer $n \geqq 2$, there is at least one set of $M(n) - 1$ mutually orthogonal latin of side n (thus, a net of order n, degree $M(n) + 1$) and he conjectured (incorrectly) that there could be no more. In [4], using a direct product construction essentially due to MacNeish, we showed rather more: *For each positive integer n there exists at least one net N of order n, degree $k = M(n) + 1$, deficiency $d = n - M(n)$, for which $\phi(N) = M(n) > 1$.* Such a net N, of course, has no transversals. Thus, if G is the complementary graph of N, G is a pseudo net-graph of order n, degree $D(n) = n - M(n)$ with no lines whatever. If n is a prime-power, $D(n) = 0$ and the result has no interest. If $n = PQ$ where P is a prime-power, Q is prime to P, and P is the least prime-power dividing n, then $M(n) = P$ and $n/D(n) = Q/(Q - 1)$. Hence, for Q large, $D(n)$ is close to n. For example, $D(20) = 16$. Thus these examples are of little help with the theorems of §§ 3, 4, though they do show that some conditions are necessary.

We call attention to other examples briefly discussed in § 3.

The results of the present paper, especially Theorems 3.1 and Theorems 4.1, 4.2, 4.3, suggest that a further study of pseudo net-graphs of order n, degree d subject to

$$d > 1, n > d + 1, (d - 1)^2 < n \leqq p(d - 1) \tag{6.2}$$

would be rewarding. We offer the following:

Conjecture. Every pseudo net-graph of order n, degree d, deficiency k, subject to (6.2), *is either the graph of a net of order n, degree d or the complementary graph of a net of order n, degree k or both.*

When $d = 2$, (6.2) yields $n = 4$, whence $k = 3$. As Shrikhande shows, there are just two pseudo net-graphs of order 4, degree 2, deficiency 3. One comes from the plane of order 4 and is thus both a net-graph and a complementary net-graph. The other is not a net-graph but is the complementary graph of the net of order 4, degree 3 defined by the cyclic group of order 4. The situation for $d > 2$ is completely unknown to the author, except for $n \leqq 7$.

One of the difficulties in dealing with pseudo net-graphs is the lack of a method of forming a "direct product" of two of them in such a way as to end up with a pseudo net-graph. The direct product of two nets (and hence of two net-graphs) of the same degree is easily defined (cf. [4]) but uses the existence of line-classes. This construction is unavailable for pseudo net-graphs. The direct-product construction for nets of the same degree allows a direct-product construction for complementary net-graphs of the same deficiency—but here we require too much knowledge of the nets to permit a generalization.

ADDENDUM. Dale Mesner, in his unpublished Ph. D. thesis ("An investigation of certain combinatorial properties of partially balanced incomplete block experimental designs and association schemes, with a detailed study of designs of Latin squares and related topics," Michigan State University, 1956) has results allied to Theorem 4.3. Essentially, he proves Theorem 4.3 with the hypothesis $n > p(d-1)$ replaced by a stronger hypothesis $n > d_0$. Here we may define d_0 to be the greatest integer in the largest of the real roots obtained from the quadratic equations

$$\text{(I)} \qquad 4x^2 - (d-1)(9d^2 - 9d + 7)x + (d-1)^2(9d^2 - 9d + 7) = 0,$$

$$\text{(II)} \qquad 2dx^2 - (d^5 - 2d^4 + 3d^3 - d^2 - 2d + 1)x - (d^6 - 3d^5 + 3d^4 + 2d^3 - 3d^2 + d + 1) = 0\,.$$

With a little labor we may verify the inequalities

$$\tfrac{1}{2}d^4 - d^3 < p(d-1) < d_0 < \tfrac{1}{2}d^4\,, \qquad d \geqq 2\,,$$

which show that Mesner's result is close to Theorem 4.3 but not as sharp.

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THE INVERSE OF THE ERROR FUNCTION

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1. Introduction. In a recent paper [3] J. R. Philip has discussed some properties of the function inverfc θ defined by means of

$$\theta = \operatorname{erfc}\,(\operatorname{inverfc}\,\theta)\,. \tag{1.1}$$

Since

$$\tfrac{1}{2}\pi^{1/2}(1 - \operatorname{erfc}\,x) = x - \frac{x^3}{3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \frac{x^9}{4!9} \cdots \tag{1.2}$$

it follows that

$$\operatorname{inverfc}\,\theta = u + \frac{1}{3}u^3 + \frac{7}{30}u^5 + \frac{127}{630}u^7 + \frac{4369}{22680}u^9 + \cdots, \tag{1.3}$$

where

$$u = \tfrac{1}{2}\pi^{1/2}(1 - \theta)\,.$$

The coefficients in (1.3) are rational numbers. It is therefore of some interest to look for arithmetic properties of these numbers.

It will be convenient to change the notation slightly. Put

$$f(x) = \int_0^{\infty} e^{-t^2/2}\,dt\,, \tag{1.4}$$

so that

$$f(x) = \left(\frac{\pi}{2}\right)^{1/2}(1 - \operatorname{erfc}\,2^{1/2}x)$$

and let $g(x)$ denote the inverse function:

$$f(g(u)) = g(f(u)) = u\,, \tag{1.5}$$

where

$$g(u) = \sum_{n=0}^{\infty} A_{2n+1}\frac{u^{2n+1}}{(2n+1)!} \qquad (A_1 = 1)\,. \tag{1.6}$$

It follows from (1.4) and (1.5) that

$$g'(u) = \exp\left(\tfrac{1}{2}g^2(u)\right)\,. \tag{1.7}$$

Differentiating again, we get

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$$g''(u) = g(u)(g'(u))^2 . \tag{1.8}$$

It follows from (1.6) and (1.8) that

$$A_{2n+3} = \sum_{r+s\leqq n} \frac{(2n+1)!}{(2r)!\,(2s)!\,(2n-2r-2s+1)!} A_{2r+1}A_{2s+1}A_{2n-2r-2s+1} . \tag{1.9}$$

Since $A_1 = 1$ it is evident from (1.9) that all the coefficients A_{2n+1} are positive integers. It is easily verified that the first few values of A_{2n+1} are

$$A_1 = A_3 = 1,\ A_5 = 7,\ A_7 = 127,\ A_9 = 4369 = 17.257 .$$

We shall show that

$$A_{2n+p} \equiv -\,2.4.6 \cdots (p-1)A_{2n+1} \pmod{p} , \tag{1.10}$$

where p is an arbitrary prime and that

$$A_{2n+5} \equiv -\,A_{2n+1} \pmod{8} \tag{1.11}$$

and indeed

$$A_{2n+9} \equiv A_{2n+1} \pmod{16} . \tag{1.12}$$

We also find certain congruences (mod p) for a sequence of integers e_{2n} related to the A_{2n+1} (see Theorems 2 and 3 below).

Finally we put

$$\frac{u}{g(u)} = \sum_0^\infty \beta_{2n} \frac{u^{2n}}{(2n)!}$$

and obtain a theorem of the Staudt-Clausen type for the β_{2n}, namely

$$\beta_{2n} = G_{2n} - \frac{b}{3} - \sum_{p-1/2n} \frac{1}{p} A_p^{2n/(p-1)} ,$$

where G_{2n} is an integer, $b = 2$ or 1 according as $n \equiv 1$ or $\not\equiv 1 \pmod{3}$ and the summation is over all primes $p > 3$ such that $p - 1/2n$. Moreover

$$A_p \equiv -\,2.4.6 \cdots (p-1) \pmod{p} .$$

2. A series of the form [2]

$$H(x) = \sum_{n=0}^\infty a_n \frac{x^n}{n!} , \tag{2.1}$$

where the a_n are rational integers, is called a Hurwitz series, or briefly an H-series. It is easily verified that sum, difference and product of two H-series is again an H-series. Also the derivative and the definite integral of the H-series define by (2.1):

$$H'(x) = \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!}, \quad \int_0^x H(t)dt = \sum_{n=1}^{\infty} a_{n-1} \frac{x^n}{n!}$$

are H-series. If $H_1(x)$ denotes an H-series without constant term then $H_1^k(x)/k!$ is an H-series for $k = 1, 2, 3, \cdots$; it follows that $H(H_1(x))$ is an H-series, where $H(x)$ is an arbitrary series of the form (2.1).

By the statement

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \equiv \sum_{n=0}^{\infty} b_n \frac{x^n}{n!} \pmod{m},$$

where the a_n, b_n are integers, is meant the system of congruences

$$a_n \equiv b_n \pmod{m} \qquad (n = 0, 1, 2, \cdots).$$

Thus the above statement about $H_1^k(x)/k!$ can be written in the form

$$H_1^k(x) \equiv 0 \pmod{k!}. \tag{2.2}$$

Returning to (1.4) it is evident that

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^n(2n+1)n!} = \sum_{n=0}^{\infty} c_{2n+1} \frac{x^{2n+1}}{(2n+1)!}, \tag{2.3}$$

where

$$c_{2n+1} = (-1)^n \frac{(2n)!}{2^n n!} = (-1)^n\, 1.\, 3.\, 5 \cdots (2n-1), \tag{2.4}$$

so that $f(x)$ is an H-series without constant term.

If p is an odd prime, it follows from (2.4) that

$$c_{2n+1} \equiv 0, \pmod{p} \qquad (2n + 1 > p). \tag{2.5}$$

Thus (1.5) implies

$$\sum_{n=0}^{1/2(p-1)} c_{2n+1} \frac{g^{2n+1}(u)}{(2n+1)!} \equiv u \pmod{p}. \tag{2.6}$$

We now compute the coefficient of $u^p/p!$ in the left member of (2.6). Clearly the terms with $1 \leqq n < (p-1)/2$ contribute nothing. Hence (2.6) yields

$$A_p + c_p \equiv 0 \pmod{p}.$$

Using (2.4) this becomes

$$A_p \equiv -(-1)^m\, 1.\, 3.\, 5 \cdots (p-2) \pmod{p}, \tag{2.7}$$

or if we prefer

$$A_p \equiv -2.\ 4.\ 6 \cdots 2m \equiv -\left(\frac{2}{p}\right)m! \pmod p, \tag{2.8}$$

where $p = 2m + 1$ and $(2/p)$ is the Legendre symbol. For example we have

$$\begin{aligned} A_5 &\equiv -1.\,3 \equiv 2 \pmod 5, \\ A_7 &\equiv 1.\,3.\,5 \equiv 1 \pmod 7, \\ A_{11} &\equiv 1.\,3.\,5.\,7.\,9 \equiv -1 \pmod{11}. \end{aligned}$$

We consider next the residue $\pmod p$ of A_{p+2n}. If $2n < p$ we have

$$\frac{(p+2n)!}{(2r)!\,(2s)!\,(p+2n-2r-2s)!} \equiv \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} \pmod p$$

by a familiar property of multinomial coefficients. Thus (1.9) implies (for $2n < p$)

$$A_{p+2n+2} \equiv \sum_{r+s\leqq n} \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} \cdot A_{2r+1}A_{2s+1}A_{p+2n-2r-2s} \pmod p. \tag{2.9}$$

Since $A_p \not\equiv 0 \pmod p$ we may put

$$A_{p+2n} \equiv A_p e_{2n} \pmod p \quad (2n \leqq p+1). \tag{2.10}$$

Then (2.9) becomes

$$e_{2n+2} \equiv \sum_{r+s\leqq n} \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} \cdot A_{2r+1}A_{2s+1}e_{2n-2r-2s} \pmod p \tag{2.11}$$

provided $2n < p$.

We now define a set of positive integers e_{2n} by means of $e_0 = 1$,

$$e_{2n+2} = \sum_{r+s\leqq n} \frac{(2n)!}{(2r)!\,(2s)!\,(2n-2r-2s)!} A_{2r+1}A_{2s+1}e_{2n-2r-2s} \quad (n = 0, 1, 2, \cdots). \tag{2.12}$$

If we put

$$\phi(x) = \sum_{n=0}^{\infty} e_{2n} \frac{x^{2n}}{(2n)!},$$

then (2.12) is equivalent to

$$\phi''(x) = \phi(x)(g'(x))^2. \tag{2.13}$$

Comparing (2.13) with (1.8) we get

$$\frac{\phi''(x)}{\phi(x)} = \frac{g''(x)}{g(x)} . \tag{2.14}$$

It follows that

$$\phi(x)g'(x) - g(x)\phi'(x) = 1 .$$

A little manipulation yields

$$\phi(x) = - g(x) \int \frac{dx}{g^2(x)} = - g(x) \int \frac{g'(x) \exp(- \frac{1}{2}g^2(x))dx}{g^2(x)}$$

and we get

$$\phi(x) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n!} \frac{g^{2n}(x)}{2n-1} . \tag{2.15}$$

Since

$$\frac{(2n)!}{2^n(2n-1)n!} = 1.\ 3.\ 5 \cdots (2n-3) ,$$

it follows from (2.2) and (2.15) that

$$\phi(x) \equiv 1 - \sum_{n=1}^{m+1} \frac{(-1)^n}{2^n n!} \frac{g^{2n}(x)}{2n-1} \pmod p , \tag{2.16}$$

where $p = 2m + 1$.

We notice also that (1.7) gives

$$g'(u) \equiv \sum_{n=0}^{\infty} \frac{g^{2n}(x)}{2^n n!} \pmod p , \tag{2.17}$$

while (1.8) yields

$$g''(u) \equiv \sum_{n=0}^{m-1} \frac{g^{2n+1}(x)}{n!} \pmod p . \tag{2.18}$$

3. We may rewrite (1.8) as

$$g''(u) = g(u) \exp g^2(u) . \tag{3.1}$$

Differentiating again and using (1.7) we get

$$g'''(u) = (1 + 2g^2(u)) \exp\left(\frac{3}{2}g^2(u)\right) . \tag{3.2}$$

Since

$$\exp\left(\frac{3}{2}g^2(u)\right) \equiv 1 \pmod 3 ,$$

it is clear that (3.2) implies

$$g'''(u) \equiv 1 + 2g^2(u) \pmod 3 .$$

On the other hand (1.7) gives

$$g'(u) \equiv 1 + \tfrac{1}{2}g^2(u) \equiv 1 + 2g^2(u) \pmod 3 .$$

We have therefore

$$g'''(u) \equiv g'(u) \pmod 3 . \tag{3.3}$$

Comparison with (1.6) yields

$$A_{2n+1} \equiv 1 \pmod 3 \qquad (n = 0, 1, 2, \cdots) . \tag{3.4}$$

If we differentiate (3.2) two more times we get

$$\begin{cases} D^4g(u) = (7g(u) + 6g^3) \exp(2g^2(u)) , \\ D^5g(u) = (7 + 46g^2(u) + 24g^4(u)) \exp\left(\frac{5}{2}g^2(u)\right) , \end{cases} \tag{3.5}$$

where $D = d/du$. From the last equation it follows easily that

$$D^5g(u) \equiv 2 + g^2(u) + 4g^4(u) \pmod 5 .$$

Since by (1.7)

$$Dg(u) \equiv 1 + \frac{1}{2}g^2(u) + \frac{1}{8}g^4(u) \equiv 1 + 3g^2(u) + 2g^4(u) \pmod 5 ,$$

it follows that

$$(D^5 - 2D)g(u) \equiv 0 \pmod 5 . \tag{3.5}$$

This is equivalent to

$$A_{2n+5} \equiv 2A_{2n+1} \pmod 5 \qquad (n = 0, 1, 2, \cdots) . \tag{3.6}$$

Since $A_1 = A_3 = 1$, (2.6) implies

$$A_{4n+1} \equiv A_{4n+3} \equiv 2^n \pmod 5 \qquad (n = 0, 1, 2, \cdots) . \tag{3.7}$$

It is clear from (3.1), (3.2) and (3.5) that

$$D^ng(u) = \psi_{n-1}(g(u)) \exp\left(\frac{n}{2}g^2(u)\right) , \tag{3.8}$$

where $\psi_n(z)$ is a polynomial of degree n in z with positive integral coefficients. Differentiating (3.8) we find that $\psi_n(z)$ satisfies the recurrence

$$\psi_n(z) = \psi'_{n-1}(z) + nz\psi_{n-1}(z) . \tag{3.9}$$

We shall require the residue (mod p) of $\psi_{p-1}(z)$. It is not evident how to obtain this residue using (3.8) and (3.9). We shall therefore use a different method.

The writer has proved [1, §6] that if

$$g(x) = \sum_1^\infty a_n \frac{x^n}{n!} \quad (a_1 = 1)$$

is an H-series without constant term, if

$$\lambda(x) = \sum_1^\infty b_n \frac{x^n}{n!} \quad (b_1 = 1)$$

is the inverse of $g(x)$ and in addition

$$b_n \equiv 0 \pmod{p} \qquad (n > p) , \tag{3.10}$$

where p is an arbitrary prime, then

$$a_{n+p} \equiv a_p a_{n+1} \pmod{p} \qquad (n \geqq 0) . \tag{3.11}$$

Clearly (3.10) is satisfied in the present case and therefore (3.11) implies

$$A_{2n+p} \equiv A_p A_{2n+1} \pmod{p} . \tag{3.12}$$

Making use of (2.8) we may now state

THEOREM 1. *The coefficients of $g(u)$ defined by* (1.6) *satisfy*

$$A_{2n+p} \equiv -2.4.6 \cdots (p-1) A_{2n+1} \pmod{p} \ (n = 0, 1, 2, \cdots) , \tag{3.13}$$

where p is an arbitrary odd prime.

It is easily verified that (3.4) and (3.6) are in agreement with (3.13).

Since (3.12) is equivalent to

$$(D^p - A_p D) g(u) \equiv 0 \pmod{p} ,$$

comparison with (3.8) yields

$$\psi_{p-1}(g(u)) \equiv A_p \exp\left(\tfrac{1}{2} g^2(u)\right) \equiv A_p \sum_{n=0}^{m} \frac{g^{2n}(u)}{2^{n} n!} \pmod{p} ,$$

where $p = 2m + 1$.

If we put

$$(g(u))^k = \sum_{n=k}^{\infty} A_n^{(k)} \frac{u^n}{n!} \qquad (k = 1, 2, 3, \cdots),$$

we can show [1, Theorem 10] that $A_n^{(k)}$ satisfies

$$A_{n+p}^{(k)} \equiv A_p A_{n+1}^{(k)} \pmod{p} \quad (n \geqq 0) \tag{3.14}$$

for all $k \geqq 1$.

We shall apply this result to the series $\phi(u)$ defined by (2.15). Since (3.14) is equivalent to

$$(D^p - A_p D)g^k(u) \equiv 0 \pmod{p},$$

it is clear that (2.16) implies

$$\begin{aligned}(D^p - A_p D)\phi(u) &\equiv \frac{(-1)^m}{2^{m+1}(m-1)!} \frac{g^{p+1}(u)}{p} \\ &\equiv A_p(D^p - A_p D)\frac{g^{p+1}(u)}{p} \pmod{p},\end{aligned} \tag{3.15}$$

where $p = 2m + 1$.

Now by [1, (6.12)] we have

$$g(u) \equiv \sum_{n=0}^{m} A_{2n+1} \frac{g_1^{2n+1}(u)}{(2n+1)!} \pmod{p},$$

where

$$g_1(u) = u + A_p \frac{g^p(u)}{p!}; \tag{3.16}$$

moreover

$$\frac{g_1^p(u)}{p!} \equiv \sum_{n=0}^{\infty} A_p^n \frac{x^{n(p-1)+1}}{(n(p-1)+1)!} \pmod{p}. \tag{3.17}$$

It follows from (3.16) and (3.17) that

$$(D^p - A_p D)\frac{g^p(u)}{p!} \equiv 1 \pmod{p}.$$

Thus (3.15) becomes

$$(D^p - A_p D)\phi(u) \equiv -A_p g(u) \pmod{p},$$

which is equivalent to

$$e_{2n+p+1} \equiv A_p(e_{2n+2} - A_{2n+1}) \pmod{p} \qquad (n = 0, 1, 2, \cdots). \tag{3.18}$$

We may state

THEOREM 2. *The coefficients* e_{2n} *defined by* (2.12) *satisfy* (3.18).

In view of (2.10) we may rewrite (3.18) as

$$A_{2n+p+2} \equiv A_p A_{2n+1} + e_{2n+p+1} \quad (2n < p) . \tag{3.19}$$

Since

$$A_p A_{2n+1} \equiv A_{2n+p} ,$$

(3.19) is equivalent to

$$A_{2n+p+2} \equiv A_{2n+p} + e_{2n+p+1} \pmod p \quad (2n < p) . \tag{3.20}$$

We notice also that repeated application of (3.18) yields

$$e_{2n+k}(p-1) \equiv A_p^k e_{2n} - kA_{2n+k}(p-1) - 1 \pmod p ; \tag{3.21}$$

in particular we have for $k = p$

$$e_{2n+p(p-1)} \equiv A_p e_{2n} \pmod p . \tag{3.22}$$

It is also easy to extend (3.20) to

$$A_{2n+k(p-1)+1} \equiv kA_{2n+k(p-1)-1} + e_{2n+k(p-1)} \pmod p \quad (0 < 2n \leqq p+1 ;\ k = 1, 2, 3, \cdots) . \tag{3.23}$$

Indeed it follows from (3.23) and (3.18) that

$$\begin{aligned} e_{2n+(k+1)(p-1)} &\equiv A_p(e_{2n+k(p-1)} - A_{2n+k(p-1)-1}) \\ &\equiv A_p e_{2n+k(p-1)} - A_{2n+(k+1)(p-1)-1} \\ &\equiv A_p(A_{2n+k(p-1)+1} - kA_{2n+k(p-1)-1}) - A_{2n+(k+1)(p-1)-1} \\ &\equiv A_{2n+(k+1)(p-1)+1} - (k-1)A_{2n+(k+1)(p-1)-1} . \end{aligned}$$

Note that (3.23) does not hold for $k = 0$.

We may state the following theorem which supplements Theorem 2.

THEOREM 3. *The coefficients* e_{2n} *defined by* (2.12) *satisfy* (3.21), (3.22) *and* (3.23).

4. We now derive congruences for A_{2n+1} (mod 8). From the first of (3.5) we have

$$\begin{aligned} D^4 g(u) &\equiv (-g(u) + 6g^3(u)) \exp(2g^2(u)) \\ &\equiv (-g(u) + 6g^3(u))(1 + 2g^2(u)) \\ &\equiv -g(u) + 4g^3(u) + 4g^5(u) \pmod 8 , \end{aligned}$$

so that

$$D^4 g(u) \equiv -g(u) \pmod 8 . \tag{4.1}$$

This is equivalent to

$$A_{2n+5} \equiv -A_{2n+1} \pmod 8 \qquad (n = 0, 1, 2, \cdots) , \tag{4.2}$$

which implies

$$A_{4n+1} \equiv A_{4n+3} \equiv (-1)^n \pmod 8 \qquad (n = 0, 1, 2, \cdots) . \tag{4.3}$$

This result can however be improved without much difficulty. Working modulo 16 we find that the $\psi_n(z)$ defined by (3.8) and (3.9) satisfy

$$\begin{aligned} &\psi_3(z) \equiv 7z + 6z^3 , &&\psi_4(z) \equiv 7 - 2z^2 , \\ &\psi_5(z) \equiv -z + 6z^3 , &&\psi_6(z) \equiv -1 + 12z^2 , \\ &\psi_7(z) \equiv z + 4z^3 ; && \end{aligned}$$

note that the $\psi_n(z)$ are here treated as finite H-series. Then by (3-8)

$$\begin{aligned} D^8 g(u) &\equiv (g(u) + 4g^3(u)) \exp (4g^2(u)) \\ &\equiv (g(u) + 4g^3(u))(1 + 4g^2(u)) , \end{aligned}$$

so that

$$D^8 g(u) \equiv g(u) \pmod{16} . \tag{4.4}$$

This is equivalent to

$$A_{2n+9} \equiv A_{2n+1} \pmod{16} . \tag{4.5}$$

Since $A_1 = A_3 = 1$, $A_5 = 7$, $A_7 \equiv 7 \pmod{16}$, (4.5) implies

$$\begin{cases} A_{8n+1} \equiv A_{8n+3} \equiv 1 \pmod{16} , \\ A_{8n+5} \equiv A_{8n+7} \equiv 7 \pmod{16} . \end{cases} \tag{4.6}$$

We may state

THEOREM 4. *The coefficients A_{2n+1} satisfy* (4.2), (4.3), (4.5), (4.6).

5. We now put

$$\frac{u}{g(u)} = \sum_{n=0}^{\infty} \beta_{2n} \frac{u^{2n}}{(2n)!} , \tag{5.1}$$

so that

$$\sum_{r=0}^{n} \binom{2n+1}{2r} A_{2n-2r+1} \beta_{2r} = 0 \quad (n > 0) . \tag{5.2}$$

It follows from (5.2) that the β_{2n} are rational numbers with odd denominators.

From (5.1) and (2.3) we have

(5.3) $$\frac{u}{g(u)} = \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+1} \frac{g^{2n}(u)}{(2n)!} .$$

By (2.4)

(5.4) $$c'_{2n+1} = \frac{c_{2n+1}}{2n+1} = (-1)^n \frac{1.3.5 \cdots (2n-1)}{2n+1} .$$

Let p be an odd prime. Then for $2n+1 > p$, c'_{2n+1} is integral (mod p) except possibly when $p/2n+1$. Let

$$2n+1 = kp^r , \quad p \nmid k , \quad r \geqq 1 .$$

If $k > 1$ it is obvious from (5.4) that c'_{2n+1} is integral (mod p). If $k = 1$, the numerator of c'_{2n+1} is divisible by at least p^w, where $w = (p^{r-1} - 1)/2$. But since

$$\tfrac{1}{2}(p^{r-1} - 1) \geqq r$$

except when $p = 3$, $r = 2$, it follows that

(5.5) $$p \frac{u}{g(u)} \equiv c_p \frac{g^{p-1}(u)}{(p-1)!} \pmod{p} \quad (p > 3) ,$$

(5.6) $$3 \frac{u}{g(u)} \equiv -\frac{g^2(u)}{2!} - \frac{g^8(u)}{8!} \pmod{3} .$$

In the next place we have [1, (6.2)]

(5.7) $$\frac{g^{p-1}(u)}{(p-1)!} \equiv \sum_{n=1}^{\infty} A_p^{n-1} \frac{u^{n(p-1)}}{(n(p-1))!} \pmod{p}$$

for all p. As for $g^8(u)/8!$, we have by (3.16)

$$\frac{g^3(u)}{3!} g_1(u) - u \equiv \sum_{1}^{\infty} \frac{u^{2n+1}}{(2n+1)!} ,$$

$$g_1'(u) \equiv 1 + \tfrac{1}{2} g^2(u) g'(u) \equiv 1 + \tfrac{1}{2} g^2(u) \equiv g'(u) \pmod{3} .$$

It follows that

$$\frac{g^4(u)}{4!} \equiv \sum_{2}^{\infty} (n-2) \frac{u^{2n}}{(2n)!} \pmod{3}$$

and a little manipulation leads to

(5.8) $$\frac{g^8(u)}{8!} \equiv \sum_{1}^{\infty} \frac{u^{6n+2}}{(6n+2)!} \pmod{3} .$$

If we recall that

$$c_p \equiv -A_p \pmod{p}$$

and make use of (5.1), (5.3), (5.5), (5.6), (5.7) and (5.8) we get the following analog of the Staudt-Clausen theorem:

THEOREM 5. *The coefficients β_{2n} defined by* (5.1) *satisfy*

$$\beta_{2n} = G_{2n} - \frac{b}{3} - \sum_{\substack{p-1/2n \\ p>3}} \frac{A_p^{2n/(p-1)}}{p}, \tag{5.9}$$

where G_{2n} is an integer,

$$b = \begin{cases} 2 & n \equiv 1 \pmod 3 \\ 1 & n \not\equiv 1 \pmod 3 \end{cases}$$

and the summation is over all primes $p > 3$ such that $p - 1 \mid 2n$.

6. The following values of A_n were computed by R. Carlitz in the Duke University Computing Laboratory.

$$\begin{aligned}
A_5 &= 7, \quad A_7 = 127, \\
A_9 &= 17.257, \\
A_{11} &= 7.34807, \\
A_{13} &= 20036983, \\
A_{15} &= 17.134138639, \\
A_{17} &= 7.49020204823, \\
A_{19} &= 127.163.467.6823703, \\
A_{21} &= 23.109.6291767620181, \\
A_{23} &= 7.655889589032992201^*, \\
A_{25} &= 17.94020690191035873697^*,
\end{aligned}$$

The numbers marked with an asterisk have not been factored completely but at any rate have no prime divisors $< 10^4$.

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SOME DEGENERATE CAUCHY PROBLEMS WITH OPERATOR COEFFICIENTS

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1. Motivated in part by connections with problems in transonic gas dynamics there has been considerable interest in equations of the form

$$u_{tt} - K(t)u_{xx} + bu_x + eu_t + du - h = 0 \tag{1.1}$$

where d, b, e and h are functions of (x, t) (see here Bers [4] for a bibliography and discussion). In particular there arises the Cauchy problem for (1.1) in the hyperbolic region with data given on the parabolic line $t = 0$ (see in particular Protter [20], Conti [9], Bers [3], Berezin [2], Hellwig [12; 13], Frankl [10], Weinstein [25], Krasnov [15; 16], Carroll [8], Germain and Bader [11], and Barancev [1]). Protter assumes that $K(t)$ is a monotone increasing function of t, $K(0) = 0$, and shows that the Cauchy problem for (1.1) with initial data $u(x, 0)$ and $u_t(x, 0)$ prescribed on a finite x-interval, is correctly set (under suitable regularity assumptions) if $tb(x, t)/\sqrt{K(t)} \to 0$ as $t \to 0$. Thus in particular if $b \equiv 0$ the condition is automatically true. Krasnov considers generalized solutions and the equation

$$u_{tt} - \Sigma \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) + \Sigma b_i \frac{\partial u}{\partial x_i} + e\frac{\partial u}{\partial t} + du = h\,. \tag{1.2}$$

Again the presence of first order terms b_i complicates the matter and (as with Protter for $K(t) \sim t^\alpha$) it is assumed that $b_i = O(t^{\alpha/2-1}\beta(t))$ where $\beta(t) \to 0$ (additional assumptions are also made). Krasnov supposes $\Sigma a_{ik}\xi_i\xi_k \geqq ct^\alpha \Sigma \xi_i^2$ with $h/t^{\frac{\alpha-1+\delta_0}{2}} \in L^2$ ($\delta_0 > 0$ is a number for which bounds are determined in the proof) and finds solutions u such that $u_t/t^{\frac{\alpha+1+\delta_0}{2}} \in L^2$ and $u_{x_i}/t^{\frac{1+\delta_0}{2}} \in L^2$. Thus the growth of h appears to play an important role in determining a solution in this more general equation (1.2). Slightly more general degeneracies for $\Sigma a_{ik}\xi_i\xi_k$ are mentioned by Krasnov but always in some comparison to a power of t.

It is one of the aims of the present paper to give a more precise estimate of the allowable degeneracy in relation to the growth of h and to give estimates for the solution. In particular we will not require that $K(t)$ be monotone. For simplicity we omit here first order terms in $\partial u/\partial x_i$; this will be dealt with, in an abstract framework, in a subsequent article. A summary of some of the present work was

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given in [8]. We remark that an operational treatment of the type of degenerate problems considered by Tersenov [24] and Hu Hsien Sun [14] is also contemplated (this involves an equation of the form $K(t)u_{tt} - u_{xx} + bu_x + eu_t + du - h = 0$ with data given for $t = 0$). As indicated above our results generalize in certain respects those of Krasnov, however the methods employed here are quite different; for example Krasnov relies heavily on a Galerkin type method for existence whereas we employ an energy method based on work of Lions [17]. Further generalizations in our framework are clearly possible (see [16]).

2. Following Lions (see [18] for an extensive bibliography and treatment of operational differential equations) we reformulate (1.2) as follows. Let V and H, $V \subset H$, be Hilbert spaces, V dense in H, with the topology of V being finer than that induced by H.* The norms in V and H are denoted by $\| \ \|$ and $| \ |$ respectively. Let $(u, v) \to a(t, u, v)$ be a continuous sesquilinear form on $V \times V$ for t fixed, $0 \leq t \leq b < \infty$, with $a(t, u, v) = \overline{a(t, v, u)}$. Assume that $t \to a(t, u, v) \in C^1[0, b]$ for (u, v) fixed. We recall (see [18]) that the form $a(t, u, v)$ defines an unbounded operator $A(t)$: $D(A(t)) \to H$ by defining $D(A(t))$ to be the set of $u \in V$ such that $v \to a(t,u,v)$ is continuous on V in the topology of H. Then we can write for $u \in D(A(t))$, $(A(t)u, v) = a(t, u, v)$ for $v \in V$. Now let $\{B(t)\}$ be a family of bounded Hermitian operators in H with $t \to B(t) \in \mathscr{E}^1(\mathscr{L}_s(H, H))$ (here $\mathscr{E}^m(G)$ is the space of m-times continuously differentiable functions of t with values in G and $\mathscr{L}_s(H, H)$ is the space of continuous linear maps $H \to H$ with the topology of simple convergence—see [5]).

Let now $\psi > 0$ be a numerical function with $\psi \uparrow$ as $t \to 0$, $\psi \in C^0(0, b]$. Here ψ does not necessarily approach ∞. We assume q is another numerical function such that $q > 0$ on $(0, b]$ with $q \to 0$ as $t \to 0$ (in what follows all limits such as $q \to 0$ will refer to $t \to 0$). Let f be given such that $\psi f \in L^2(H)$ (for the spaces $L^p(H)$ and the integration of vector valued functions see [6; 7]). We assume $q \in C^1(0, b]$. Let $\mathscr{F}_s$ be the Hilbert space of functions u on $[0, s]$ such that $u(0) = 0$, $\psi u' \in L^2(H)$, and $\omega u \in L^2(V)$ with

$$\|u\|^2_{\mathscr{F}_s} = \int_0^s \{\|\omega u\|^2_V + |\psi u'|^2_H\}\, dt \tag{2.1}$$

(ω is a numerical function to be determined, $\omega > 0$, $\omega \to \infty$). Here all derivatives are taken in the sense of vector valued distributions in $\mathscr{D}'(H)$(see [23]) and $\mathscr{F}_s$ may be proved complete by standard arguments. Let now $\mathscr{H}_s$ be the space of functions h which satisfy $h(s) = 0$, $h/\psi \in L^2(H)$, $h'/\psi \in L^2(H)$, and $qh/\omega \in L^2(V)$. Set

* H is also assumed to be separable for simplicity in a later argument; this condition is not necessary however.

$$\tilde{E}_s(u,h) = \int_0^s \{qa(t, u, h) + (B(t)u', h) - (u', h')\}dt \tag{2.2}$$

and define

$$\tilde{L}_s(h) = \int_0^s (f, h)dt . \tag{2.3}$$

We note that (2.2) and (2.3) are well defined for $u \in \mathscr{F}_s, h \in \mathscr{H}_s$, and f as described. Thus assume ω as indicated has been given; then we pose

Problem 1. Find s and $u \in \mathscr{F}_s$ such that for all $h \in \mathscr{H}_s$

$$\tilde{E}_s(u, h) = \tilde{L}_s(h) . \tag{2.4}$$

Naturally we wish to find the best ω in some sense when posing problem 1. Here best will be left vague for the present in remarking only that ω furnishes a measure of how rapidly the solution u tends to 0 as $t \to 0$. We define now $\mathscr{K}_s$ to be the space of functions k such that $k = \int_0^t \varphi h d\xi$ for $h \in \mathscr{H}_s$ where φ is a numerical function to be determined (in general $\varphi \in C'[0, s]$, $\varphi > 0$ on $(0, s]$, and $\varphi \to 0$ as $t \to 0$). Clearly $k' = \varphi h$ and thus $k'/\varphi\psi = h/\psi \in L^2(H)$. For suitable choice of the numerical function $\delta > 0, \delta \to \infty$, we define $\mathscr{K}_s$ as a prehilbert space with norm

$$||k||^2_{\mathscr{K}_s} = \int_0^s \left\{ ||\delta k||^2_V + \left| \frac{k'}{\varphi\psi} \right|^2_H \right\} dt \tag{2.5}$$

LEMMA 1. *Define $v = \varphi/q$ and assume*
(i) $\varphi\psi^2 \in L^\infty$
(ii) $\omega \leqq \delta$
(iii) $\omega^2 v^2 \in L^1$
(iv) $\delta^2 \int_0^t \omega^2 v^2 d\xi \in L^1$ *with* $\varphi, q, \omega, \psi, \delta \in C^0(0, s]$ *all positive on* $(0, s]$.
Then $\mathscr{K}_s \subset \mathscr{F}_s$ algebraically and topologically.

Proof. The following estimates are straightforward

$$|\psi k'| = \left| \frac{\varphi\psi^2 k'}{\varphi\psi} \right| \leqq c \left| \frac{k'}{\varphi\psi} \right| \tag{2.6}$$

$$||\delta k||^2 = \left| \left| \delta \int_0^t \frac{q}{\omega} \omega v h d\xi \right| \right|^2 \leqq \delta^2 \int_0^t \omega^2 v^2 d\xi \int_0^t \left| \left| \frac{qh}{\omega} \right| \right|^2 d\xi . \tag{2.7}$$

Thus by (2.7) for $k \in \mathscr{K}_s$ and δ satisfying the hypotheses we have $\int_0^s ||\delta k||^2 d\xi < \infty$; also by (2.6) and the fact $\omega \leqq \delta$ it follows that $||k||_{\mathscr{F}_s} \leqq \tilde{c}||k||_{\mathscr{K}_s}$. From (2.7) we obtain also the result that $||k||^2 \to 0$ as $t \to 0$ which proves that in fact $\mathscr{K}_s \subset \mathscr{F}_s$.

LEMMA 2. *Assume* (i)–(iv) *and*

(v) $1/v\int_0^t \omega^2 v^2 d\xi \in L^\infty$

(vi) $\varphi'\psi^2 \in L^\infty$

(vii) $1/v\delta^2 \in L^\infty$

(viii) $-(1/v)'\,1/\delta^2 \in L^\infty$, $v' \geqq 0$. *Assume also that* $a(t, u, u) \geqq \alpha \|u\|^2$, *then*

$$2ReE_s(k, k) \geqq \int_0^s \|\delta k\|^2 \left\{-\alpha\left(\frac{1}{v}\right)' \frac{1}{\delta^2} - \frac{c_1}{v\delta^2}\right\} dt + \int_0^s \left|\frac{k'}{\varphi\psi}\right|^2 \{\varphi'\psi^2 - 2\beta\varphi\psi^2\}\, dt \tag{2.8}$$

where, for $k = \int_0^t \varphi h d\xi$, $E_s(u, k) = \tilde{E}_s(u, h)$.

Proof. Formally we have

$$2ReE_s(k, k) = \frac{q}{\varphi} a(t, k, k)\Big|_0^s - \int_0^s \left\{\left(\frac{q}{\varphi}\right)' a(t, k, k) - \left(\frac{q}{\varphi}\right) a'(t, k, k)\right\} dt\Big\} + 2Re\int_0^s \frac{1}{\varphi}(Bk', k')dt - \varphi|h|^2\Big|_0^s + \int_0^s \varphi'|h|^2 dt . \tag{2.9}$$

Noting that $\lim \varphi|h|^2 = \lim 1/\varphi|k'|^2 = \theta^2 \geqq 0$ will exist if all the other terms make sense we have

$$\frac{q}{\varphi} a(t, k, k) \leqq \frac{c}{v}\|k\|^2 \leqq \frac{c}{v}\int_0^t \omega^2 v^2 d\xi \int_0^t \left\|\frac{qh}{\omega}\right\|^2 d\xi \tag{2.10}$$

which vanishes as $t \to 0$. Note by the Banach Steinhaus theorem it follows that (see [18])

$$|a(t, u, h)| \leqq c\|u\|\,\|h\| \tag{2.11}$$

$$|a'(t, u, h)| \leqq c_1\|u\|\,\|h\| \tag{2.12}$$

$$\left|\int_0^s \frac{1}{\varphi}(Bk', k')dt\right| \leqq \beta\int_0^s \left|\frac{k'}{\varphi\psi}\right|^2 \varphi\psi^2 dt < \infty . \tag{2.13}$$

Moreover under the hypotheses above

$$\int_0^s \frac{\varphi'}{\varphi^2}|k'|^2 dt = \int_0^s \varphi'\psi^2 \left|\frac{k'}{\varphi\psi}\right|^2 dt < \infty \tag{2.14}$$

$$\left|\int_0^s \frac{q}{\varphi} a'(t, k, k)dt\right| \leqq c_1\int_0^s \frac{1}{v\delta^2}\|\delta k\|^2 dt < \infty \tag{2.15}$$

$$-\int_0^s \left(\frac{q}{\varphi}\right)' a(t, k, k)dt \leqq c\int_0^s -\left(\frac{1}{v}\right)' \frac{1}{\delta^2}\|\delta k\|^2 dt < \infty \tag{2.16}$$

Thus (2.9) is valid and (2.8) follows.

The formula (2.8) indicates the properties desired of δ and φ in order to obtain an estimate $ReE_s(k, k) \geqq \Omega \| k \|^2_{\mathscr{H}_s}$ thus enabling us to apply the Lions projection theorem (see [18]). We will give here a natural choice for δ, φ etc. without seeking the best possible result. To this end set

$$\varphi = \hat{c} \int_0^t \frac{d\xi}{\psi^2} . \tag{2.17}$$

Then $\varphi \in C^1[0, b]$, $\varphi \to 0$, and since ψ is monotone $\varphi/\varphi' = \psi^2 \int_0^t d\xi/\psi^2 \leqq Nt$. Hence $\varphi\psi^2 = \hat{c}\varphi/\varphi' \to 0$ also and thus $1/\varphi\psi \to \infty$. Next let $R \neq 0$ be a constant and

$$-\left(\frac{1}{v}\right)' \frac{1}{\delta^2} = R; \; v = \frac{1}{\left[\delta_1 + \int_t^s R\delta^2 d\xi\right]} \tag{2.18}$$

where $\delta_1 > 0$ is determined by $v(s)$. Thus $v \to 0$ corresponds to $\delta \notin L^2$ and in any case, noting $v' = Rv^2\delta^2$,

$$\frac{1}{v}\int_0^t \omega^2 v^2 d\xi \leqq \frac{1}{v}\int_0^t \delta^2 v^2 d\xi = \frac{1}{R}\left[1 - \frac{v(0)}{v(t)}\right] = \frac{1}{R}\left\{1 - \frac{\delta_1 + \int_t^s R\delta^2 d\xi}{\delta_1 + \int_0^s R\delta^2 d\xi}\right\}. \tag{2.19}$$

(This shows that $\int_0^t \omega^2 v^2 d\xi < \infty$ and that $1/v \int_0^t \omega^2 v^2 d\xi \leqq M$. The last term in (2.19) is taken to be zero if $\delta \notin L^2$ or $v(0) = 0$, and $v(0)/v(t)$ is seen to be bounded by one in all other cases.) Thus (i), (ii) (by assumption), (iii), (v), (vi), and (viii) hold. Also the $\varphi'\psi^2$ term dominates in the second integral of (2.8) for s small. Now for (vii) we note that $1/v\delta^2 = (v/v')R$ and $v' = (\varphi/q)'$; thus

$$\frac{v'}{v} = \frac{\varphi'}{\varphi} - \frac{q'}{q} = \frac{\varphi'}{\varphi}\left[1 - \frac{q'\psi^2}{q}\int_0^t \frac{d\xi}{\psi^2}\right]. \tag{2.20}$$

If we assume for example that $(q'\psi^2/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$ for t small then $v'/v \geqq \varepsilon_1\varphi'/\varphi \to \infty$ since $\varphi, \varphi' > 0$ on $(0, b]$ and $\varphi/\varphi' \to 0$. In any case if $v'/v \to \infty$ then $v/v' \to 0$ and $1/v\delta^2 \to 0$ which means not only that (vii) holds but that the $-\alpha(1/v)' 1/\delta^2$ term dominates in the first integral of (2.8) for s small. Note here that φ and hence v are defined on $[0, b]$ independently of s by say (2.17) whereas (2.18) determines δ^2 on any interval $(0, s]$ for v given. Finally with regard to (iv) there are various hypotheses on ω and v which would work but we assume simply that

$$\omega^2 = \frac{v'}{v^{2-\varepsilon}}, \; 0 < \varepsilon < 1 \tag{2.21}$$

Then if say $v \in C^0[0, b]$

$$\begin{aligned}\int_0^s \delta^2\left(\int_0^t \omega^2 v^2 d\xi\right) dt &= \int_0^s \frac{v'}{Rv^2}\left(\int_0^t v' v^\varepsilon d\xi\right) dt \\ &= \frac{1}{R(1+\varepsilon)}\int_0^s \frac{v'}{v^{1-\varepsilon}}\, dt = \frac{1}{R\varepsilon(1+\varepsilon)}\, v^\varepsilon(t)\Big|_0^s .\end{aligned} \tag{2.22}$$

It should be noted that $v \in C^0[0, b]$ now implies that $\omega \leqq c\delta$ since $\omega^2/\delta^2 = Rv^\varepsilon$ and this would be a condition equivalent to (ii). We remark that $v \to 0$ implies $\omega \notin L^2$ since $\int_t^s \omega^2 d\xi = \int_t^s v'/v^{2-\varepsilon} d\xi = 0(1/v^{1-\varepsilon})$. This proves

LEMMA 3. *Assume* $a(t, u, u) \geqq \alpha \| u \|^2$, $v'/v \to \infty$, $v \in C^0[0, b]$, $\omega^2 = v'/v^{2-\varepsilon}$, $\varphi = \hat{c}\int_0^t d\xi/\psi^2$, *and* $v = 1/\delta_1 + \int_t^s R\delta^2 d\xi$. *Then* $\omega \leqq c\delta$ *and* (i), (iii)–(viii) *hold with* $ReE_s(k, k) \geqq \Omega \| k \|^2_{\mathscr{K}_s}$ *for s sufficiently small.*

Using the above lemmas and the Lions projection theorem (see [18]) there results

THEOREM 1. *Under the hypotheses of Lemma* 3 *and the conditions on* $a(t, u, v)$, $B(t)$ *stipulated above there exist functions* ω ($\omega \notin L^2$ *if* $v \to 0$) *such that for s small problem* 1 *has a solution.*

Proof. We need only check that the map $u \to E_s(u, k)$: $\mathscr{F}_s \to \boldsymbol{C}$ is continuous for $k \in \mathscr{K}_s$ fixed and that the map $k \to L_s(k) = \tilde{L}_s(h)$: $\mathscr{K}_s \to \boldsymbol{C}$ is continuous. This verification is immediate.

Now since $q > 0$ on $(0, b]$ we can treat $qa(t, u, v)$ as a nondegenerate form on say $[s/2, b]$ and apply Lions' results for such problems (see [17; 18]). We want to solve

Problem 2. Find $u \in \mathscr{F}_b$ such that $\tilde{E}_b(u, h) = \tilde{L}_b(h)$ for all $h \in \mathscr{H}_b$.

Thus suppose the problem has been solved for $[0, s]$, that is suppose problem 1 has been solved with solution u_1. Then following [17] let $p \in C^1$ with $p = 1$ on $[0, 2/3\, s]$ and $p = 0$ in a neighborhood of s. Set $u_2 = u - pu_1$; then $u_2 = 0$ on $[0, 2/3\, s]$ and $u_2 = u$ for $t \geqq s$. The problem 2 for u becomes

$$\begin{aligned}\tilde{E}_b(u_2, h) &= \int_0^b (f, h)dt - \int_0^b p'[(Bu_1, h) + (u_1', h)]dt \\ &\quad - \int_0^b \{qa(t, u_1, ph) + (Bu_1', ph) - (u_1', (ph)')\}dt .\end{aligned} \tag{2.23}$$

Now if $h \in \mathscr{H}_b$ we see that $ph \in \mathscr{H}_s$; hence

$$\tilde{E}_b(u_2, h) = \int_0^b (f, h - ph)dt - \int_0^b p'[(Bu_1, h) + (u_1', h)]dt . \tag{2.24}$$

In particular we see that everything vanishes on say $[0, s/2]$; hence we pose the Cauchy problem with initial data given at $s/2$ as follows. Let $\mathscr{F}_{s/2\ s_1}$ be the space of u such that $\omega u \in L^2(v)$ and $\psi u' \in L^2(H)$ on $[s/2, s/2 + s_1]$ with $u(s/2) = 0$. The space $\mathscr{H}_{s/2\ s_1}$ corresponding to $\mathscr{H}_s$ is defined similarly on $[s/2, s/2 + s_1]$. We extend ω and δ to be constant on $[s, b]$; then since ψ, ω, δ etc. are positive and continuous we may define say $\mathscr{F}_{s/2, s_1}$ in terms of $u \in L_2(V)$ and $u' \in L^2(H)$. Let $\tilde{E}_{s/2\ s_1}$ denote the terms in $\tilde{E}_b$ integrated over $[s/2, s/2 + s_1]$, and denote the right side of (2.24) integrated from $s/2$ to $s/2 + s_1$ by $\tilde{\tilde{L}}_{s/2\ s_1}(h)$. Then consider

Problem 3. Find $u_2 \in \mathscr{F}_{s/2\ s_1}$ such that $\tilde{E}_{s/2, s_1}(u_2, h) = \tilde{\tilde{L}}_{s/2\ s_1}(h)$ for all $h \in \mathscr{H}_{s/2, s_1}$.

Problem 3 has a (unique) solution for s_1 sufficiently small by [17] and the above extension procedure may be repeated in steps of length $s_1/2$. Thus u will eventually be determined on $[0, b]$ satisfying problem 2. Hence

THEOREM 2. *Under the hypotheses of Theorem* 1 *there exists a solution of problem* 2.

3. Suppose now that $\tilde{E}_s(u, h) = 0$ for all $h \in \mathscr{H}_s$. Let $h = -\int_t^s Ju d\xi$, $h' = Ju$, $J \to \infty$. Then

LEMMA 4. *Assume*

(a) $J^2/\omega^2 \int_0^t d\xi/\psi^2 \in L^1$

(b) $J/\omega\psi \in L^\infty$

(c) $J^2/\omega^2 \int_0^t (q^2/\omega^2)\, d\xi \in L^1$. *Then* $h \in \mathscr{H}_s$ *if* $u \in \mathscr{F}_s$ *and* $h = -\int_t^s Ju d\xi$.

Proof. Clearly $h'/\psi = (J/\omega\psi)\omega u \in L^2(V)$ (hence certainly $h'/\psi \in L^2(H)$) and $h(s) = 0$; also

$$\left|\frac{h}{\psi}\right|^2 \leqq c\left|\left|\frac{h}{\psi}\right|\right|^2 \leqq \left(\frac{1}{\psi}\int_t^s \frac{J}{\omega}\,||\omega u||\, d\xi\right)^2 \leqq \frac{1}{\psi^2}\int_t^s \frac{J^2}{\omega^2}\, d\xi \int_t^s ||\omega u||^2 d\xi \tag{3.1}$$

$$\int_0^s \left|\left|\frac{q}{\omega} h\right|\right|^2 d\xi \leqq \int_0^s \frac{q^2}{\omega^2}\left(\int_t^s \frac{J^2}{\omega^2}\, d\xi\right) dt \int_0^s ||\omega u||^2\, d\xi\,. \tag{3.2}$$

Using the Fubini and Tonelli theorems (see e.g. [19]) the lemma follows.

We note now explicitly the fact that if $u \in L^2(H)$ and $u' \in L^2(H)$ (u' taken in $\mathscr{D}'(H)$ on $(0, s)$) then u may be identified with a continuous function and $u(0) = 0$ makes sense. Indeed for u, determined almost everywhere, we see that $u' \in L^1(H)$ on $[0, s]$ and clearly $D\tilde{u} = u'$ in $\mathscr{D}'(H)$ where $\tilde{u} = \int_0^t u' d\xi \in \mathscr{E}^0(H)$ (see [23]). Thus $D(\tilde{u} - u) = 0$ and by [21] for any $h \in H$, $(\tilde{u} - u, h) = c_h$ in $\mathscr{D}'$. Hence $(\tilde{u} - u, h) = c_h$

almost everywhere as a function and thus u may be identified scalarly with the continuous function $\tilde{u}$. Since H is separable we may then identify u with a continuous function and $u(0) = 0$ is meaningful (see [23], [22]). Hence $u = \tilde{u}$ follows. Thus setting $u = \int_0^t u' d\xi$, $h = -\int_t^s h' d\xi$

$$(3.3)\quad |(u, h)| = \left| -\int_0^t \int_t^s (u'(\xi), h'(\eta)) d\eta d\xi \right|$$
$$\leqq \sup \left| \frac{\psi(\eta)}{\psi(\xi)} \right| \int_0^t \int_t^s |\psi u'| \left| \frac{h'}{\psi} \right| d\eta d\xi \leqq \frac{N}{2} \int_0^t \int_t^s \left\{ |\psi u'|^2 + \left| \frac{h'}{\psi} \right|^2 \right\} d\eta d\xi$$
$$\leqq \frac{N}{2} \left\{ \int_0^t (s - t) |\psi u'|^2 d\xi + t \int_t^s \left| \frac{h'}{\psi} \right|^2 d\eta \right\} .$$

Thus $(u, h) = 0$ at $t = 0$ and we note that $\int_0^s (Bu', h) dt = -\int_0^s (B'u, h) dt - \int_0^s (Bu, h')\, dt$. Hence $\tilde{E}_s(u, h) = 0$ becomes, with h as above

$$(3.4)\qquad \int_0^s \left\{ \frac{q}{J} a(t, h', h) - (B'u, h) - J(Bu, u) - J(u', u) \right\} dt = 0 .$$

Set now $\tilde{\theta}^2 = \lim q/J\, a(t, h, h)$ which will exist if everything else makes sense in the following. Then we have

LEMMA 5. *Assume* (a)–(c) *from Lemma* 4 *and*
(d) $J \int_0^t d\xi/\psi^2 \in L^\infty$
(e) $-J'/\omega^2 \in L^\infty$; $J' < 0$
(f) $J \to \infty$; $J/J' \to 0$
(g) $(q/J)'/(q/J) \to \infty$. *Then if* $h = -\int_t^s Jud\xi$, $u \in \mathscr{F}_s$, *and if* $a(t, h, h) \geqq \alpha \|h\|^2$ *it follows that*

$$(3.5)\qquad \int_0^s \left\{ \alpha \left(\frac{q}{J} \right)' \frac{\omega^2}{q^2} - c_1 \left(\frac{q}{J} \right) \frac{\omega^2}{q^2} \right\} \left\| \frac{qh}{\omega} \right\|^2 dt$$
$$+ \int_0^s \left\{ -\frac{J'}{\omega^2} - \frac{2\beta J}{\omega^2} - \frac{\hat{\beta}}{\omega^2} \int_t^s J d\xi - \frac{\hat{\beta} t J}{\omega^2} \right\} |\omega u|^2 dt \leqq 0$$

Proof. By (d) we have

$$J|u|^2 \leqq J \left(\int_0^t |\psi u'| \frac{d\xi}{\psi} \right)^2 \leqq J \int_0^t \frac{d\xi}{\psi^2} \int_0^t |\psi u'|^2 d\xi \to 0$$

whereas from (e) there results $-J'|u|^2 = -J'/\omega^2 |\omega u|^2 \in L^1$. Next by (f) and (e) it follows that $\lim Jq/\omega^2 = \lim (J/-J')(-J'q/\omega^2) = 0$; hence $Jq/\omega^2 \in L^\infty$ and

$$(3.6)\quad \int_0^s \left(\frac{q}{J} \right)' \|h\|^2 d\xi \leqq \int_0^s \left(\frac{q}{J} \right)' \left(\int_t^s J \|u\| d\xi \right)^2 dt$$
$$\leqq \int_0^s \left(\frac{q}{J} \right)' \left(\int_t^s \frac{J^2}{\omega^2} d\xi \int_t^s \|\omega u\|^2 d\xi \right) dt \leqq \left(\int_0^s \|\omega u\|^2 d\xi \right) \int_0^s \frac{Jq}{\omega^2} d\xi .$$

Note here $q/J \to 0$ and $q/J = \int_0^t (q/J)' \, d\xi$; also by (g) surely $\int_0^s q/J \, \| h \|^2 \, d\xi < \infty$. Now by (f) it follows that $J|u|^2 = (J/J')\, J'|u|^2 \in L^1$ and finally we remark that

$$(3.7) \qquad \left| 2Re\int_0^s (B'u, h)\, d\xi \right| \leqq \hat{\beta}\int_0^s\int_t^s J(\xi)\,\{|u(t)|^2 + |u(\xi)|^2\}\, d\xi dt$$
$$\leqq \hat{\beta}\left\{\int_0^s |\omega u|^2 \left(\frac{1}{\omega^2}\int_t^s J d\xi\right) dt + \int_0^s \frac{Jt}{\omega^2}\, |\omega u|^2\, dt\right\} .$$

Here the Jt/ω^2 term makes sense since $Jt/\omega^2 = (J/-J')(-J't/\omega^2) \to 0$ by (e) and (f). Then we note that

$$\frac{1}{\omega^2}\int_t^s J d\xi = \left(\frac{-J'}{\omega^2}\right)\left(\frac{J}{-J'}\right)\left(\frac{1}{J}\int_t^s J d\xi\right) ;$$

but by 1' Hospital's rule $\lim 1/J \int_t^s J d\xi = \lim J/-J^1 = 0$ (here note that $J' \neq 0, J \neq 0$ for $t > 0$). Hence we may write

$$(3.8) \qquad \tilde{\theta}^2 + \int_0^s \left\{\left(\frac{q}{J}\right)' a(t, h, h) + \left(\frac{q}{J}\right) a'(t, h, h)\right\} dt$$
$$+ 2Re\int_0^s (B'u, h) dt + 2Re\int_0^s J(Bu, u) dt$$
$$- \int_0^s J'\, |u|^2\, dt + J\, |u(s)|^2 = 0 .$$

The lemma follows immediately.

Now let $\omega^2 = v'/v^{2-\varepsilon}$ as before and consider the following choice for the function J

$$(3.9) \qquad J = j + \check{c}\int_t^s \omega^2 d\xi; \; -\frac{J'}{\omega^2} = \check{c} .$$

It follows that (e) holds (we assume ω, v etc. are as before) and since $v = \varphi/q$ (d) is a consequence of the fact that

$$(3.10) \qquad \check{c}\int_t^s \omega^2 d\xi \int_0^t \frac{d\eta}{\psi^2} \leqq \check{c}\varphi\int_t^s \delta^2 d\xi = \check{c}\varphi\int_t^s -\left(\frac{1}{v}\right)' \frac{d\xi}{R}$$
$$= \check{c}\,\frac{\varphi}{R}\left[\frac{1}{v(t)} - \frac{1}{v(s)}\right] = \frac{\check{c}}{R}\left[q(t) - \varphi(t)\,\frac{q(s)}{\varphi(s)}\right] .$$

Note now that with the above choice of ω we can write J in the form $J = j + \check{c}\int_t^s v'/v^{2-\varepsilon}\, d\xi = j - (\check{c}/1-\varepsilon)\,(1/v(s))^{1-\varepsilon} + (\check{c}/1-\varepsilon)\,(1/v(t))^{1-\varepsilon}$. If j is taken to be $j = (\check{c}/1-\varepsilon)\,(1/v(s))^{1-\varepsilon}$ then

$$(3.11) \qquad J = \frac{\check{c}}{1-\varepsilon}\left(\frac{1}{v}\right)^{1-\varepsilon}; \frac{J}{J'} = \frac{-1}{1-\varepsilon}\left(\frac{v}{v'}\right) .$$

Thus if $v/v' \to 0$ then $J/-J' \to 0$. Moreover since $\omega^2 = (v'/v)\,(1/v)^{1-\varepsilon}$ it follows that $\omega \to \infty$ if $v \to 0$ and $v/v' \to \infty$ and also by (3.11) $J \to \infty$ if $v \to 0$. Hence if $v'/v \to \infty$ and $v \to 0$ then (f) holds and $\omega \to \infty$.

Consider now condition (a); using (d) we have $J^2/\omega^2 \int_0^t d\xi/\psi^2 \leqq c\, J/\omega^2 = -\check{c}c\, J/J' \to 0$ which implies (a). For (c) we note

$$(3.12) \quad \int_0^s \frac{J^2}{\omega^2} \left(\int_0^t \frac{q^2}{\omega^2} d\xi \right) dt$$

$$\leqq \int_0^s \left\{ \frac{j^2 + 2j\check{c} \int_t^s \omega^2 d\xi + \left(\check{c} \int_t^s \omega^2 d\xi \right)^2}{\omega^2} \right\} \left(\int_0^t \frac{q^2}{\omega^2} d\xi \right) dt \,.$$

However $1/\omega^2 \int_t^s \omega^2 d\xi = v^{2-\varepsilon}/v' \int_t^s v'/v^{2-\varepsilon}\, d\xi = (1/1-\varepsilon)\{v/v' - c/\omega^2\}$ and if $v/v' \to 0$ and $\omega \to \infty$ it follows that the first two integrals in (3.12) exist. The last integral in (3.12) is bounded by

$$c \int_0^s \left[\frac{1}{\omega^2} \int_t^s \omega^2 d\xi \right] \left[\int_t^s \omega^2 d\xi \int_0^t \frac{d\eta}{\omega^2} \right] dt \,.$$

The first term in the integrand vanishes as $t \to 0$ by the above remarks and using 1' Hospital's rule on the second term we note that $\lim \int_t^s \omega^2 d\xi \int_0^t d\eta/\omega^2 = \lim \left(\int_t^s \omega^2 d\xi \right)^2 \Big/ \omega^4$ which is zero by the above (note here if $\omega \in L^2$ (3.12) is seen immediately to exist and no recourse to the preceding argument is intended). Thus if $v'/v \to \infty$ and $\omega \to \infty$ (c) surely holds.

Now since $J/\omega\psi = (\check{c}/1-\varepsilon)\, 1/\omega\psi v^{1-\varepsilon}$ it follows that (b) holds if $\omega^2 v^{2-2\varepsilon} > c/\psi^2$ or $(v'/v)\varepsilon > c/\psi^2$. It is not necessary that $\psi \uparrow \infty$ in general; when $v \to 0$ (b) will hold if $v' > c/\psi^2$. Thus (b) holds if $v \to 0$ and

$$(3.13) \qquad 1 - \left(\frac{\psi^2 q'}{q} \right) \int_0^t \frac{d\xi}{\psi^2} > \tilde{c} q$$

since $v' = \varphi'/q - \varphi q'/q^2$ and $\varphi = \hat{c} \int_0^t d\xi/\psi^2$. In particular (3.13) holds if for example $(\psi^2 q'/q) \int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$, since $q \to 0$ (see here also equation (2.20)). This proves

LEMMA 6. *Assume* (h) $(q'\psi^2/q) \int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$ *for t small. Then if* $J = (\check{c}/1-\varepsilon)\, 1/v^{1-\varepsilon}$ $(J' = -\check{c}\omega^2)$ *and* $v \to 0$ *it follows that* $v'/v \to \infty$ *and* (a)–(f) *hold.*

We recall that φ and v are defined independently of s (see (2.17)) and our constructions and proofs have shown that for t small enough the $(q/J)'\,\omega^2/q^2$ and $-J'/\omega^2$ terms will dominate in the first and second integrals respectively of (3.5). It remains to check only a few terms in order to see whether by suitable choice of s this domination prevails over $[0, s]$. Now by (3.11) J/J' is independent of s as is J/ω^2 (indeed a priori ω^2 and δ^2 depend only on v). Now since $-J' = \check{c}\omega^2 > 0$ we have J monotone decreasing and clearly

$$\frac{1}{J(t)}\int_t^s J(\xi)d\xi \leqq s - t \leqq b\ .$$

Hence referring to the proof of Lemma 5 we can establish domination over an interval $[0, s]$ in the second integral of (3.5). There remains the $(q/J)'$ term for which we may write

$$\frac{\left(\dfrac{q}{J}\right)'}{\left(\dfrac{q}{J}\right)} = \frac{q'}{q} + (1-\varepsilon)\frac{v'}{v} = \frac{\varphi'}{\varphi}\left\{1 - \varepsilon\left[1 - \frac{q'\varphi}{q\varphi'}\right]\right\}, \tag{3.14}$$

Thus in particular the ratio in (3.14) is a priori independent of s and the desired domination may be obtained on an interval $[0, s]$ by choosing s sufficiently small. Thus we have proved

LEMMA 7. *If the hypotheses of Lemma* 6 *hold and* (g) *is true it follows that for suitably small* s, $\int_0^s |\omega u|^2\,dt \leqq 0$.

Clearly the condition (h) in Lemma 6 is much stronger than is necessary but it gives a manageable criterion. We note now that if $q' \geqq 0$ then by (h) $\varepsilon_1 \leqq [1 - q'\varphi/q\varphi'] \leqq 1$ and from (3.14) it results that $(q/J)'/(q/J) \geqq (1-\varepsilon)\,\varphi'/\varphi \to \infty$. Thus if q is monotone, for any ε, $0 < \varepsilon < 1$, (g) is a consequence of (h). Another case of interest would be if $1 - q'\varphi/q\varphi' \leqq \tilde{Q}$; then if $\varepsilon \leqq 1/\tilde{Q}$ (g) holds. A somewhat better result may be obtained as follows. We note that

$$\frac{q'\varphi}{q\varphi'} = \frac{q'\psi^2}{q}\int_0^t \frac{d\xi}{\psi^2} = \frac{(\text{loq}\ q)'}{\left(\text{loq}\int_0^t \dfrac{d\xi}{\psi^2}\right)'}\ .$$

Then assume that $Q = \lim\,(q'\psi^2/q)\int_0^t d\xi/\psi^2$ exists as $t \to 0$. We note that the conditions needed to apply l'Hospital's rule hold and thus $Q = \lim \text{loq}\ q/\text{loq}\int_0^t d\xi/\psi^2$. Therefore for t small (h) implies that

$$\log q/\log\int_0^t \frac{d\xi}{\psi^2} \leqq 1 - \varepsilon_2\ , \quad 0 < \varepsilon_2 < \varepsilon_1\ .$$

But for t small the logarithms are negative and thus loq $q \geqq \text{loq}\left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ or $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2} = c\varphi^{1-\varepsilon_2}$. Conversely if $q \geqq c\varphi^{1-\varepsilon_2}$ and if $Q = \lim q'\varphi/q\varphi'$ exists then $Q \leqq 1 - \varepsilon_3$ for some ε_3, $0 < \varepsilon_3 < \varepsilon_2$.

Hence if Q exists as defined and $q \geqq c\varphi^{1-\varepsilon_2}$ then (h) holds and moreover $v = \varphi/q \leqq \varphi/c\varphi^{1-\varepsilon_2} = (1/c)\varphi^{\varepsilon_2} \to 0$. We note that by construction if Q exists then $Q = \lim \log q/\log \int_0^t d\xi/\psi^2 \geqq 0$; hence $\varepsilon[1 - q'\varphi/q\varphi'] < \varepsilon(1+\varepsilon_4)$ for t small enough and $\varepsilon_4 > 0$ given. Choose now ε_4 such that $\varepsilon(1+\varepsilon_4) < 1$ or $\varepsilon_4 < (1-\varepsilon)/\varepsilon$ then from (3.14) $(q/J)'/(q/J) \geqq c\varphi'/\varphi$ for t small. This proves

THEOREM 3. *Assume* $Q = \lim (q'\psi^2/q)\int_0^t d\xi/\psi^2$ *exists and that* $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$, $0 < \varepsilon_2 < 1$. *Then* (h) *holds,* $v \to 0$, *and* $(q/J)'/(q/J) \to \infty$ *for* $J = c/v^{1-\varepsilon}$ *as above. Hence for s small enough the solution of problem 1 is unique.*

Again using [17] we conclude

THEOREM 4. *Assume* $a(t, u, u) \geqq \alpha\|u\|^2$, $t \to a(t, u, v) \in C^1[0, b]$, $t \to B(t) \in \mathscr{E}^1(\mathscr{L}_s(H, H))$, $a(t, u, v) = \overline{a(t, v, u)}$, $q \in C^1(0, b]$, $q > 0$ *for* $t > 0$, $q \to 0$ *as* $t \to 0$, $\psi \in C^0(0, b]$, $\psi > 0$, $\psi \uparrow$ *as* $t \to 0$, $\psi f \in L^2(H)$, $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ $(0 < \varepsilon_2 < 1)$, *and* $Q = \lim (q'\psi^2/q)\int_0^t d\xi/\psi^2$ *exists. Then there exists a unique solution of problem 2 for spaces* $\mathscr{F}_b$, $\mathscr{H}_b$ *based on functions* $\omega \notin L^2$ $(\omega \in C^0(0, b])$.

We note now that if $Q \neq 0$ then $q' < 0$ for t small is not possible. Moreover if $\log q/\log \int_0^t d\xi/\psi^2 \geqq \varepsilon_4 > 0$ then $q \leqq \left(\int_0^t d\xi/\psi^2\right)^{\varepsilon 4}$ and we may assume $\varepsilon_4 < 1$ since if $q \leqq \gamma^{1+\eta}$, $\eta \geqq 0$, $\gamma \to 0$, then $q \leqq \gamma^{\varepsilon 4}$ for any $\varepsilon_4 < 1$ when t is small. In fact $\varepsilon_4 < 1$ is necessary if we are to have $q \geqq c\varphi^{1-\varepsilon_2}$ and thus the case $Q \neq 0$ with $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ amounts to an estimate of the form $\left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2} \leqq q \leqq \left(\int_0^t d\xi/\psi^2\right)^{\varepsilon 4}$, $0 < \varepsilon_2 < 1$, $\varepsilon_2 + \varepsilon_4 \leqq 1$. Finally we remark that under the hypotheses of Theorem 4 if $\lim q'\psi^2$ exists then by l'Hospital's rule $\lim q'\psi^2 = \lim q\Big/\int_0^t d\xi/\psi^2 = \lim \check{c}\, q/\varphi = \infty$. This implies that $\psi \uparrow \infty$ if q' is bounded but in a case such as $q = t^{1/2}$, $\psi \uparrow \infty$ is not required.

4. Let now $\hat{\mathscr{K}}_s$ be the completion of $\mathscr{K}_s$ for the norm $\|\ \|_{\mathscr{K}_s}$. Then we may pose problem 1 for $\hat{\mathscr{K}}_s$ instead of $\mathscr{F}_s$ (call this problem 1′) and repeating the procedures of §§ 2 and 3 there will exist a function $\hat{u} \in \hat{\mathscr{K}}_s$ solving problem 1′ if s is small enough. It may be easily seen that the elements adjoined to $\mathscr{K}_s$ by completion correspond to functions $\hat{k}$ such that $\delta\hat{k} \in L^2(V)$, $\hat{k}'/\varphi\psi \in L^2(H)$, and $\hat{k}(0) = 0$. Moreover the injection i: $\mathscr{K}_s \to \mathscr{F}_s$ may be extended by continuity to a continuous map $\hat{i}$: $\hat{\mathscr{K}}_s \to \mathscr{F}_s$.

LEMMA 8. $\hat{\mathscr{K}}_s \subset \mathscr{F}_s$ *algebraically and topologically.*

Proof. We need only show, after the above remarks, that $\hat{i}$ is an injection. Let $k_n \to \hat{k}$ in $\hat{\mathscr{K}}_s$, $k_n \in \mathscr{K}_s$, and assume that $i(k_n) = k_n \to 0 = \hat{i}(\hat{k})$. We want to show that $\hat{k} = 0$ in $\hat{\mathscr{K}}_s$. First $k_n = i(k_n) \to 0$ in $\mathscr{F}_s$ means in particular that $\omega k_n \to 0$ in $L^2(V)$. Hence (see [6], p. 133) there is a subsequence $\|\omega k_{n_p}\|^2 \to 0$ almost everywhere. Therefore $\|\delta k_{n_p}\|^2 \to 0$ almost everywhere and by the assumption $k_n \to \hat{k}$ in $\hat{\mathscr{K}}_s$ we know $\delta k_{n_p} \to \delta\hat{k}$ in $L^2(V)$. Theorefore we must have (see [6], p. 133 again) $\delta k_{n_p} \to 0$ in $L^2(V)$, and $\delta\hat{k} = 0$ in $L^2(V)$ (similarly $\hat{k}'/\varphi\psi = 0$ in $L^2(H)$); thus in particular $\hat{k} = 0$ which shows that $\hat{i}(\hat{k}) = 0$ implies $\hat{k} = 0$.

Let now $\hat{u} \in \hat{\mathscr{K}}_s$ be the solution of problem 1′ above. Then $\hat{u} \in \mathscr{F}_s$ by Lemma 8 and by the uniqueness Theorem 3 we must have $\hat{u} = u$ for s small where u is the solution of problem 1. Hence

THEOREM 5. *Let the hypotheses of Theorem 4 hold. Then there exists a unique solution u of problem 2 which belongs to $\hat{\mathscr{K}}_b$.*

Now consider the proof of the Lions projection theorem given say in [17] (see also [18]). We have $ReE_s(k, k) \geqq \Omega\|k\|^2_{\hat{\mathscr{K}}_s}$ for $k \in \mathscr{K}_s$ and wish to solve $E_s(u, k) = L_s(k)$ for $u \in \hat{\mathscr{K}}_s$ (the equation holding for all $k \in \mathscr{K}_s$). Then we write, following Lions, $L_s(k) = ((\chi, k))_{\hat{\mathscr{K}}_s}$, $\chi \in \hat{\mathscr{K}}_s$, and $E_s(u, k) = ((u, Lk))_{\hat{\mathscr{K}}_s}$, $Lk \in \hat{\mathscr{K}}_s$. Here $L\colon \mathscr{K}_s \to \hat{\mathscr{K}}_s$ is a densely defined linear operator in $\hat{\mathscr{K}}_s$. But $k \in \mathscr{K}_s$

$$\Omega\|k\|^2_{\hat{\mathscr{K}}_s} \leqq |((k, Lk))_{\hat{\mathscr{K}}_s}| \leqq \|k\|_{\hat{\mathscr{K}}_s}\|Lk\|_{\hat{\mathscr{K}}_s} \tag{4.1}$$

which implies L is one-to-one. Moreover if $R_0 = L(\mathscr{K}_s)$ then L^{-1} is a bounded operator on R_0 and may be extended by continuity to $\bar{R}_0$ defining $\hat{L}^{-1}\colon \bar{R}_0 \to \hat{\mathscr{K}}_s$. Let $P\colon \hat{\mathscr{K}}_s \to \bar{R}_0$ be the projection and set $R = \hat{L}^{-1}P$ which is thus everywhere defined and continuous on $\hat{\mathscr{K}}_s$. Then we want to find u such that $((u, Lk)) = ((\chi, L^{-1}Lk)) = ((\chi, RLk)) = ((R^*\chi, Lk))$ for all $k \in \mathscr{K}_s$. Thus a solution is $u = R^*\chi$ and by the subsequent uniqueness result $u = R^*\chi$ is the only solution. Using this sketch of the proof of the projection theorem we can bound u. Indeed $\|u\|_{\hat{\mathscr{K}}_s} \leqq \|R^*\chi\|_{\hat{\mathscr{K}}_s} \leqq c\|\chi\|_{\hat{\mathscr{K}}_s}$ since R^* is bounded. Moreover

$$\begin{aligned} |((\chi, k))| &= \left|\int_0^s\left(\psi f, \frac{h}{\psi}\right)dt\right| \leqq \left(\int_0^s|\psi f|^2\,dt\int_0^s\left|\frac{h}{\psi}\right|^2 dt\right)^{1/2} \\ &\leqq \left(\int_0^s|\psi f|^2\,dt\int_0^s|k'/\varphi\psi|^2\,dt\right)^{1/2} \leqq \left(\int_0^s|\psi f|^2\,dt\right)^{1/2}\|k\|_{\hat{\mathscr{K}}_s} = F\|k\|_{\hat{\mathscr{K}}_s}\,. \end{aligned} \tag{4.2}$$

This means (see [5], p. 111) since $\mathscr{K}_s$ is dense in $\hat{\mathscr{K}}_s$ that $\|\chi\| \leqq F = \left(\int_0^s|\psi f|^2\,dt\right)^{1/2}$. Therefore we have proved

THEOREM 6. *Under the hypotheses of Theorem 4 and for s suf-*

ficiently small the (unique) solution of problem 1 *satisfies the estimate* $\|u\|_{\hat{\mathscr{H}}_s} \leqq c\left(\int_0^s |\psi f|^2 \, dt\right)^{1/2}$.

The estimate can clearly be extended to $[0, b]$ which given

COROLLARY. *Under the hypotheses of Theorem* 6 *the unique solution of problem* 2 *satisfies the estimate* $\|u\|_{\hat{\mathscr{H}}_b} \leqq c\left(\int_0^b (\psi f|^2 \, dt\right)^{1/2}$.

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A THEOREM ON MATRICES OF 0'S AND 1'S

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In this note we define two types of matrices, called "special" and "quasi-special", which we first discuss in their own rights; it turns out that the quasi-special matrices have a canonical representation (under permutational similarity) in terms of special matrices. We show how this fact can, essentially, be expressed in the language of graph theory, and we also use it to give a new proof of a theorem of Goldberg [**1**] on matrices with real roots. We shall be concerned, specifically, with the following properties of an $n \times n$ matrix $A = (a_{ij})$:

DEFINITION 1. We call A *special* if $a_{ij} \neq 0$ implies $a_{ji} \neq 0$.

DEFINITION 2. Given any integer s with[1] $3 \leqq s \leqq n$, we call A *s-special* if, for every ordered set $(i) = (i_1, \cdots, i_s)$ of integers i_r in the range $1 \leqq i_r \leqq n$ $(r = 1, \cdots, s)$, the statement

$$\mathrm{N}_A(i): \quad a_{i_1 i_2} \neq 0, \ \cdots, \ a_{i_{s-1} i_s} \neq 0 \ , \quad a_{i_s i_1} \neq 0$$

implies

$$\mathrm{N}_{A'}(i): \quad a_{i_2 i_1} \neq 0, \ \cdots, \ a_{i_s i_{s-1}} \neq 0 \ , \quad a_{i_1 i_s} \neq 0 \ .$$

For example, every symmetric matrix is special (and the same is true of hermitian matrices over any ring with involution). Also, obviously, every special $n \times n$ matrix is s-special for each $s = 3, \cdots, n$, and it will be convenient to call any matrix with this latter property *quasi-special*. Thus every special matrix is quasi-special. The converse of this is easily seen to be false: e.g.

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

is 3-special (since $\mathrm{N}_A(i_1, i_2, i_3)$ is always false), hence quasi-special, but this A is evidently not special. Nevertheless, every quasi-special matrix does have certain special matrices associated with it. More precisely, our main result is

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[1] Clearly $s = 1, 2$ would lead to properties enjoyed by every matrix A, and so we need not consider these values of s.

THEOREM. (1) *Given any $n \times n$ matrix of the lower-triangular block form*

$$B = \begin{pmatrix} B_{11} & 0 & 0 & \cdots & 0 \\ B_{21} & B_{22} & 0 & & 0 \\ \vdots & & & \ddots & \vdots \\ B_{m1} & B_{m2} & B_{m3} & \cdots & B_{mm} \end{pmatrix}$$

where each block B_{kk} occuring on the diagonal of B is special (in particular, square), and given any $n \times n$ permutation matrix P, then the matrix $A = PBP^{-1}$ is always quasi-special.

(2) *Conversely, every quasi-special $n \times n$ matrix A can be expressed in the form $A = PBP^{-1}$, with B, P as in* (1).

For matrices over any integral domain, of course $\mathrm{N}_A(i)$ becomes simply $a_{i_1 i_2} \cdots a_{i_s i_1} \neq 0$. However, our Theorem is essentially combinatorial, in that its proof involves no genuinely algebraic operations on the elements of the matrices A, B, which may consequently be of quite arbitrary nature. All that we need is that there be given some classification of these elements into two disjoint subsets, say Z and N (standing for "zero" and "nonzero"), in which case we must replace each inequality $a_{i_r i_{r+1}} \neq 0$ occurring in $\mathrm{N}_A(i)$ by a corresponding statement $a_{i_r i_{r+1}} \in N$ (or, equivalently, by a relational statement $i_r R i_{r+1}$). Since our arguments will not require any further properties of Z or N we might, with no real loss of generality, equally well have stated the theorem for matrices whose elements are all 0 or 1 (hence our title). Nevertheless, for the sake of its application in a Corollary below, where the elements will be complex numbers, we have preferred to state the result in the apparently (but rather illusorily) more general from above.

Proof of (1). This is relatively trivial. Since the property of being quasi-special (or not) is clearly preserved under similarity transformation by any permutation matrix P, we need only prove that a matrix of the type B must itself be quasi-special. To this end, let $(i) = (i_1, \cdots, i_s)$ (where $3 \leqq s \leqq n$) be any sequence for which $\mathrm{N}_B(i)$ holds. We shall show first that this can happen only if each of the $b_{i_r i_{r+1}}$ (where we define $i_{s+1} = i_1$ conventionally) lies in some diagonal block B_{kk} (and indeed all in the same block, though this is not vital to our argument).

For, since $\mathrm{N}_B(i)$ requires all the $b_{i_r i_{r+1}}$ to be nonzero, each $b_{i_r i_{r+}}$

must lie in some block B_{uv} with $v \leqq u$ (where u, v depend on r). Among all those B_{uv} which contain a $b_{i_r i_{r+1}}$, choose one with minimum u; without loss of generality, we may suppose that the corresponding $r = 1$, i.e. that $b_{i_1 i_2} \in B_{uv}$ with u minimum and $v \leqq u$. Then, since all the B_{kk} are square, $b_{i_2 i_3} \in B_{vw}$ for some w, and, by the minimality of u, we must have $v = u$, i.e. $b_{i_1 i_2} \in B_{uu}$. Repeating the argument, we see that $b_{i_2 i_3} \in B_{uu}, \cdots, b_{i_{s-1} i_s} \in B_{uu}, b_{i_s i_1} \in B_{uu}$.

Thus all the $b_{i_r i_{r+1}}$ corresponding to any sequence (i) for which $N_B(i)$ holds must belong to the same diagonal block B_{uu}. Since each such B_{uu} is given to be special (even quasi-special would be enough for our present purpose) and since all the $b_{i_r i_{r+1}}$ are nonzero (by $N_B(i)$), it follows that all the $b_{i_{r+1} i_r}$ are nonzero too, i.e. $N_{B'}(i)$ holds. To summarize, $N_B(i)$ implies $N_B(i)$, so that B, and hence A, is indeed quasi-special, as required.

Proof of (2). If A is not itself special, i.e. if for some u, v we have $a_{uv} = 0$, $a_{vu} \neq 0$, then, since of course $u \neq v$, by applying a suitable permutational similarity (specifically, the one that interchanges the first row with the uth and the vth with the nth, and the columns similarly), we may take $u = 1$, $v = n$, i.e. we may suppose throughout that

$$(*) \qquad a_{1n} = 0 \ , \qquad a_{n1} \neq 0 \ .$$

We now apply a double induction, first on the order n of A and secondly on the row index i within A. Thus, supposing the theorem already proved for all square matrices of order $<n$, we let A be as stated, assume by way of contradiction that A can *not* be transformed to the form B by permutation, and take as our "inner" inductive hypothesis the proposition

H_i: *there exist an* $n \times n$ *permutation matrix* Q_i, *and integers* $k_1, \cdots, k_i$ *satisfying* $1 \leqq k_1 \leqq k_2 \leqq \cdots \leqq k_i < n$ *such that, for each* $h = 1, \cdots, i$, *we have*

$$c_{hj} \neq 0 \, (k_{h-1} < j \leqq k_h) \ , \qquad c_{hj} = 0 \quad (k_h < j \leqq n) \ ,$$

and also $c_{n1} \neq 0$, *where* $C = (c_{hj})$ *denotes the matrix* $Q_i^{-1} A Q_i$ *and we interpret* $k_0 = 1$.

We wish to prove first that H_i is true for each $i = 1, \cdots, n-1$, and our chief task in so doing will be to deduce H_i from H_{i-1}. Suppose then, for some i with $1 < i < n$, that H_{i-1} holds. Since the property of being quasi-special is unaffected under similarity transformation by a permutation matrix, and since any product of permutation matrices is itself a permutation matrix, we may assume with no loss of generality that Q_{i-1} is just the unit matrix (so that we may speak of A rather than C).

Given H_{i-1}, if $k_{i-1} \leqq i-1$, then A would have an $(i-1) \times (n-i+1)$ block of zeroes in its upper right hand corner. Also the leading $(i-1) \times (i-1)$ block of A and its complementary $(n-i+1) \times (n-i+1)$ submatrix are both quasi-special, of order at most $n-1$, and so, by our inductive hypothesis on n, we could find an $n \times n$ permutation matrix P (of the form $P = \mathrm{diag.}\,(P_1, P_2)$, where P_1, P_2 are permutation matrices of orders $i-1$, $n-i+1$ respectively) transforming A to the form B, which is contrary to assumption.

Thus the only possibility is that each $k_{i-1} \geqq i$. Let us now permute the columns of A to the right of the k_{i-1}th, but omitting the nth (i.e. $n - k_{i-1} - 1$ columns in all), among themselves so that, in the set of elements where these columns intersect the ith row, the nonzero elements (if any) are brought to the left, and the zeroes (if any) to the right (while, by the definition of $k_1, \cdots, k_{i-1}$, such a permutation of columns leaves the first $i-1$ rows unaffected); and define an integer k_i (clearly in the desired range $k_{i-1} \leqq k_i < n$) by writing the number of these nonzero elements as $k_i - k_{i-1}$.. Then, since $k_{i-1} \geqq i$, we may perform a corresponding permutation on the $(k_{i-1}+1)$th through $(n-1)$st rows without interfering with any of the first i rows (or the nth, so that a_{n1} is left nonzero), i.e., with this k_i, we have constructed a permutational similarity taking A into just the form prescribed in H_i, *provided only that* $a_{in} = 0$.

To prove that we do always in fact have $a_{in} = 0$, we proceed indirectly, and shall first consider the elements of the ith column which lie above the ith row. For $i > 1$, if $a_{pi} = 0$ for each $p = 1, \cdots, i-1$, then this would imply $i > k_{i-1}$, a contradiction. Hence there must be some integer i_1 in the range $i > i_1 \geqq 1$ such that $a_{i_1 i} \neq 0$. By repeating this argument, we can find a sequence of integers $i > i_1 > i_2 > \cdots > i_t > i_{t+1} = 1$ such that $a_{i_1 i} \neq 0, a_{i_2 i_1} \neq 0, \cdots, a_{i_t i_{t-1}} \neq 0$, $a_{1 i_t} \neq 0$ (where we interpret $t = 0$ if $i_1 = 1$, in which case we need only the fact that $a_{1i} \neq 0$). But then, if $a_{in} \neq 0$, we should have (since $a_{n1} \neq 0$ by H_{i-1}) a $(t+3)$-cycle of nonzero elements

$$a_{1i_t} \neq 0\,, \quad a_{i\ i\ _{-1}} \neq 0\,, \quad \cdots\,, \quad a_{i_2 i_1} \neq 0\,, \quad a_{i_1 i} \neq 0\,, \quad a_{in} \neq 0\,, \quad a_{n1} \neq 0\,,$$

whence, since A is quasi-special and $t+3 \leqq i+1 \leqq n$, it would follow that (in particular) $a_{1n} \neq 0$, contrary to H_{i-1} (at least for $n \geqq 3$); hence $a_{in} \neq 0$ cannot occur, i.e. H_i holds in its entirety.

Thus, to sum up, given the truth of (2) for all matrices A of order $<n$, where $n \geqq 3$, we have proved, for each i with $1 < i < n$, that H_{i-1} implies H_i. Since H_1 always holds (as is easily verified, given $(*)$), it follows that H_{n-1} holds. But, since $k_{n-1} < n$, this would imply that (after a suitable permutational transformation) the nth column of A (excepting perhaps the (n, n) element) consisted entirely of zeroes, so that, by our ("outer") inductive hypothesis on n, it would follow that A could, after all, be permuted into the form B, which contradiction completes our inner induction argument.

Thus for $n \geqq 3$, the required assertion (2) about any $n \times n$ quasi-special matrix A is implied by the corresponding assertion about all quasi-special matrices of order $<n$; the cases $n = 1, 2$ being trivial, (2) now follows at once by induction on n.

Though the proof we have given is in a sense quite direct, it is also possible to regard our Theorem as being just an algebraic formulation of a geometrically almost self-evident result in the theory of graphs; and, in the process, our apparently somewhat exotic Definitions 1, 2 above will now appear in a more natural light.

We suppose given a directed graph G, i.e. a set of *vertices* (denoted $p, q, p_1, p_2, \cdots$) and a binary relation R on this set (so that, for given vertices p, q, then pRq may or may not hold); we may think of the vertices of G as points in a plane, with a directed segment from p to q for each pair p, q satisfying pRq. By convention, pRp is always false[2]. By a *cycle* of G we shall mean any ordered subset $p_1, \cdots, p_s$ of its vertices such that $p_1Rp_2, \cdots, p_{s-1}Rp_s, p_sRp_1$; we call such a cycle *reversible* if $p_s, p_{s-1}, \cdots, p_1$ is also a cycle. If G has no cycles, we call G *acyclic*. If, for arbitrary $p, q \in G$, pRq implies qRp, then we call G *symmetric*. If, for arbitrary $p, q \in G$ with $p \neq q$, there is always a sequence $q_1, \cdots, q_s$ of vertices of G such that $q_1 = p, q_s = q$ and also, for each $i = 2, \cdots, s$, either $q_{i-1}\, Rq_i$ or q_iRq_{i-1}, then we call G *connected*.

The concept of a subgraph is clear, and we can also define quotient graphs by factoring G with respect to any prescribed identifications of its vertices. More precisely, given any equivalence relation S on the vertices of G, inducing equivalence classes denoted by $G_{h(h=1,2\ \cdots)}$, then we may regard the G_h as vertices of a new graph $\mathfrak{G}$ by defining $\mathfrak{R}$ on $\mathfrak{G}$ by the rule that $G_h\, \mathfrak{R}\, G_k$ (for $h \neq k$) if and only if there exist $p \in G_h$, $q \in G_k$ such that pRq. We call $\mathfrak{G}$ the *quotient graph of G by S*, and write $\mathfrak{G} = G/S$. We can now state

LEMMA. (1) *Given any directed graph G and a quotient graph G/S of it which is acyclic and of which every vertex is a symmetrical subgraph of G, then every cycle of G is reversible.*

[2] For definiteness, it is desirable to adopt either this convention or its opposite, and in the present connection this alternative seems the more convenient. However, there is no general agreement on the point: e.g. Harary uses the opposite convention in [**2**], but in effect also uses ours in [**3**].

(2) *Conversely, if every cycle of a directed graph G is reversible, then there is a factoring G/S such that G/S is acyclic and each vertex G_h of G/S is a symmetrical connected subgraph of G.*

Proof of (1). By the acyclic nature of G/S, any cycle in G can involve only vertices from a single equivalence class G_h under S; and, since each such G_h is given to be symmetric, any cycle in G_h is certainly reversible.

Proof of (2). We define a binary relation S on G by the rule that, for $p, q \in G$, we have pSq whenever either $p = q$ or there is a cycle of G containing both p and q. We see at once that S is an equivalence. It is also a trivial matter to check that the induced equivalence classes are connected and (by our hypothesis on G) symmetric with respect to the given relation R on G, and, finally, that G/S is acyclic.

So transparent a lemma as this deserves stating only for the sake of its applications, and presumably various forms of the same result have appeared in the literature; for example, a somewhat more general version is implicit in [3]. However, it seems desirable here to have an explicit account in a terminology adapted to our present concerns.

In both parts of the Lemma, clearly G is connected if and only if G/S and all its vertices G_h are. The two parts of the Lemma are in close analogy with those of our Theorem, and in fact we can set up a one-one correspondence between directed graphs of n vertices (numbered in some specified order) on the one hand, and $n \times n$ matrices of 0's and 1's with zero diagonal on the other (we shall suppose n finite, for conformity with our statement of the Theorem, but this is not really necessary). Specifically, given G, with vertices $p_1 \cdots$, p_n, we define $a_{ij} = 1$ if $p_i R p_j$, and $a_{ij} = 0$ otherwise; conversely, given any $n \times n$ matrix A of 0's and 1's with zero diagonal, we can reverse this to obtain a unique numbered graph G of order n. Thus we may write $A = M(G)$, $G = M^{-1}(A)$. We verify at once that A is special if and only if G is symmetric, that A is quasi-special if and only if the cycles of G are reversible, and that A is lower-triangular (i.e. $a_{ij} = 0$ whenever $i < j$) if and only if $p_i R p_j$ implies $i > j$ (in particular, this makes G acyclic). The restriction that A have zero diagonal is purely a technicality, since the diagonal elements have no effect on the properties of being special or quasi-special.

Also, given any equivalence S on G, there is a simple relationship between the matrix A corresponding to G and those corresponding to G/S and its vertices G_h. For the matrix $M(G_h)$, relative to the ordering induced on G_h by the prescribed numbering of G, is just the submatrix of A formed by the intersections of the rows and columns of A corresponding to those vertices of G which lie in G_h. Also, if the numbering of G is chosen so that all the vertices of G_1 come first (in some arbitrary order), then those of G_2, and so on, and if we partition A accordingly, then each G_h will have as its matrix the hth diagonal block of A; and G/S will have as its matrix (e_{hk}), where $e_{hk} = 1$ if $h \neq k$ and there exist $p \in G_h$, $q \in G_k$ with pRq, and where $e_{hk} = 0$ otherwise.

Finally, similarity transformation of A by a permutation matrix P corresponds to re-numbering the vertices of G according to the permutation defined by P, while G is connected if and only if there is no such P transforming A into a diagonal sum of smaller matrices.

Thus our correspondence $A = M(G) \leftrightarrow G = M^{-1}(A)$ embraces all the concepts involved in the Theorem and the Lemma, and it is a routine matter (the only point constituting a minor exception is that we need to show that any finite acyclic graph can be numbered in such a way that its matrix is triangular) to check that the various hypotheses and conclusions of the two parts of the Theorem translate, under this correspondence, into those of the Lemma. Thus (at the cost of introducing several additional concepts and definitions) our lemma and its proof provide an alternative and more intuitive proof for the Theorem. This second approach shows also that the set of diagonal blocks B_{kk} appearing in the Theorem is uniquely determined by A (up to permutational similarities applied to the B_{kk} themselves).

We conclude with our promised application: this could be established as a direct consequence of the Lemma, but seems more naturally obtained from the theorem. We first need some terminology analogous to that in Definition 2 above.

Definition 3. Given any complex $n \times n$ matrix A and an integer s with $3 \leqq s \leqq n$, we call A *s-hermitian* if, for every ordered index set (i) (as in Definition 2), we have

$$a_{i_1 i_2} \cdots a_{i_{s-1} i_s} a_{i_s i_1} = \overline{a_{i_2 i_1} \cdots a_{i_s i_{s-1}} a_{i_1 i_s}} \,.$$

If A is s-hermitian for each $s = 3, \cdots, n$, then we call A *quasi-hermitian*. Thus every quasi-hermitian matrix is quasi-special.

Corollary. *If, for a given $n \times n$ quasi-hermitian matrix A, we have*

(P): *all $a_{ij}a_{ji}$ are real and non-negative $(i, j = 1, \cdots, n)$, then A has all its eigenvalues real.*

This result is due to Goldberg [1], whose own proof was by explicitly exhibiting a certain hermitian matrix having the same principal minors (and hence the same characteristic function) as A.

Proof. By part (2) of our Theorem, A is permutationally similar to a matrix of the form B. Since (P) and the property of being quasi-hermitian are preserved under any permutational similarity, consequently B, and hence each of its diagonal blocks B_{kk}, again satisfies the hypotheses of the Corollary; thus, since the eigenvalues of the B_{kk} are collectively just those of A, it will be enough to prove that all of the eigenvalues of these B_{kk} are real. In other words, we need only prove the Corollary for the case of a *special* matrix; accordingly, we may suppose from the outset that A is itself special.

We now introduce an $n \times n$ matrix D coinciding with A except where A has zeroes, in which places we let D have 1's; i.e., more formally, let

$$d_{ij} = \begin{cases} a_{ij} & \text{when } a_{ij} \neq 0 , \\ 1 & \text{when } a_{ij} = 0 . \end{cases}$$

Since A is special and satisfies (P), we have $d_{ij}d_{ji}$ real and strictly positive ($i, j = 1, \cdots, n$). Define also, for all u, v with $1 \leqq u < v \leqq n$,

$$\begin{aligned} f_{uv} &= d_{u,u+1}d_{u+1,u+2} \cdots d_{v-1,v} , \\ g_{uv} &= d_{u+1,u}d_{u+2,u+1} \cdots d_{v,v-1} , \\ f_{uu} &= g_{uu} = 1 , \end{aligned}$$

and write

$$t_i = |g_{1i}|^2\overline{f_{in}g_{in}} \qquad (i = 1, \cdots, n)$$

Now, since the d_{ij} are all nonzero by definition, certainly each $g_{1i} \neq 0$, while also, for any u, v with $u < v$, we have

$$f_{uv}g_{uv} = (d_{u,u+1}d_{u+1,u}) \cdots (d_{v-1,v}d_{v,v-1}) ,$$

so that each $f_{uv}g_{uv}$ is real and strictly positive. In particular $f_{in}g_{in} > 0$ for all $i < n$, while this is trivially true for $i = n$. Thus all the t_i are strictly positive real numbers.

We wish to show next that $t_j a_{ij} = t_i\overline{a_{ji}}$ ($i, j = 1, \cdots, n$), to which end it will be convenient to write the t_i in the equivalent form $t_i = \overline{g_{1n}}g_{1i}\overline{f_{in}}$. There being no loss of generality (since the t_i are real) in supposing that $i < j$, it will suffice to prove that

$$g_{1j}\overline{f_{jn}}a_{ij} = g_{1i}\overline{f_{in}a_{ji}} \qquad (1 \leqq i < j \leqq n) .$$

But, under our assumption $i < j$, clearly $\overline{g_{1j} = g_{1i}g_{ij}, f_{in} = f_{ij}f_{jn}}$, and so we need only verify that $g_{ij}a_{ij} = \overline{f_{ij}a_{ji}}$, which, on being translated back in terms of the a_{ij}, is an immediate consequence of our assumption that A is special, quasi-hermitian and satisfies (P).

Thus we have produced positive real $t_1, \cdots, t_n$ such that $t_j a_{ij} = t_i \overline{a_{ji}}$ $(i, j = 1, \cdots, n)$, i.e. $AT^2 = T^2 A^*$, where $T = \text{diag.}\,(t_1^{1/2}, \cdots, t_n^{1/2})$ is hermitian and non-singular, and we use an asterick to denote the conjugate transposed. Thus $T^{-1}AT = (T^{-1}AT)^*$, so that A is similar to the hermitian matrix $T^{-1}AT$; in particular, the eigenvalues of A must be real, as required.

In conclusion, we note that, by considering matrices of the form

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix},$$

with characteristic function $x(x^2 - a_{12}a_{21} - a_{13}a_{31})$, it is clear that a special quasi-hermitian matrix A can have its eigenvalues all real even if (P) fails (in particular, A need not be hermitian).

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ON COMPLEX APPROXIMATION

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1. Let C denote the set of complex numbers and G the set of Gaussian integers. In this note we prove the following theorem which is a two-dimensional analogue of Theorem 2 in [3].

THEOREM 1. *If* $\beta, \gamma \in C$, *then there exists* $u \in G$ *such that* $|\beta - u| < 2$ *and*

$$|\beta - u||\gamma - u| < \begin{cases} 27/32 & \text{if } |\beta - r| < \sqrt{11/8} \\ \sqrt{2}\,|\beta - \gamma|/2 & \text{if } |\beta - \gamma| \geq \sqrt{11/8}\,. \end{cases}$$

As an illustration of the application of Theorem 1 to complex approximation, we use it to prove the following result.

THEOREM 2. *If* $\theta \in C$ *is irrational and* $a \in C$, $a \neq m\theta + n$ *where* $m, n \in G$, *then there exist infinitely many pairs of relatively prime integers* $x, y \in G$ *such that*

$$|x(x\theta - y - a)| < 1/2\,.$$

The method of proof of Theorem 2 is due to Niven [6]. Also in [7], Niven uses Theorem 1 to obtain a more general result concerning complex approximation by nonhomogeneous linear forms.

Alternatively, Theorem 2 may be obtained as a consequence of a theorem of Hlawka [5]. This was done by Eggan [2] using Chalk's statement [1] of Hlawka's Theorem.

2. Theorem 1 may be restated in an equivalent form. For $u, b, c \in C$, define

$$g(u, b, c) = |u - (b + c)||u - (b - c)|\,.$$

Then Theorem 1 may be stated as follows.

THEOREM 1′. *If* $b, c \in C$, *then there exist* $u_1, u_2 \in G$ *such that*
(i) $|u_1 - (b + c)| < 2$, $|u_2 - (b - c)| < 2$
and for $i = 1,2$,

(ii) $$g(u_i, b, c) < \begin{cases} 27/32 & \text{if } |c| < \sqrt{11/32} \\ \sqrt{2}\,|c| & \text{if } |c| \geqq \sqrt{11/32}\,. \end{cases}$$

It is clear that Theorem 1′ implies Theorem 1 by taking

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$$b = (\beta + \gamma)/2, \; c = (\beta - \gamma)/2 .$$

To see that Theorem 1 implies Theorem 1′, first apply Theorem 1 with $\beta = b + c$, $\gamma = b - c$ and then apply Theorem 1 with $\beta = b - c$, $\gamma = b + c$.

3. We precede the proof of Theorem 1′ with a few remarks concerning the nature of the proof.

Given $b, c \in C$, introduce a rectangular coordinate system for the complex plane such that b has coordinates $(0, 0)$ and $b + c$ has coordinates $(k, 0)$ where $k = |c|$. Then if $u \in C$ has coordinates (x, y)

$$\begin{aligned} g^2(u,b,c) &= |u - b - c|^2|u - b + c|^2 \\ &= ((x - k)^2 + y^2)((x + k)^2 + y^2) \\ &= (x^2 + y^2 + k^2)^2 - 4k^2x^2 . \end{aligned}$$

Now for k a positive real number let $R(k)$ be the set of all points (x, y) such that

$$(x^2 + y^2 + k^2)^2 - 4k^2x^2 < \begin{cases} (27/32)^2 & \text{if } k < \sqrt{11/32} \\ 2k^2 & \text{if } k \geqq \sqrt{11/32} . \end{cases}$$

Theorem 1′ depends upon showing that $R(k)$ under any rigid motion always contains two lattice points, not necessarily distinct. These lattice points correspond to the integers u_1 and u_2 of the theorem.

For $k > 1/\sqrt{2}$, $R(k)$ contains two circles with centers at

$$(\pm\sqrt{k^2 - 1/2}, \; 0)$$

and each of radius $1/\sqrt{2}$. Each of these circles contains a lattice point no matter how $R(k)$ is displaced in the plane. In this case, u_1 and u_2 correspond to these lattice points.

For $k < \sqrt{11/32}$, $R(k)$ contains the circle with center at $(0, 0)$ and radius $1/\sqrt{2}$. In this case, $u_1 = u_2$ corresponds to a lattice point in this circle. Finally if $\sqrt{11/32} \leqq k \leqq 1/\sqrt{2}$ $R(k)$ contains a region described by Sawyer [8] which always contains a lattice point no matter how it is displaced and $u_1 = u_2$ corresponds to a lattice point in this region.

4. We turn now to the proof of Theorem 1′. As above, for given $b, c \in C$, introduce a coordinate system so that b has coordinate $(0, 0)$ and $b + c$ has coordinates $(k, 0)$ where $k = |c|$. Then if $u \in C$ has coordinates (x, y),

$$g^2(u, b, c) = (x^2 + y^2 + k^2)^2 - 4k^2x^2 . \tag{1}$$

Suppose that $|c| = k > 1/\sqrt{2}$. For $i = 1, 2$ let

$$d_i = (\delta_i \sqrt{k^2 - 1/2}, 0)$$

where $\delta_i = (-1)^{i+1}$ and let $u_i \in G$ be a closest Gaussian integer to d_i (i.e. $|d_i - u_i| \leqq |d_i - t|$, $t \in G$). Then, omitting the subscripts,

$$|d - (b + \delta c)| = |\delta\sqrt{k^2 - 1/2} - \delta k| = k - \sqrt{k^2 - 1/2} < 1/\sqrt{2}.$$

Hence

$$|u - (b + \delta c)| \leqq |u - d| + |d - (b + \delta c)| < 2(1/\sqrt{2}) < 2$$

and condition (i) is satisfied.

Now let u_i have coordinates (x_i, y_i). Then, again omitting subscripts, since $|d - u| \leqq 1/\sqrt{2}$, we have

$$(x - \delta\sqrt{k^2 - 1/2})^2 + y^2 \leqq 1/2, \tag{2}$$

equality holding if and only if d is the center of a unit square with Gaussian integers as vertices. Also, since for any two real numbers a and b, $2ab \leqq a^2 + b^2$, equality holding if and only if $a = b$, we have

$$2\delta x\sqrt{k^2 - 1/2} \leqq x^2 + k^2 - 1/2, \tag{3}$$

equality holding if and only if $x = \sqrt{k^2 - 1/2}/\delta$. Thus

$$\begin{aligned}(1 + 2\delta x\sqrt{k^2 - 1/2})^2 &= 4\delta x\sqrt{k^2 - 1/2} + 4x^2(k^2 - 1/2) + 1 \\ &\leqq 2x^2 + 2k^2 - 1 + 4x^2(k^2 - 1/2) + 1 \\ &= k^2(2 + 4x^2)\end{aligned}$$

and since k and $k^2(2 + 4x^2)$ are positive,

$$1 + 2\delta x\sqrt{k^2 - 1/2} \leqq k\sqrt{2 + 4x^2}.$$

Hence

$$\begin{aligned}1/2 - (x - \delta\sqrt{k^2 - 1/2})^2 &= 1 + 2\delta x\sqrt{k^2 - 1/2} - x^2 - k^2 \\ &\leqq k\sqrt{2 + 4x^2} - x^2 - k^2.\end{aligned} \tag{4}$$

Using (4) and (2), we have

$$\begin{aligned}x^2 + k^2 + y^2 &\leqq k\sqrt{2 + 4x^2} + (x - \delta\sqrt{k^2 - 1/2})^2 - 1/2 + y^2 \\ &\leqq k\sqrt{2 + 4x^2},\end{aligned}$$

$$(x^2 + k^2 + y^2)^2 \leqq 2k^2 + 4k^2x^2. \tag{5}$$

Thus, from (1) and (5), $g^2(u, b, c) \leqq 2k^2$, the equality holding if and only if equality holds in both (2) and (3). If equality holds in (2), then there exist four possible choices for u, at least two of these choices having unequal first coordinates. Now equality holds in (3) if and only if, for fixed k, x is unique. Thus if equality holds in (2), u may be chosen so that equality does not hold in (3). For this choice of u, $g^2(u,b,c) < 2k^2$ which establishes condition (ii).

Next suppose $|c| = k < \sqrt{11/32}$. Now there exists $u \in G$ such that $|u - b| \leqq 1/\sqrt{2}$. Thus

$$|u - (b \pm c)| \leqq |u - b| + |c| < 2(1/\sqrt{2}) < 2 .$$

Also, if u has coordinates (x, y), $x^2 + y^2 \leqq 1/2$ and thus

$$\begin{aligned} g^2(u, b, c) &= (x^2 + y^2)^2 + 2k^2(y^2 - x^2) + k^4 \\ &< \frac{1}{4} + 2\left(\frac{11}{32}\right)\frac{1}{2} + \left(\frac{11}{32}\right)^2 = \left(\frac{27}{32}\right)^2 \end{aligned}$$

which establishes the theorem for $|c| < \sqrt{11/32}$.

Finally, for $\sqrt{11/32} \leqq |c| = k \leqq 1/\sqrt{2}$, we use a result due to Sawyer [8] which states that the region defined by $|x| \leqq 3/4 - y^2$, $|y| \leqq 1/2$ always contains a lattice point no matter how it is displaced in the plane. Thus there exists $u \in G$ with coordinates (x, y) such that $|x| \leqq 3/4 - y^2$, $|y| \leqq 1/2$.

If $|x| < 1/2$, then

$$|u - (b \pm c)| \leqq |u - b| + |c| = \sqrt{x^2 + y^2} + |c| \leqq \sqrt{2} .$$

Also since $|x^2 - k^2| \leqq 1/2$,

$$\begin{aligned} g^2(u, b, c) &= (x^2 - k^2)^2 + 2y^2(x^2 + k^2) + y^4 \\ &< \frac{1}{4} + 2\frac{1}{4}\left(\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{16} = \frac{11}{16} \leqq 2|c|^2 . \end{aligned}$$

If $1/2 \leqq |x| \leqq 3/4 - y^2$, then

$$x^2 + y^2 \leqq \frac{9}{16} - \frac{1}{2}y^2 + y^4 = \frac{1}{2} + \left(y^2 - \frac{1}{4}\right)^2 \leqq \frac{9}{16} .$$

Hence

$$|u - (b + c)| \leqq \sqrt{x^2 + y^2} + |c| \leqq \frac{3}{4} + \frac{1}{\sqrt{2}} < 2 .$$

Also $-x^2 \leqq -1/4$ so $y^2 - x^2 \leqq 0$. Thus

$$\begin{aligned} g^2(u, b, c) &= (x^2 + y^2)^2 + 2k^2(y^2 - x^2) + k^4 \\ &\leqq \left(\frac{9}{16}\right)^2 + 0 + \frac{1}{4} < \frac{11}{16} \leqq 2|c|^2 . \end{aligned}$$

This completes the proof of Theorem 1′.

5. To prove Theorem 2, we require a well-known result of Ford [4] which states that for any irrational $\theta \in C$, there exist infinitely many pairs of relatively prime $h, k \in G$ such that

$$|k(k\theta - h)| < 1/\sqrt{3}\,. \tag{6}$$

For θ and a is in the statement of Theorem 2, choose h, k satisfying (6) and let $t \in G$ be such that $|t - ka| \leqq 1/\sqrt{2}$. Since h and k are relatively prime, there exist $r, s \in G$ such that $rh - sk = t$ and hence

$$|rh - sk - ka| \leqq 1/\sqrt{2}\,. \tag{7}$$

Now, in Theorem 1, let

$$\beta = \frac{r\theta - s - a}{k\theta - h}\,, \qquad \gamma = \frac{r}{k}$$

and set

$$x = r - ku,\ y = s - hu$$

where u is the Gaussian integer whose existence is guaranteed by the theorem. Then $x, y \in G$ and

$$|x\theta - y - a|\,|x| = |\beta - u|\,|\gamma - u|\,|k|\,|k\theta - h|\,.$$

Hence if $|\beta - \gamma| < \sqrt{11/8}$ we have, using Theorem 1 and (6),

$$|x\theta - y - a|\,|x| < \frac{27}{32}|k(k\theta - h)| < \frac{27}{32} \qquad \frac{1}{\sqrt{3}} < \frac{1}{2}\,.$$

If $|\beta - \gamma| \geqq \sqrt{11/8}$, using Theorem 1 and (7), we have

$$\begin{aligned} |x\theta - y - a|\,|x| &< \frac{1}{2}\sqrt{2}\,|\gamma - \beta|\,|k(k\theta - h)| \\ &= \frac{1}{2}\sqrt{2}\left|\frac{hr - ks - ka}{k(k\theta - h)}\right||k(k\theta - h)| \leqq \frac{1}{2}\,. \end{aligned}$$

Thus for each pair h, k satisfying (6) we have a solution in G of

$$|x(x\theta - y - a)| < 1/2\,. \tag{8}$$

To show that there are infinitely many solutions to (8), we note that since $|\beta - u| < 2$ and $a \neq m\theta + n$, $m, n \in G$, we have with the use of (6).

$$0 < |x\theta - y - a| = |\beta - u|\,|k\theta - h| < 2/(\sqrt{3}\,|k|)\,. \tag{9}$$

If there are only a finite number of solutions of (8), let M be the minimum of $|x\theta - y - a|$ for these solutions. Then from (9), for every h, k satisfying (6) we have $|k| < 2/(\sqrt{3}M)$ and

$$|h| \leqq |h - k\theta| + |k\theta| < 1/(\sqrt{3}\,|k|) + |k||\theta| < N,$$

say. But this is impossible since there are infinitely many pairs $h, k \in G$ which satisfy (6).

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WEAK CONTAINMENT AND KRONECKER PRODUCTS OF GROUP REPRESENTATIONS

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Introduction. Throughout this paper G is a fixed locally compact group. Let us recall some concepts bearing on the representation theory of G. The family of all unitary equivalence classes of unitary representations of G will be called $\mathscr{T}(G)$. A function φ of positive type on G is *associated* with a subset $\mathscr{S}$ of $\mathscr{T}(G)$ if there is an S in $\mathscr{S}$, and a vector ξ in the space $H(S)$ of S, such that $\varphi(x) = (S_x\xi, \xi)$ for all x in G. An element T of $\mathscr{T}(G)$ is *weakly contained* in a subset $\mathscr{S}$ of $\mathscr{T}(G)$ if every function of positive type on G associated with T can be approximated uniformly on compact sets by sums of functions of positive type associated with $\mathscr{S}$. The notion of weak containment leads to that of the *inner hull-kernel topology* of $\mathscr{T}(G)$: A net $\{T^i\}$ of elements of $\mathscr{T}(G)$ converges to T in this topology if and only if every subnet of $\{T^i\}$ weakly contains T. Relativized to the subset $\hat{G}$ of $\mathscr{T}(G)$ consisting of the irreducible representations of G, this topology becomes the ordinary *hull-kernel topology* of $\hat{G}$. (For these notions and facts see [1] and [2]).

If H is a Hilbert space, the *adjoint space* $\bar{H}$ of H can be defined as the Hilbert space whose underlying set is the same as that of H, and which is conjugate-isomorphic with H under the identity map. If T is a unitary representation of G, the *adjoint representation* $\bar{T}$ is defined by the requirements: $H(\bar{T}) = H(T)^-$, $\bar{T}_x = T_x (x \in G)$. The *Kronecker product* $S \otimes T$ of two unitary representations S and T of G is that representation whose space is $H(S) \otimes H(T)$, and for which $(S \otimes T)_x(\xi \otimes \eta) = (S_x\xi) \otimes (T_x\eta)$. We can also describe the Kronecker product $S \otimes \bar{T}$ as follows: $H(S \otimes \bar{T})$ is the Hilbert space of all Hilbert-Schmidt operators on $H(T)$ to $H(S)$, and $(S \otimes \bar{T})_x(A) = S_x A T_x^{-1}$.

If $\mathscr{S} \subset \mathscr{T}(G)$ and $\mathscr{T} \subset \mathscr{T}(G)$, let $\mathscr{S} \otimes \mathscr{T}$ denote $\{S \otimes T \mid S \in \mathscr{S}, T \in \mathscr{T}\}$.

Throughout this paper I will be the one-dimensional identity representation of G. It is well known and easily verified that if S and T are finite-dimensional unitary representations of G and T is irreducible, $S \otimes \bar{T}$ contains I if and only if S contains T. Can this be generalized to the case where S and T are infinite-dimensional and 'containment' is replaced by 'weak containment'? The main object of this note is to answer this question affirmatively for the case that S is infinite-dimensional but T is still finite-dimensional (Theorem 4). In preparation for this we shall show (Theorem 2) that the Kronecker product oper-

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ation is continuous with respect to the inner hull-kernel topology of $\mathscr{T}(G)$.

Another by-product of the main result is the following strenthening (Theorem 3) of a remark of Godement ([4], p. 77): If the regular representation R of G weakly contains some finite-dimensional irreducible unitary representation of G, then R weakly contains all unitary representations of G.

1. The continuity of the Kronecker product.

LEMMA 1. *Suppose that $\mathscr{S} \subset \mathscr{T}(G)$ and $T \in \mathscr{T}(G)$; and let K be the set of all those ξ in $H(T)$ such that the function φ defined by $\varphi(x) = (T_x\xi, \xi)(x \in G)$ can be approximated, uniformly on compact sets, by sums of functions of positive type associated with $\mathscr{S}$. Then K is a closed T-invariant linear subspace of $H(T)$.*

Proof. Obviously K is closed in the norm and under scalar multiplication. By the easy argument of [1], p. 368, (ii′), $\sum_{i=1}^{n} a_i T_{x_i}\xi$ is in K whenever $\xi \in K$, the x_i are in G, and the a_i are complex; in particular K is T-invariant. It remains only to show K closed under addition.

Let ξ and η be elements of K; let L and M be the closed invariant subspaces of $H(T)$ generated by ξ and η respectively; and let Q be the closure of $L + M$. By the preceding paragraph

$$L \subset K \quad \text{and} \quad M \subset K . \tag{1}$$

If A is projection onto $L^\perp$, $A(M)$ is a dense subspace of $Q \cap L^\perp$. So by Mackey's form of Schur's Lemma ([7], Theorem 1.2), the restriction of T to the invariant subspace $Q \cap L^\perp$ is equivalent to a subrepresentation of the restriction of T to M. This and (1) show that

$$Q \cap L^\perp \subset K . \tag{2}$$

Putting $\zeta = \xi + \eta$, we have $\zeta = \xi' + \eta'$, where $\xi' \in L$ and $\eta' \in Q \cap L^\perp$. Since L and $Q \cap L^\perp$ are orthogonal and T-invariant,

$$(T_x\zeta, \zeta) = (T_x\xi', \xi') + (T_x\eta', \eta') \tag{3}$$

$(x \in G)$. By (1) and (2) ξ' and η' are in K; so by (3) $\zeta \in K$, and K is closed under addition.

REMARK 1. If A is a C^*-algebra, $\mathscr{T}(A)$ is defined as the set of all equivalence classes of *-representations of A. Exactly the same proof shows that Lemma 1 is valid for C^*-algebras, provided that we replace functions of positive type by positive functionals, and uniform approximation on compact sets by weak* approximation.

REMARK 2. According to Lemma 1, T will be weakly contained in $\mathscr{S}$ provided $H(T)$ is generated (under T) by those ξ in $H(T)$ whose associated functions of positive type are approximated by sums of functions of positive type associated with $\mathscr{S}$. For example, we have immediately:

THEOREM 1. *Suppose that* $\mathscr{S}_k \subset \mathscr{T}(G)$ *and* $\mathscr{S}_k$ *weakly contains* $T_k\,(k = 1, 2)$. *Then* $\mathscr{S}_1 \otimes \mathscr{S}_2$ *weakly contains* $T_1 \otimes T_2$.

THEOREM 2. *The map* $\langle S, T\rangle \rightarrow S \otimes T$ *(of* $\mathscr{T}(G) \times \mathscr{T}(G)$ *into* $\mathscr{T}(G)$*) is continuous with respect to the inner hull-kernel topology of* $\mathscr{T}(G)$.

Proof. Let $S^i \rightarrow S$ and $T^i \rightarrow T$ in $\mathscr{T}(G)$. By the definition of the topology of $\mathscr{T}(G)$, we have only to show that the net $\{S^i \otimes T^i\}$ (and hence by the same argument every subnet of it) weakly contains $S \otimes T$. But Theorem 2.2 of [2] clearly shows that the function of positive type associated with each product vector $\xi \otimes \eta$ in $H(S) \otimes H(T)$ can be approximated by functions of positive type associated with the $S^i \otimes T^i$. Hence by Lemma 1 $S \otimes T$ is weakly contained in $\{S^i \otimes T^i\}$.

It should be mentioned that the "easy verification" of the proposition used in the proof of [2], p. 260, Corollary 1, actually requires the above Theorem 1.

2. When does $S \otimes \bar{T}$ weakly contain I? In this section G is assumed to satisfy the second axiom of countability; and we shall consider only unitary representations acting in a separable space.

Suppose that $T \in \hat{G}$ and $S \in \mathscr{T}(G)$. Is it true that $S \otimes \bar{T}$ weakly contains I if and only if S weakly contains T? In general, as we next show, the implication is false in both directions, even if S is assumed irreducible.

Let R be the regular representation of G, and T some irreducible representation weakly contained in R. Clearly $R \cong \bar{R}$. By [6], Theorem 12.2, $R \otimes R$ is a multiple of R. So $R \otimes \bar{R}$ weakly contains I if and only if R does. Choose G so that R does not weakly contain I; for example G might be the free group on two generators, or a non-compact connected semisimple Lie group (see [8]). Then $R \otimes \bar{R}$ does not weakly contain I, and hence, by Theorem 1, nor does $T \otimes \bar{T}$.

For an easy counter-example in the other direction take G to be the "$ax + b$" group, and T to be one of the two infinite-dimensional irreducible representations of G. Then $\bar{T} = I \otimes \bar{T}$ weakly contains I (see [2], Theorem 5.1), but I does not weakly contain T. A "better" example, in which $S \otimes \bar{T}$ weakly contains I but neither S nor T weakly contains the other, will be given in § 3.

However, if T is finite-dimensional, the answer to the question posed above is affirmative (Theorem 4).

LEMMA 2. *If $\mathscr{S} \subset \mathscr{T}(G)$ and $\mathscr{S}$ weakly contains a finite-dimensional irreducible unitary representation T of G, then $\mathscr{S} \otimes \bar{T}$ weakly contains I.*

Proof. $\mathscr{S} \otimes \bar{T}$ weakly contains $T \otimes \bar{T}$ by Theorem 1. Since T is finite-dimensional, $T \otimes \bar{T}$ contains I.

Here is an interesting consequence of Lemma 2:

THEOREM 3. *If the regular representation R of G weakly contains some finite-dimensional irreducible representation T of G, it weakly contains all unitary representations of G.*

Proof. By Lemma 2 $R \otimes \bar{T}$ weakly contains I. But by [2], Lemma 4.2, $R \otimes \bar{T}$ is a multiple of R. Hence R weakly contains I, and the conclusion follows from Godement's remark ([4], p. 77, or [2], p. 260).

LEMMA 3. *Let T be an irreducible finite-dimensional unitary representation of G. To each $\delta > 0$, there is a finite subset F of G and an $\varepsilon > 0$ such that, whenever A is a positive linear operator on $H(T)$ satisfying* (i) $\|A\| = 1$ *and* (ii) $\|AT_x - T_xA\| < \varepsilon$ *for all x in F, then $\|A - E\| < \delta$ (E being the identity operator on $H(T)$).*

Proof. Assume the lemma false. Then there is a $\delta > 0$ and a net $\{A_i\}$ of positive operators in Q such that $A_iT_x - T_xA_i \xrightarrow[i]{} 0$ for all x in G; here Q is the compact set of those positive operators A on $H(T)$ for which $\|A\| = 1$ and $\|A - E\| \geqq \delta$. Replacing $\{A_i\}$ by a subnet, we may assume that $A_i \rightarrow A$ in Q. Passing to the limit, we deduce that $AT_x = T_xA$ for all x, whence $A = \lambda E$. Since A is positive and of norm 1, we must have $\lambda = 1$; but this contradicts $\|A - E\| \geqq \delta$.

LEMMA 4. *Suppose that $\mathscr{S} \subset \mathscr{T}(G)$, and T is a finite-dimensional irreducible unitary representation of G such that $\mathscr{S} \otimes \bar{T}$ weakly contains I. Then $\mathscr{S}$ weakly contains T.*

Proof. The family of all finite direct sums of elements of $\mathscr{S}$ weakly contains T if and only if $\mathscr{S}$ does; hence we may assume without loss of generality that $\mathscr{S}$ is closed under finite direct sums. But then I belongs to the quotient closure of $\mathscr{S} \otimes \bar{T}$ ([2], Theorem 1.1).

Let C be a compact subset of G. For fixed $\delta > 0$, choose F and ε as in Lemma 3. Let r be the dimension of $H(T)$; and put $C' = (C \cup F) \cup (C \cup F)^{-1}$.

By [2], Lemma 1.1, there is an S in $\mathscr{S}$ and a unit vector ζ in $H(S \otimes \bar{T})$ such that

$$\|(S \otimes \bar{T})_x \zeta - \zeta\| < \frac{\varepsilon}{2r^4} \tag{4}$$

for all x in C'. Fixing an orthonormal basis $\xi_1, \cdots, \xi_r$ of $H(T)$, let us write $\zeta = \sum_{i=1}^{r} \eta_i \otimes \xi_i (\eta_i \in H(S))$, where

$$1 = \|\zeta\|^2 = \sum_{i=1}^{r} \|\eta_i\|^2 . \tag{5}$$

If the matrix of T_x in the basis $\{\xi_i\}$ is $\{\tau_{ij}(x)\}$, we have $\bar{T}_x \xi_i = \sum_{j=1}^{r} \overline{\tau_{ji}(x)} \xi_j$. So $(S \otimes \bar{T})_x \zeta = \sum_j (\sum_i \overline{\tau_{ji}(x)} S_x \eta_i) \otimes \xi_j$, whence

$$\|(S \otimes \bar{T})_x \zeta - \zeta\|^2 = \sum_j \left\| \left(\sum_i \overline{\tau_{ji}(x)}\, S_x \eta_i \right) - \eta_j \right\|^2 . \tag{6}$$

By (4) and (6),

$$\left\| \left(\sum_i \overline{\tau_{ji}(x)}\, S_x \eta_i \right) - \eta_j \right\| < \frac{\varepsilon}{2r^4} \tag{7}$$

$(x \in C', j = 1, \cdots, r)$. From (7) and the unitariness of $\tau(x)$,

$$\begin{aligned} &\left\| S_x \eta_k - \sum_j \tau_{jk}(x) \eta_j \right\| \\ &\leqq \sum_j |\tau_{jk}(x)| \left\| \left(\sum_i \overline{\tau_{ji}(x)}\, S_x \eta_i \right) - \eta_j \right\| \\ &< \frac{\varepsilon}{2r^3} . \end{aligned} \tag{8}$$

Let A be the linear map of $H(T)$ into $H(S)$ sending ξ_i into $\eta_i (i = 1, \cdots, r)$. Then (8) gives

$$\|S_x A - A T_x\| < \frac{\varepsilon}{2r^2} \qquad (x \in C') . \tag{9}$$

From this and the symmetry of C',

$$\|A^* S_x - T_x A^*\| < \frac{\varepsilon}{2r^2} \qquad (x \in C') . \tag{10}$$

By (5), $\|A\| = \|A^*\| \leqq r$ and also

$$\|A^* A\| \geqq \frac{1}{r} . \tag{11}$$

Hence, denoting $A^*A/\|A^*A\|$ by B, we obtain from (9) and (10) $\|BT_x - T_x B\| < \varepsilon$ $(x \in C')$. Since B is positive, $\|B\| = 1$, and $F \subset C'$, Lemma 3 asserts that $\|B - E\| < \delta$. From this, setting $\eta'_i = \eta_i/\|A\|$, we get

$$(12) \qquad |(\eta'_i, \eta'_j) - \delta_{ij}| < \delta$$

for all i, j. Let $\varphi(x) = (S_x\eta'_1, \eta'_1)(x \in G)$. By (8) and (11) $\|S_x\eta'_1 - \sum_j \tau_{j1}(x)\eta'_j\| < \varepsilon/2r^2$. Combining this with (12) we have for x in C

$$\begin{aligned} |\varphi(x) - \tau_{11}(x)| &\leqq \left|\left(\left(S_x\eta'_1 - \sum_j \tau_{j1}(x)\eta'_j\right), \eta'_1\right)\right| \\ &\quad + \left|\left(\sum_j \tau_{j1}(x)\eta'_j, \eta'_1\right) - \tau_{11}(x)\right| \\ &\leqq \sum_j |\tau_{j1}(x)|\,|(\eta'_j, \eta'_1) - \delta_{j1}| + \frac{\varepsilon}{2r^2}\|\eta'_1\| \\ &\leqq r\delta + \frac{\varepsilon}{2r}, \end{aligned}$$

which is as small as we wish. Thus we have an S in $\mathscr{S}$ and a function φ of positive type associated with S which differs from τ_{11} on C by an arbitrarily small quantity. So $\mathscr{S}$ weakly contains T.

Combining Lemmas 2 and 4 we get:

THEOREM 4. *Let $\mathscr{S}$ be a family of unitary representations of G and T a finite-dimensional irreducible unitary representation of G. Then $\mathscr{S}$ weakly contains T if and only if $\mathscr{S} \otimes \bar{T}$ weakly contains I.*

As a corollary we mention the following weak "Frobenius-like" proposition. As usual, U^S denotes the representation of G induced from the representation S of a subgroup.

COROLLARY. *Let K be a closed subgroup of G, and J and I the identity representations of K and G respectively. We assume that U^J weakly contains I. If $\mathscr{S} \subset \mathscr{T}(K)$, T is a finite-dimensional irreducible unitary representation of G, and $\mathscr{S}$ weakly contains some irreducible component of $T|K$, then $\{U^S \mid S \in \mathscr{S}\}$ weakly contains T.*

Proof. By Theorem 4 $\mathscr{S} \otimes \bar{T}|K$ weakly contains J. Hence by [2], Theorem 4.2, $\{U^{S \otimes \bar{T}|K} \mid S \in \mathscr{S}\}$ weakly contains U^J. By hypothesis the latter weakly contains I; so $\{U^{S \otimes \bar{T}\,K} \mid S \in \mathscr{S}\}$ weakly contains I. But by [2], Lemma 4.2, $U^{S \otimes \bar{T}.K} \cong U^S \otimes \bar{T}$. Hence another application of Theorem 4 gives the required conclusion.

3. A counter-example. Let G be the proper Euclidean group in three-dimensional real space R^3. We observe that the hull-kernel topology of $\hat{G}$ is T_1 (i.e. points are closed). Indeed, the results of [5] show that T_f is completely continuous whenever $T \in \hat{G}$ and $f \in L_1(G)$. So, by [1], Lemma 1.11, $\hat{G}$ is T_1. Thus, if S and T are inequivalent elements of $\hat{G}$, neither weakly contains the other. We shall now construct two inequivalent elements S and T of $\hat{G}$ such that $S \otimes \bar{T}$ weakly contains I (see the beginning of § 2).

Let N and K be the translation and rotation subgroups of G respectively; τ_u will denote translation by u: $\tau_u(v) = u + v (u, v \in R^3)$. Let χ be the fixed character of N defined by $\chi(\tau_u) = e^{iu_1}$. The "stationary subgroup" for χ (consisting of those σ in G such that $\chi(\sigma\tau_u\sigma^{-1}) = \chi(\tau_u)$ for all u) is $Z = HN$, where $H = \{\rho \in K \mid \rho(1, 0, 0) = (1, 0, 0)\}$. Thus, by [6], Theorem 14.1, to each character φ of the Abelian group H we get an irreducible representation T^φ of G, namely, that induced from the character ψ of Z, where

$$\psi(\rho\tau_u) = \varphi(\rho)\chi(\tau_u) \qquad (\rho \in H, u \in R^3) . \tag{13}$$

Further, if φ and φ' are distinct characters of H, T^φ and $T^{\varphi'}$ are inequivalent.

Now let φ and φ' be distinct characters of H. Let $0 < \theta < \pi/2$ and let ρ be the element of K consisting of rotation through an angle θ about the third axis. We verify easily that $Z \cap \rho Z \rho^{-1} = N$. Hence by [3], Theorem 5.4 (the 'weak containment' version of Mackey's Kronecker Product Theorem), $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains the representation of G induced from the character χ_θ of N given by $\chi_\theta(\tau_u) = \chi(\tau_{\rho(u)})\chi(\tau_u)$. (Here $(T^{\varphi'})^-$ is the adjoint of $T^{\varphi'}$). Since this is true whenever $0 < \theta < \pi/2$, we can use [2], Theorem 4.2, to pass to the limit as $\theta \to 0$; we then conclude that $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains U^{χ_0}, where χ_0 is the identity character of N. But U^{χ_0} is obtained by lifting to G the regular representation of the compact group K; hence it contains I as a direct summand. Thus we conclude that $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains I. This is the desired example, since we have already observed that T^φ and $T^{\varphi'}$ are inequivalent irreducible representations of G.

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SCHAUDER ESTIMATES UNDER INCOMPLETE HÖLDER CONTINUITY ASSUMPTIONS

PAUL FIFE

Dedicated to Charles Loewner on the occasion of his 70th birthday

1. Introduction. In 1934 Schauder [6], [7] obtained a priori pointwise estimates for solutions to general second order linear elliptic differential equations. These estimates have been generalized and simplified by many authors, but by far the most general estimates of this type so far are the interior estimates of Douglis and Nirenberg [3] and the estimates up to the boundary of Agmon, Douglis, and Nirenberg [2]. In the latter paper the boundary-value problem

$$\begin{aligned} &L(x, D)u = f \quad \text{in a domain } \mathscr{D}\,, \\ &B_j(x, D)u = \varphi_j \quad \text{on a portion of the boundary} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \dot{\mathscr{D}}\ (j = 1, 2, \cdots, m)\,, \end{aligned}$$

is considered, where L is uniformly elliptic of order $2m$ and the B_j satisfy the "complementing condition" with respect to L. Roughly speaking, under certain smoothness assumptions on the coefficients of L and B_j, on $\dot{\mathscr{D}}$, and on the functions u, f, φ_j, a priori bounds on certain derivatives of u and their Hölder difference quotients are obtained in terms of the maximum values in $\mathscr{D}$ (or $\dot{\mathscr{D}}$) of certain derivatives of f and φ_j and their Hölder difference quotients. As a byproduct at one stage near the beginning, an estimate is obtained (their Theorem 2.2) for the case of constant coefficients and a half-space domain, in which no Hölder difference quotients occur. This estimate leads to a maximum principle. The history of this latter kind of estimate is also extensive, but maximum principles of greatest generality seem to have been obtained by Agmon [1].

The present paper explores the possibility of obtaining a priori pointwise estimates involving Hölder difference quotients not with respect to all, but only with respect to some of the independent variables x_i. With a few exceptions, the argument follows in basic outline the argument in [2]. Also the notation of [2] is preserved where possible. Throughout the paper $n + 1$ denotes the number of independent variables, and q of them ($0 \leqq q \leqq n + 1$) are distinguished from the others in that relevant functions are considered to be Hölder

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continuous only in the distinguished variables.

The first step is the derivation of certain potential theoretic results in § 2. Results of this nature go back to Hölder, Petrini, Korn, and Lichtenstein (see the survey in [5]). These are applied in § 3 to functions given by convolutions with a fundamental solution to an elliptic operator as kernel, and in § 4 to solutions of the basic boundary value problem with compact support when the operators have constant coefficients and $\mathscr{D}$ is a half-space. These results are in the form of sufficient conditions on the operator $P(D)$ in order that $P(D)u$ may be estimated in terms of certain derivatives and "distinguished" Hölder difference quotients of Lu and B_ju. Also a necessary condition on $P(D)$ for such estimates to hold is given. Let $\hat{L}$ and $\hat{B}_j$ denote the operators obtained from L and B_j respectively by deleting all differentiations with respect to distinguished variables, and $\hat{u}$ a solution to the basic boundary-value problem with L and B_j replaced by $\hat{L}$ and $\hat{B}_j$. As a corollary it is found (in the constant coefficient, half-space case) that u and $\hat{u}$ differ by a function whose appropriate derivatives have estimable Hölder difference quotients in all variables.

In §§ 5 and 6 the results are extended to a class of problems with variable coefficients and domains with curved boundaries by the method [2, 3]. The distinguished variables are now certain local curvilinear coordinates. When $q < n$ this method appears to be inapplicable to the general class treated in [2, § 7]; in addition to the assumptions made there, we must impose the requirement that coordinate transformations exist which map small neighborhoods adjoining $\dot{\mathscr{D}}$ into hemispheres and which transform L and B_j into operators L' and B_j' such that, on the flat boundary of the hemisphere, $\hat{L}'(x, D) = \lambda(x)\hat{L}_0(D)$ and $\hat{B}_j'(x, D) = \beta_j(x)\hat{B}_{j_0}(D)$ (the notation $\hat{L}'$, $\hat{B}_j'$ is explained above). In § 6 the case $q = n$ is given special attention. It is shown that essentially every result in the area of the usual Schauder estimates ($q = n + 1$); i.e., every result in §§ 1–7 of [2], has its analog with $q = n$. In particular, existence and uniqueness occurs in the classes of functions corresponding to $q = n$ exactly when it occurs in the classes corresponding to $q = n + 1$. In §§ 5 and 6 the coefficients in the operators L and B_j are assumed to be completely Hölder continuous.

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2. Potential theory. Let x be a point in n-space. We shall distinguish its first q $(0 \leqq q \leqq n)$ from its last $n - q$ coordinates and write $x = (\tilde{x}, \hat{x})$, where $\tilde{x} = (x_1, \cdots, x_q)$ and $\hat{x} = (x_{q+1}, \cdots, x_n)$. If $q = n$ we write $x = \tilde{x}$, and if $q = 0, x = \hat{x}$. The concern in this section will be with functions $u(x, t)$ defined in the $(n+1)$-dimensional half-space $t > 0$ by a singular integral

$$u(x, t) = \int K(x - y; t) g(y) dy \ . \tag{2.1}$$

In certain cases u may be extended to be a continuous function in the closed half-space $t \geqq 0$; then we shall use the notation $u(x, 0)$ without further explanation. Our object is to exhibit conditions on the kernel K under which certain boundedness and/or continuity properties of u will be implied by similar properties of g.

Explicitly, we assume $K(x; t)$ to be continuous except for $x = t = 0$, and that there is a constant C_1 such that

$$D^{\mu} K(x; t) > C_1(|x|^2 + t^2)^{-(1/2)(n+\mu)} \qquad (\mu = 0, 1) \tag{2.2}$$

where here and below D^{μ} denotes any μth order derivative. We also assume that

$$\left| \int_{\tilde{y}-\text{space}} K(y; t) d\tilde{y} \right| \leqq C_2 t(|\hat{y}|^2 + t^2)^{-(1/2)(n-q+1)} \tag{2.3a}$$

if $1 \leqq q \leqq n - 1$,

$$|K(\hat{x}; t)| \leqq C_2 t(|\hat{x}|^2 + t^2)^{-(1/2)(n+1)} \tag{2.3b}$$

if $q = 0$, and

$$\left| \int_{|y|>\delta} K(y; t) dy \right| \leqq C_2 \qquad \text{for all } \delta > 0 \tag{2.3c}$$

if $q = n$. In certain important cases the integral in (2.3a) will vanish; then we shall simply say that $C_2 = 0$.

Concerning $g(x)$ we assume that it is in L_∞, has compact support, and is uniformly Hölder continuous for some exponent $\alpha\,(0 < \alpha < 1)$ with respect to the variables $\tilde{x}$ (in case $q > 0$); i.e.,

$$\operatorname*{lub}_{\hat{x}, \tilde{x}\, \tilde{x}'} \frac{|g(\tilde{x}, \hat{x}) - g(\tilde{x}', \hat{x})|}{|\tilde{x} - \tilde{x}'|^{\alpha}} < \infty \ . \tag{2.4}$$

It will be convenient to use the norm

$$\begin{aligned} [g]^q_\alpha &= \text{true max}\,|g| + \text{the above l.u.b. for } q > 0 \ ; \\ &= \text{true max}\,|g| \text{ for } q = 0 \ . \end{aligned}$$

THEOREM 2.1. *Under these assumptions the norm $[u]^q_t$ exists for all $t \geqq 0$ and*

$$[u]^q_\alpha < C_3 [g]^q_\alpha \ , \qquad 0 \leqq q \leqq n \ , \tag{2.5}$$

where C_3 depends only on C_1, C_2, n, q, and α.

If in addition $C_2 = 0$, then $u(x, t)$ is Hölder continuous in all variables including t, and

$$\text{(2.6)} \qquad \underset{x,x',t,t'}{\text{l.u.b.}} \frac{|u(x, t) - u(x', t')|}{(|x - x'|^2 + |t - t'|^2)^{\alpha/2}} = [u]_\alpha \leqq C_3[g]^q_\alpha .$$

This theorem, in the case $q = n$, yields the results proven in [2, § 3] (under slightly different hypotheses on K). Its proof is trivial in the case $q = 0$, so we assume $q > 0$. We shall employ the representation

$$\text{(2.7)} \qquad \begin{aligned} u(x, t) = &\int d\hat{y}\int K(x - y; t)[g(y) - g(\tilde{x}, \hat{y})]d\tilde{y} \\ &+ \int d\hat{y}g(\tilde{x}, \hat{y})\int K(x - y; t)d\tilde{y} , \end{aligned}$$

which is equivalent to (2.1). If $q = n$ it is understood that the symbols $\int d\hat{y}$ and $\hat{y}$ are to be omitted where they occur. Let $x = (\tilde{x}, \hat{x})$ and $x' = (\tilde{x}', \hat{x}')$ be any two points in x-space. Let $\delta = |x - x'|$, S the set of points y with $|\tilde{y} - \tilde{x}'| < 2\delta$, $|\hat{y} - \hat{x}'| < 2\delta$, and E the exterior of S. Then using (2.7) we write

$$u(x, t) - u(x', t) = I_1 + \cdots + I_7 ,$$

where

$$\begin{aligned}
I_1 &= \int_S K(x - y; t)[g(y) - g(\tilde{x}, \hat{y})]dy , \\
I_2 &= -\int_S K(x' - y; t)[g(y) - g(\tilde{x}'; y)]dy , \\
I_3 &= \int_E [K(x - y; t) - K(x' - y; t)][g(y) - g(\tilde{x}, \hat{y})]dy , \\
I_4 &= -\int_{|\hat{y}-\hat{x}'|>2\delta} d\hat{y}[g(\tilde{x}, \hat{y}) - (\tilde{x}', \hat{y})]\int K(x' - y; t)d\tilde{y} , \\
I_5 &= -\int_{|\hat{y}-\hat{x}'|<2\delta} d\hat{y}[g(\tilde{x}', \hat{y}) - g(\tilde{x}, \hat{y})]\int_{|\tilde{y}-\tilde{x}'|>2\delta} K(x' - y; t)d\tilde{y} , \\
I_6 &= \int d\hat{y}[g(\tilde{x}, \hat{y}) - g(\tilde{x}', \hat{y})]\int K(x - y; t)d\tilde{y} , \\
I_7 &= \int d\hat{y}\, g(\tilde{x}', \hat{y})\int [K(x - y; t) - K(x' - y; t)]d\tilde{y} .
\end{aligned}$$

In case $q = n$ we set $I_4 = 0$ and disregard the integration with respect to $\hat{y}$ in I_{5-7}.

Since $|g(y) - g(\tilde{x}, \hat{y})| \leqq [g]^q_\alpha |\tilde{y} - \tilde{x}|^\alpha \leqq [g]^q_\alpha |y - x|^\alpha$, it follows that $|I_3| <$ const. $[g]^q_\alpha \delta^\alpha$. Using (2.2) again we see by the usual argument that I_1 and I_2 are subject to the same estimate. I_4 and I_6 may be estimated by (2.3a):

$$|I_4|, |I_6| \leqq C_2[g]^q_\alpha \delta^\alpha t \int (|\hat{y}|^2 + t^2)^{(1/2)(-n+q-1)} d\hat{y} \leqq \text{const.}\ [g]^q_\alpha \delta^\alpha .$$

To estimate I_5 we set $r = |\tilde{y} - \tilde{x}'|$ so that $|K(x' - y)| < C_1 r^{-n}$, and obtain, if $q < n$, $|I_5| \leqq \text{cont.}\ [g]^q_\alpha \delta^\alpha \int_{|\hat{y}=\hat{x}'|<2\delta} \int_\delta^\infty r^{-n+q-1} dr \leqq \text{const.}\ [g]^q_\alpha \delta^\alpha$. If $q = n$ we use (2.3c) to obtain the same estimate.

The estimates obtained so far tell us that

$$|u(x, t) - u(x', t)| < \text{const.}\ [g]^q_\alpha |x - x'|^\alpha + |I_7| . \tag{2.8}$$

Now I_7 will vanish provided that either (*a*) $C_2 = 0$, or (b) x and x' differ only in their first q components; i.e., $x = (\tilde{x}, \hat{x})$, $x' = (\tilde{x}', \hat{x})$. Condition (b) is sufficient because

$$\int [K(x - x; t) - K(x' - y; t) d\tilde{y} = \int K(\tilde{x} - \tilde{y}, \hat{x} - \hat{y}; t) d\tilde{y}$$
$$- \int K(\tilde{x}' - \tilde{y}, \hat{x} - \hat{y}; t) d\tilde{y} = 0 .$$

Now assume condition (b) to hold, so that the last term in (2.8) does not appear. Taking the l.u.b. of the left side, (2.5) is proven for the case $1 \leqq q \leqq n$. It is easily extended, however, to the case $q = 0$ by using (2.1) and (2.3b).

To prove the second part of Theorem 2.1 we assume condition (a); i.e., $C_2 = 0$, so that again the last term in (2.8) disappears. The only thing left to prove is Hölder continuity with respect to t. Let t, t' be two numbers such that $0 \leqq t < t'$. Since the last integral in (2.7) also vanishes we may write

$$u(x, t') - u(x, t) = \int d\tilde{y} \int \left(\int_t^{t'} K_t(x - y; \tau) d\tau \right) [g(y) - g(\tilde{x}, \hat{y})] d\hat{y} .$$

Again (2.2) tells us that this integral is absolutely convergent, so we write it as

$$\int_t^{t'} \int_{\text{all } y} K_t(x - y; \tau)[g(y) - g(\tilde{x}, \hat{y})] dy d\tau = I_8 + I_9 ,$$

where

$$I_8 = \int_t^{t'} \int_{|x-y|<t'-t} \cdots dy d\tau ,$$

and

$$I_9 = \int_t^{t'} \int_{|x-y|>t'-t} \cdots dy d\tau .$$

Setting $\rho^2 = |x - y|^2 + \tau^2$, we may estimate

$$|I_8| \leqq \text{const}\,[g]^q_\alpha \int_t^{2t'-t} \rho^{-1+\alpha} d\rho \leqq \text{const}\,[g]^q_\alpha[(t')^\alpha - t^\alpha + (t'-t)^\alpha]$$
$$\leqq \text{const}\,[g]^q_\alpha\,|t'-t|^\alpha\,,$$

and

$$|I_9| \leqq \text{const}\,[g]^q_\alpha\,|t'-t|\left|\int_{t'-t}^\infty r^{-2+\alpha} dr \leqq \text{const}\,[g]^q_\alpha\,|t'-t|^\alpha\,.\right.$$

Combining these results with (2.8), (2.6) is easily obtained, completing the proof of Theorem 2.1.

Since the above constants do not depend on t or t', this last argument yield an immediate corollary:

COROLLARY 2.1: *Let*

$$\bar{U}(x, t) = \int d\hat{y}\int K(x - y; t)[g(y) - g(\tilde{x}, \hat{y})]d\tilde{y}\,,$$

the first term in (2.7). *Then* $\bar{U}$ *may be extended as a completely Hölder-continuous function to the closed region* $t \geqq 0$, *in which it satisfies the estimate* (2.6).

3. **Interior-type estimates.** In using Theorem 2.1 to obtain Schauder estimates the kernel K will be interpreted as a derivative of a fundamental solution or of a Poisson kernel for an elliptic boundary value problem. In this section we treat the case when K is a derivative of a fundamental solution.

The following norms and pseudonorms will be employed extensively. They refer to functions defined in the half-space $t > 0$ (or on the hyperplane $t = 0$). The differentiability properties needed for the quantities below to be well-defined will be obvious. These norms and pseudonorms will correspond to those in [2, § 5]. Subscripts will always denote the order of differentiation, and superscripts the independent variables with respect to which the Hölder difference quotients are to be taken.

$$\text{(3.1a)}\quad \begin{aligned}[]
[\varphi]^q_{l+\alpha} &= \underset{\tilde{x}\,\tilde{x}',\hat{x},t}{\text{l.u.b.}} \frac{|D^l\varphi(\tilde{x}, \hat{x}, t) - D^l\varphi(\tilde{x}', \hat{x}, t)|}{|\tilde{x} - \tilde{x}'|^\alpha} + \text{l.u.b.}\,|D^l\varphi|\,,\\
[\varphi]^{q,t}_{l+\alpha} &= \underset{\tilde{x},\tilde{x}',\hat{x},t,t'}{\text{l.u.b.}} \frac{|D^l\varphi(\tilde{x}, \hat{x}, t) - D^l\varphi(\tilde{x}', \hat{x}, t')|}{(|t - t'|^2 + |\tilde{x} - \tilde{x}'|^2)^{\alpha/2}} + \text{l.u.b.}\,|D^l\varphi|\,,
\end{aligned}$$

where, as before, $\tilde{x} = (x_1, \cdots, x_q)$. In particular

$$[\varphi]^{0,t}_{l+\alpha} = \underset{x,t,t'}{\text{l.u.b.}} \frac{|D^l\varphi(x, t) - D^l\varphi(x, t')|}{|t - t'|^\alpha} + \text{l.u.b.}\,|D^l\varphi|\,,$$

$$[\varphi]^0_{l+\alpha} = \text{l.u.b.}\,|\,D^l\varphi\,|\;, \tag{3.1b}$$

and

$$[\varphi]^{n\ t}_{l+\alpha} = [\varphi]^{n+1}_{l+\alpha} = [\varphi]_{l+\alpha}$$

in the sense the latter is used in [2], for instance. Of course, in all of these the l.u.b. is taken over all derivatives of order l. Also we define

$$\begin{aligned} |\,\varphi\,|^{q\ t}_{l+\alpha} &= \sum_{j=0}^{l} \text{l.u.b.}\,|\,D^j\varphi\,| + [\varphi]^{q\ t}_{l+\alpha}\;, \\ |\,\varphi\,|^{q}_{l+\alpha} &= \sum^{l} \text{l.u.b.}\,|\,D^j\varphi\,| + [\varphi]^{q}_{l+\alpha}\;. \end{aligned} \tag{3.1c}$$

Corresponding to these norms we define $\mathscr{C}^q_{l+\alpha}$ as the class of functions φ defined in the half-space $t > 0$ with continuous and bounded derivatives of order $< l$, and piecewise continous and bounded derivatives of order l which are uniformly Hölder continuous in $\tilde{x}$. The class $\mathscr{C}^{q\ t}_{l+\alpha}$ has an analogous definition.

The symbol $\tilde{D}^\lambda$ will denote any derivative of order λ, at least one of whose differentiations is with respect to a component of $\tilde{x}$; i.e., $\tilde{D}^\lambda = (\partial/\partial x_i)D^{\lambda-1}$, where $i \leqq q$.

REMARK: *Let λ be any integer $\geqq 1$. Assume $f(x)$ has absolutely continuous derivatives of order $\lambda - 1$, that $q > 0$, and that $[f]^q_{\lambda+\alpha}$ is finite. Then every derivative $\tilde{D}^\lambda f$ is Hölder continuous with respect to all variables, and*

$$[\tilde{D}^\lambda f]^n_\alpha \leqq C(\alpha)[f]^q_{\lambda+\alpha}\;, \tag{3.2}$$

where C depends only on α.

Proof. It is sufficient to consider the case $\lambda = 1, q = 1, n = 2$, for the general case may be reduced to this case by freezing all but two of the independent variables and replacing f in the proof by some $D^{\lambda-1}f$. By assumption, then $\tilde{x}$ and $\hat{x}$ have single components; call them x and y for simplicity, so that $f = f(x, y)$. The absolute continuity guarantees the identity

$$\int_x^{x+h} [f_x(\xi, y + k) - f_x(\xi, y)]d\xi = \int_y^{y+k} [f_y(x + h, \eta) - f_y(x, \eta)]d\eta$$

to hold for all values of $x, y, h,$ and k. It follows that

$$\begin{aligned} h[f_x(x, y + k) - f_x(x, y)] = &\int_y^{y+k} [f_y(x + h, \eta) - f_y(x, h)]d\eta \\ &- \int_x^{x+h} [f_x(\xi, y + k) - f_x(x, y + k)]d\xi + \end{aligned}$$

$$\int_x^{x+h}[f_x(\xi, y) - f_x(x, y)d\xi \ .$$

The first term on the right is bounded in absolute value by $kh^\alpha[f]^1_{1+\alpha}$, and each of the other two by

$$[f]^1_{1+\alpha}\int_0^h(\xi')^\alpha d\xi' = \frac{1}{1+\alpha}h^{1+\alpha}[f]^1_{1+\alpha} \ .$$

Dividing through by hk^α and setting $\sigma = h/k$, we have the estimate

$$|f_x(x, y+k) - f_x(x, y)|\, k^{-\alpha} \leqq [f]^1_{1+\alpha}\Big(\sigma^{\alpha-1} + \frac{2}{1+\alpha}\sigma^\alpha\Big)$$

for all values of σ. Taking the l.u.b. of the left over all x, y, and k, and the g.l.b. of the right over σ, we have $[f_x]^2_\alpha \leqq C(\alpha)[f]^1_{1+\alpha}$. As mentioned, this generalizes immediately to (3.2).

The following lemma will constitute an application of Theorem 2.1 to the case when $K(x-y; 0)$ is a fundamental solution of an elliptic differential operator in the variables x with constant coefficients, and containing only derivatives of order $2m$. The constant H will be defined as an upper bound for the ellipticity constant of L, and for the coefficients of L. It is shown in [4] that a fundamental solution $\Gamma(x)$ to L always exists having the property

$$|D^k\Gamma(x)| < \text{const}\ |x|^{2m-n-k}(1 + |\log|x||) \ , \tag{3.3}$$

the log term being omitted unless n is even and $0 \leqq k \leqq 2m - n$.

THEOREM 3.1. *Assume $1 \leqq q \leqq n$. Let l be any number $\geqq 2m$, and let $f(x)$ have derivatives of order l-$2m$ which are uniformly Hölder continuous with respect to $\tilde{x}$. If $l > 2m$ we also assume the derivatives of order $l - 2m - 1$ to be absolutely continuous, and if $l = 2m$, $f(x)$ is to be integrable. (That derivatives $D^{l-2m}f$ are integrable for $l > 2m$ follows from the absolute continuity assumption.) Also we assume f to have compact support. Then if*

$$v(x) = \int\Gamma(x-y)f(y)dy \ , \tag{3.4}$$

every derivative $\tilde{D}^l v$ exists and

$$[\tilde{D}^l v]^n_\alpha \leqq \textit{const}\, [f]^q_{l-2m+\alpha} \ . \tag{3.5}$$

The constants here depend only on H, n, m, l, and α.

Proof. The case $q = n$ is a well-known result, so we take $0 \leqq q \leqq n - 1$. Differentiating equation (3.4) $l - 1$ times while integrating by parts if necessary we have

$$\tilde{D}^{l-1}v = \int \tilde{D}^{2m-1}\Gamma(x-y)D^{l-2m}f(y)dy \,. \tag{3.6a}$$

Now let x' be a point, all except one of whose coordinates are the same as those of x. We shall derive the following representation for the corresponding difference quotient:

$$\begin{aligned}\frac{\tilde{D}^{l-1}v(x) - \tilde{D}^{l-1}v(x')}{|x-x'|} &= \int \frac{\tilde{D}^{2m-1}\Gamma(x-y) - \tilde{D}^{2m-1}\Gamma(x'-y)}{|x-x'|}[D^{l-2m}f(y) \\ &\qquad - D^{l-2m}f(\tilde{x}, \hat{y})]dy \,.\end{aligned} \tag{3.6b}$$

Let x_j be the component of $\tilde{x}$ with respect to which a differentiation occurs in the operator $\tilde{D}^{2m-1}$ in (3.6a), so that $\tilde{D}^{2m-1} = (\partial/\partial x_j)D^{2m-2}$. Then, since $D_x\Gamma(x-y) = -D_y\Gamma(x-y)$,

$$\begin{aligned}\int_{-\infty}^{\infty}&[\tilde{D}^{2m-1}\Gamma(x-y) - \tilde{D}^{2m-1}\Gamma(x'-y)]dy_j \\ &= -\lim_{y_j\to\infty}[D^{2m-2}\Gamma(x-y) - D^{2m-2}\Gamma(x'-y)] \\ &\quad + \lim_{y_j\to-\infty}[D^{2m-2}\Gamma(x-y) - D^{2m-2}\Gamma(x'-y)] = 0 \,,\end{aligned}$$

as can be seen from the behavior of Γ at infinity indicated in (3.3) (using also the mean value theorem in the case $n = 2$). It follows immediately that

$$\begin{aligned}&\frac{\tilde{D}^{l-1}v(x) - \tilde{D}^{l-1}v(x')}{|x-x'|} \\ &\quad = \int' dy \int \frac{\tilde{D}^{2m-1}\Gamma(x-y) - \tilde{D}^{2m-1}\Gamma(x'-y)}{|x-x'|} \\ &\qquad\qquad [D^{l-2m}f(y) - D^{l-2m}f(\tilde{x}, \hat{y})]dy_j \,,\end{aligned}$$

where $\int' dy$ signifies integration with respect to all variables except y_j. But this integral is absolutely convergent, as can be seen by applying the mean value theorem to the difference quotient in the integral, using (3.3), and recognizing that the integrations with respect to components of $\tilde{y}$ may be considered as only over a finite range (since $f(y)$ has compact support); the order of integration is therefore immaterial and (3.6b) is valid. Defining D' as the derivative in the direction from x to x' and $\tilde{D}^{2m} = D'\tilde{D}^{2m-1}$, we subtract the absolutely convergent integral

$$\int \tilde{D}^{2m}\Gamma(x, y)[D^{l-2m}f(y) - D^{l-2m}f(\tilde{x}, \hat{y})]dy$$

from each side of (3.6b), obtaining on the right an integral which is bounded in absolute value by const. $|x - x'|^{\alpha}[f]^{q}_{l-2m+\alpha}$. This last estimate is obtained by the usual process of splitting the region of integration into the sphere $|y - x| < 2|x - x'|$ and its exterior, and applying the mean value theorem in the latter region. Now letting $x' \to x$, this bound vanishes, and furthermore the left side of (3.6b) approaches $\tilde{D}^{l}v(x)$. Hence

$$\tilde{D}^{l}v = \int \tilde{D}^{2m}\Gamma(x - y)[D^{l-2m}f(y) - D^{l-2m}f(\tilde{x}, \hat{y})] \,. \tag{3.7}$$

This integral is reminiscent of the first term on the right of (2.7); and in fact we shall apply Corollary 2.1 directly in proving the theorem. We identify $K(x - y; 0)$ with $\tilde{D}^{2m}\Gamma(x - y)$ and $g(y)$ with $D^{l-2m} f(y)$; then according to Corollary 2.1, (3.5) will follow from (2.6) if the hypotheses (2.2) and (2.3a) with $t = 0$ are true. But (2.2) follows from (3.3) and (2.3a) from our representation of K as a derivative. Theorem 3.1 is thereby proved.

4. Boundary-type estimates. In this section $L(D)$ will again be an elliptic differential operator with constant coefficients containing only terms of order $2m$; but now it will be an operator in the $n + 1$ variables $x_1, \cdots, x_n, t$. Similarly, let $B_j(D)$ $(j = 1, \cdots, m)$ be operators with constant coefficients and only terms of order m_j. We assume L and B_j to satisfy the root condition and complementing condition stated in [2, § 1]. The concern here will be with the boundary-value problem

$$\begin{aligned} L(D_x, D_t)u &= f(x, t) && (t > 0) \\ B_j(D)u &= \varphi_j(x) && (t = 0, j = 1, \cdots, m) \,. \end{aligned} \tag{4.1}$$

We initially assume all functions to be infinitely differentiable and to have compact support; this restiction will be removed at the end of the section (Theorem 4.6).

First we review some important results from [2] concerning representations of the function $u(x, t)$. Let l be any integer with $l \geqq \max(2m, m_j)$, and $P(D)$ a differential operator, each term of which is of degree l. Then

$$\begin{aligned} P(D)u(x, t) &= P(D)v(x, t) \\ &\quad + \sum_{j=1}^{m} \int P(D)K_j(x - y; t)[\varphi_j(y) - \psi_j(y)]dy \,, \end{aligned} \tag{4.2}$$

where $v(x, t) = \int \Gamma(x - y, t - \tau)f_N(y, \tau)dyd\tau$, $\Gamma(x - y, t - \tau)$ is a fundamental solution for L, f_N is a sufficiently smooth extension of $f(x, y)$

to the whole space such that f_N has compact support, $\psi_j(x) = B_j(D)v(x, t)|_{t=0}$ and K_j are Poisson kernels given explicitly in [2].

Section 3 was concerned with estimating the first term on the right of (4.2) in terms of properties of f. We shall now consider the other terms and develop estimates for functions given by

$$w(x, t) = \int K_j(x - y; t)\varphi_j(y)dy = K_j * \varphi_j \qquad (t > 0) . \tag{4.3}$$

It is proved in [2] that

$$[w]^{n;t}_{l+\alpha} \leqq C[\varphi_j]^{n}_{l-m_j+\alpha} . \qquad (l \geqq m_j) \tag{4.4}$$

(for the notation see (3.1)). Also it is proved in [2] that

$$[P(D)w]^0_0 \leqq C[\varphi_j]^0_{l-m_j} \tag{4.5}$$

provided $P(\xi, \tau)$ (obtained from $P(D)$ by replacing $\partial/\partial x_i$ by ξ_i and $\partial/\partial t$ by τ) is of the form

$$P(\xi, \tau) = \sum_{j=1}^{m} \xi^{l-m_j} B_j(\xi, \tau) ,$$

where ξ^{l-m_j} stands for any monomial of degree $l - m_j$ in the variables ξ_i alone.

(4.4) corresponds to the case $q = n$; (4.5) to the case $q = 0$. Our primary aim in this section will be to supplement these estimates by (1) extending them to intermediate values of q, $0 < q < n$, and (2) deriving, for $q < n$, a necessary condition on $P(\xi, \tau)$ for such estimates to hold.

First we shall review and develop certain properties of the Poisson kernels. The kernels are given by

$$\begin{aligned} K_j(x; t) &= \Delta_x^{(n+s)/2} K_{j,s}(x, t) , \\ K_{j,s}(x; t) &= b_{j,s} \int_{|\xi|=1} \int_\gamma \frac{N_j(\xi, \tau)(x\cdot\xi + t\tau)^{m_j+s}}{M^+(\xi, \tau)} \\ &\qquad \left(\log \frac{x\cdot\xi + t\tau}{i} + c_{j,s}\right) d\tau . \end{aligned} \tag{4.6}$$

Here $b_{j,s}$ and $c_{j,s}$ are appropriate constants; $M^+(\xi, \tau) = \prod_{k=1}^m (\tau - \tau_k^+(\xi))$ where $\tau_k^+(\xi)$, $k = 1, \cdots, m$ are the m roots of $L(\xi, \tau) = 0$ with positive imaginary part ($L(\xi, \tau)$ is the polynomial obtained by replacing $\partial/\partial x_i$ by ξ_i and $\partial/\partial t$ by τ in $L(D_x, D_t)$); the contour γ surrounds the m roots $\tau_k^+(\xi)$ and lies entirely above the real axis; $N_j(\xi, \tau)$ are polynomials in τ such that

$$\int_\gamma \frac{N_j(\xi, \tau)B_k(\xi, \tau)}{M^+(\xi, \tau)} d\tau = \delta_{jk} . \tag{4.7}$$

In (4.6) and elsewhere below, if $n = 1$ then $\int_{|\xi|=1} dw_\xi$ is to be understood as $\sum_{\xi=\pm 1}$.

We shall state three lemmas concerning integrals such as occur in (4.6).

LEMMA 4.1. *Let $F(\xi)$ be a function of the real vector ξ continuous on the sphere $|\xi| = 1$. Let τ_0 be a complex constant with $Im\ \tau_0 \neq 0$, and k an integer $\geqq 1$. Then*

$$\left| \int_{|\xi|=1} F(\xi)(x\cdot\xi + t\tau_0)^{-k} d\omega_\xi \right| \leqq C(|x|^2 + t^2)^{-k/2} ,$$

C depending on τ_0, k, and $\max |F|$.

The proof of this lemma is given in Appendix 1 of [2]. This same estimate will clearly hold if the integrand is replaced by

$$\int_\gamma F(\xi, \tau)(x\cdot\xi + t\tau)^{-k} d\tau ,$$

where γ is a finite contour in the complex τ-plane bounded away from the real axis, and $F(\xi, \tau)$ is continuous for $\tau \in \gamma$, $|\xi| = 1$.

LEMMA 4.2. *If $\lambda \geqq m_j + s + 1$,*

$$\text{(4.8a)} \qquad | D^\lambda K_{j,s} | < C(|x|^2 + t^2)^{(1/2)(m_j+s-\lambda)} .$$

If D_x^λ is any derivative of order $\lambda \geqq 0$ in the variables x, then

$$\text{(4.8b)} \qquad | D_x^\lambda B_k(D) K_j(x; t) | < Ct(|x|^2 + t^2)^{(1/2)(m_j - m_k - n - \lambda - 1)} .$$

If $k \neq j$, $\lambda \geqq m_j - m_k + s$, then

$$\text{(4.8c)} \qquad | D_x^\lambda B_k K_{j,s}(x, t) | < Ct(|x|^2 + t^2)^{(1/2)(m_j - m_k + s - \lambda - 1)} .$$

In all these, C depends only on the ellipticity constant, bounds for the coefficients in L and B_j, the complementing condition constant, and all integers mentioned.

Proof. These estimates follow from Lemma 4.1 and the properties of N_j and are given in [2] (eqs. (2.13)′, (2.15)).

LEMMA 4.3. *Let the first q $(0 \leqq q \leqq n - 1)$ coordinates of n-space be distinguished as in § 2, and write $x = (\tilde{x}, \hat{x})$, $\xi = (\tilde{\xi}, \hat{\xi})$. Writing $L(\xi, \tau) = L(\tilde{\xi}, \hat{\xi}, \tau)$, let the polynomial $\hat{L}(\hat{\xi}, \tau) = L(0, \hat{\xi}, \tau)$, and similarly $\hat{B}_i(\xi, \tau) = B_j(0, \hat{\xi}, \tau)$. Let $\hat{K}_{j,s}$ be the Poisson kernels corresponding to $\hat{L}$ and $\hat{B}_j$ in $(n - q + 1)$-space. Let $P(D)$ be a homogeneous differential operator of order $> m_j + s + q$ and $\hat{P}$ the operator* *obtained from P by omitting all differentiations with respect to components of $\tilde{x}$. Then*

$$\int_{\tilde{x}-\text{space}} P(D)K_{j,s}(x;t)d\tilde{x} = \hat{P}(D)\hat{K}_{j,s}(\hat{x};t) . \tag{4.9}$$

This lemma is proved in Appendix A.

The following is an interesting consequence of Lemma 4.3 and the results of § 2. In this and the other theorems of this section, C denotes a constant depending only on the quantities listed in Lemma 4.2.

THEOREM 4.1. *Corresponding to the function $w(x)$ given by* (4.3) *define*

$$\hat{w}(x, t) = \hat{K}_j * \varphi_j(x) = \int_{\hat{y}-\text{space}} \hat{K}_j(\hat{x} - \hat{y}, t)\varphi_j(\tilde{x}, \hat{y})d\hat{y} ,$$

so that $\tilde{x}$ appears only as a parameter in the function φ_j. Also define $W(x, t) = w - \hat{w}$. Then if $l \geqq m_j$,

$$[\widehat{W}]_{l+\alpha} \leqq c[\varphi_j]^q_{l-m_j+\alpha} , \tag{4.10}$$

where the symbol $[\widehat{.}]_{l+\alpha}$ is defined as is $[\,.\,]^{n\,t}_{l+\alpha}$, except that the quantity inside brackets is considered a function of $\hat{x}$ alone (*and dependence on $\tilde{x}$ is ignored*).

This means that w and $\hat{w}$ differ by a function whose appropriate derivatives have estimable Hölder difference quotients with respect to all $n - q + 1$ variables $\hat{x}, t$. Actually the proof will show that only those derivatives whose order with respect to components of $\tilde{x}$ is greater than $l - m_j$ need be excluded.

Proof. Let D^l be any derivative of order l in the variables $\hat{x}$ and t. We assume $l - m_j$ to be even; a similar proof goes through for the odd case. Applying (4.6) and integrating by parts, as is done in [2], we have

$$D^l w = D^l K_j * \varphi_j = D^l \Delta^{(1/2)(n+s-l+m_j)} K_{j,s} * \Delta^{(1/2)(l-m_j)} \varphi_j . \tag{4.11}$$

From (4.8a) we know that (2.2) holds for the kernel $D^l \Delta^{(1/2)(n+s-l+m_j)} K_{j,s}$, so we may decompose the convolution into two terms as in (2.7):

$$D^l w = I_1 + I_2$$

where

$$I_1 = \int d\hat{y} \int D^l \Delta^{(1/2)(n+s-l+m_j)} K_{j,s}(x - y; t) \cdot [\Delta^{(1/2)(l-m_j)}\varphi_j(y) - \Delta^{(1/2)(l-m_j)}\varphi_j(y)\,|_{\tilde{y}=\tilde{x}}]d\tilde{y}$$

and satisfies

$$[I_1]_\alpha^{n\ t} \leqq C[\varphi_j]_{l-m_j+\alpha}^q$$

(according to Corollary 2.1); and

$$I_2 = \left(\int_{\tilde{x}-\text{space}} D^l \Delta^{(1/2)(n+s-l+m_j)} K_{j,s} d\tilde{x}\right) * \Delta^{(1/2)(l-m_j)} \varphi_j(y) \,|_{\tilde{y}=\tilde{x}} \,,$$

which, according to Lemma 4.3, is simply

$$D^l \hat{\Delta}^{(1/2)(n+s-l+m_j)} \hat{K}_{j,s} * \Delta^{(1/2)(l-m_j)} \varphi_j \,|_{\tilde{y}=\tilde{x}} = I_3 + I_4 \,,$$

where

$$I_3 = D^l \hat{\Delta}^{(1/2)(n+s-l+m_j)}) \hat{K}_{j,s} * \tilde{\Delta}^{(1/2)(l-m_j)} \varphi_j(\tilde{x}, \hat{y}) \,,$$

and

$$I_4 = D^l \hat{\Delta}^{(1/2)(n+s-l+m_j)}) \hat{K}_{j,s} * \hat{\Delta}^{(1/2)(l-m_j)} \varphi_j(\tilde{x}, \hat{y}) = D^l \hat{K}_j * \varphi_j(\tilde{x}, \hat{y}) = D^l \hat{w}(x, t) \,,$$

the operators $\hat{\Delta}$ and $\tilde{\Delta}$ denoting the Laplacian in $\hat{x}$ and $\tilde{x}$ respectively. Now (3.2) yields the estimate

$$[\tilde{\Delta}^{(1/2)(l-m_j)} \varphi_j(\tilde{x}, \hat{y})]_\alpha^n \leqq C[\varphi_j]_{l-m_j+\alpha}^q \,, \tag{4.12}$$

hence the usual boundary estimates ([2], or Theorem 2.1 with $q = n$) indicate that

$$[\hat{I}_3]_\alpha \leqq C[\varphi_j]_{l-m_j+\alpha}^q \,. \tag{4.13}$$

But since $D^l W = I_1 + I_3$, (4.10) is proven.

We are now ready to develop the two principal theorems of this section. The complementing condition states that for every $\xi \neq 0$, the m operators $B_j(\xi, \tau)$ are, as polynomials in τ, linearly independent modulo $M^+(\xi, \tau)$. It follows that every polynomial $P(\xi, \tau)$ admits a decomposition of the form

$$P(\xi, \tau) = a(\xi, \tau) M^+(\xi, \tau) + \sum_{j=1}^m a_j(\xi) B_j(\xi, \tau) \,, \tag{4.14}$$

where $a(\xi, \tau)$ is a polynomial in τ, but $a(\xi, \tau)$ and $a_j(\xi)$ are not necessarily polynomials in ξ.

THEOREM 4.2. (*Sufficient condition.*) *Let the polynomial $P(\xi, \tau)$ be normalized and homogeneous of degree $l \geqq \max[m_j]$. Let q be in the range $0 \leqq q \leqq n - 1$.* If there exists a polynomial $A_0(\xi, \tau)$ and polynomials $a_{0j}(\xi)$ (of degrees $l - m_j$) such that*

* If $q = n$ we know from [2] that (4.16) holds for every p of degree l.

$$P(0, \hat{\xi}, \tau) = A_0(\hat{\xi}, \tau)L(0, \hat{\xi}, \tau) + \sum_{j=1}^{m} a_{0j}(\hat{\xi})B_j(0, \hat{\xi}, \tau) , \tag{4.15}$$

then

$$[P(D)w]_\alpha^q \leqq C[\varphi_j]_{l-m_j+\alpha}^q . \tag{4.16}$$

THEOREM 4.3. (*Necessary condition.*) *Again let* $0 \leqq q \leqq n-1$. *A necessary condition on* $P(\xi, \tau)$ (*normalized and homogeneous of degree* l) *in order that the estimate* (4.16) *hold for all* φ_j *infinitely differentiable and with compact support is that there exist a polynomial* $A_0(\hat{\xi}, \tau)$ *and functions* $a_{0j}(\hat{\xi}), 1 \leqq j \leqq m$, *with* $a_{0j}(-\hat{\xi}) = (-1)^{l-m_j}a_{0j}(\hat{\xi})$ *such that* (4.15) *holds.*

The difference between the two conditions is that only in the first case are the $a_{0j}(\xi)$ assumed to be polynomials. The author is of the opinion that the condition in Theorem 4.2 is necessary as well as sufficient. Theorem 4.3 is proved in Appendix B.

Proof of Theorem 4.2. The case $q = 0$ is essentially the abovementioned result (4.5) obtained in [2]. Therefore assume $1 \leqq q \leqq n-1$. From (4.15) it follows that

$$P(\xi, \tau) = A_0(\hat{\xi}, \tau)L(\xi, \tau) + \sum_{j=1}^{m} a_{0j}(\hat{\xi})B_j(\xi, \tau) + \widetilde{Q}(\xi, \tau) ,$$

where $\widetilde{Q}$ is a polynomial every term of which contains as factor some component of $\tilde{\xi}$. We write

$$P(D)w = W_1 + W_2 ,$$

where (using (4.3), (4.6)),

$$W_1 = A_0(\hat{D}, D_t)L(D)K_j*\varphi_j + a_{0j}(\hat{D})B_jK_j*\varphi_j ,$$

$$W_2 = (\sum_{i\neq j} a_{0i}(\hat{D})B_j + \widetilde{Q}(D))\Delta^{(1/2)(n+s)}K_{j,s}*\varphi_j .$$

(Here s is an integer of the same parity as n such that $n + s + m_j - l > 0$.) Since $LK_j = 0$, we may write

$$W_1 = B_jK_j*a_{0j}(\hat{D})\varphi_j .$$

Also, writing $R = \sum_{i\neq j} a_{0i}B_i + \widetilde{Q}$, we follow the procedure in [2] and write

$$W_2 = R(D)\Delta^{(1/2)(n+s+m_j-l)}K_{j,s}*\Delta^{(1/2)(l-m_j)}\varphi_j$$

if $l - m_j$ is even, and

$$W = \sum_k R(D)\frac{\partial}{\partial x_k}\Delta^{(1/2)(n+s+m_j-l-1)}K_{j,s}*\frac{\partial}{\partial y_k}\Delta^{(1/2)(l-m_j-1)}\varphi_j$$

if $l - m_j$ is odd. For simplicity we consider only the even case. Theorem 2.1 may now be applied by identifying the u in it with W_1 or W_2, g with $a_{0j}\varphi_j$ or $\Delta^{(1/2)(l-m_j)}\varphi_j$, and K with B_jK_j or $R\Delta^{(1/2)(n+sm_j-l)}$ $K_{j,s}$. Conditions (2.2) and (2.3a) must be verified. The first follows from (4.8a), and for (2.3a) we use Lemma 4.3 and (4.8b):

$$\left|\int B_jK_j d\tilde{x}\right| = |\hat{B}_j\hat{K}_j| \leqq Ct(|\hat{x}|^2 + t^2)^{(1/2)(-n+q-1)} ;$$

also, using (4.8c) and the fact that $\tilde{Q}(0, \hat{\xi}, \tau) = 0$,

$$\begin{aligned}\left|\int R\Delta^{(1/2)(n+s+m_j-l)}K_{j,s}d\tilde{x}\right| &= |R(0, \hat{D}, D_t)\hat{\Delta}^{(1/2)(n+s+m_j-l)}\hat{K}_{j,s}| \\ &\leqq \sum_{i\neq j}|a_{0i}(\hat{D})\hat{\Delta}^{(1/2)(n+s+m_j-l)}\hat{B}_i\hat{K}_{j,s}| \leqq Ct(|\hat{x}|^2 + t^2)^{(1/2)(-n+q-1)} .\end{aligned}$$

This establishes Theorem 4.2.

COROLLARY 4.2. *If $\tilde{D}^l$ is any derivative of order l involving at least one differentiation with respect to a component of $\tilde{x}$, then*

$$[\tilde{D}^l w]_\alpha^{n,t} \leqq C[\varphi_j]_{l-m_j+\alpha}^q . \tag{4.17}$$

Proof. The operator $\tilde{D}^l$ is a particular case of the type treated in the theorem but in this case $W_1 = 0$ and $R(0, \hat{D}, D_t) = 0$, so that in applying Theorem 2.1 we see that $C_2 = 0$ and the second statement in that theorem holds.

We shall now return to the system (4.1). Our object will be to find operators $Q(D)$ such that $Q(D)u$ will be estimable in various senses in terms of f and φ_j. Our first result is an immediate consequence of Theorems 3.1 and 4.2. For these we shall think of t as the $(n + 1)$-st component of x, $t = x_{n+1}$, and let $[\tilde{u}]_{l+\alpha}^{n+1}$ denote

$$[\tilde{u}]_{l+\alpha}^{n+1} = \text{l.u.b.}\,|\tilde{D}^l u(x)| + \text{l.u.b.}\,\frac{|\tilde{D}^l u(x_1) - \tilde{D}^l u(x_2)|}{|x_1 - x_2|^\alpha} ,$$

where the l.u.b.'s are taken over points x, x_1, x_2 in the domain of definition of u, and over derivatives $\tilde{D}^l$ which involve at least one differentiation with respect to a component of $\tilde{x}$.

THEOREM 4.4. *Let the normalized polynomial $P(\xi, \tau)$ of degree $l \geqq 2m$ satisfy* (4.15) *and u, f, and φ_j of compact support satisfy* (4.1). *Then*

$$[P(D)u]_\alpha^q \leqq C([f]_{l-2m+\alpha}^n + \sum_j [\varphi_j]_{l-m_j+\alpha}^q) . \tag{4.18}$$

Furthermore if $l >$ max $[m_j]$ *and $q > 0$, then*

$$[\tilde{u}]^{n+1}_{l+\alpha} \leqq C([f]^{q}_{l-2m+\alpha} + \sum_{j} [\varphi_j]^{q}_{l-m_j+\alpha}) . \tag{4.19}$$

Proof. We use representation (4.2). Theorem 3.1 yields

$$[\tilde{v}]^{n+1}_{l+\alpha} \leqq C[f]^{q}_{l-2m+\alpha} ,$$

and

$$[v]^{n}_{l+\alpha} \leqq C[f]^{n}_{l-2m+\alpha} .$$

The latter is obtained directly for derivatives $D^l v$ containing at least one differentiation with respect to a component $x_i\,(1 \leqq i \leqq n)$ by setting $q = n$; but we may differentiate $Lv = f$ $l - 2m$ times with respect to x_{n+1} and solve for $\partial^l v/\partial x^l_{n+1}$ in terms of such, thus obtaining the estimate in general.

Thus it follows that

$$[\psi_j]^{n}_{l-m_j+\alpha} \leqq C[f]^{n}_{l-2m+\alpha}$$

and

$$[\tilde{\psi}_j]^{n}_{l-m_j+\alpha} \leqq C[f]^{q}_{l-2m+\alpha} \qquad \text{(for } l > m_j) .$$

The former, together with Theorem 4.2, yields (4.18). To derive (4.19) we represent

$$\tilde{D}^l K_j * \psi_j = D^{m_j} K_j * \tilde{D}^{l-m_j} \psi_j ,$$

then apply Theorem 4.2 with $q = n$ to obtain (4.19).

THEOREM 4.5. *All the interior and boundary-type estimates proved so far (i.e., Theorems 3.1 and 4.–4.4) remain true when the smoothness requirements of the functions involved are relaxed to the extent that they have only the differentiability and boundedness properties implied in the statement of the corresponding estimate. For example,* (4.18) *is true if only* $u \in \mathscr{C}^{q}_{l+\alpha}, f \in \mathscr{C}^{n}_{l-2m+\alpha}$, *and* $\varphi_j \in \mathscr{C}^{q}_{l-m_j+\alpha}$.

Proof. The theorem follows from the fact that every function $\varphi \in \mathscr{C}^{q}_{l+\alpha}$ may be approximated by functions $\varphi_\varepsilon \in \mathscr{C}^{n+1}_{\infty}$ in such a way that $\lim_{\varepsilon \to 0} |\varphi_\varepsilon|^{q}_{l+\alpha} = |\varphi|^{q}_{l+\alpha}$. The φ_ε may, for example, be defined by $\varphi_\varepsilon(x) = j_\varepsilon(x) * \varphi(x)$, where j_ε is the Friedrichs mollifier, $j_\varepsilon(x) = \varepsilon^{-n} j_1(x/\varepsilon)$, $j_1(x)$ being a function in $\mathscr{C}^{n+1}_{\infty}$ with $\int j_1(x)dx = 1$, and $j_1 = 0$ for $|x| > 1$. Then it is an easy consequence of the "smearing" action of j_ε that $|\varphi_\varepsilon|^{0}_{0} \leqq |\varphi|^{0}_{0}$. Also it is seen that at every point x where φ is continuous, $\varphi_\varepsilon(x) \to \varphi(x)$. Since for every δ we can find such a point of continuity x with $|\varphi(x)| > |\varphi|^{0}_{0} - \delta$, it follows that

$\lim_{\varepsilon\to 0}\inf|\varphi_\varepsilon|_0^0 \geqq |\varphi|_0^0$. Combining the two inequalities, we have $\lim_{\varepsilon\to 0}|\varphi_\varepsilon|_0^0 = |\varphi|_0^0$. But the same reasoning may be applied to derivatives and difference quotients of φ, since these processes commute with the convolution. Hence

$$\lim_{\varepsilon\to 0}|\varphi_\varepsilon|_{l+\alpha}^q = |\varphi|_{l+\alpha}^q$$

as stated. Now in treating a typical Theorem such as 4.4, we first continue u a short distance into the region $t \leqq 0$ as a function with the same smoothness properties as it has for $t > 0$, then define $u_\varepsilon = j_\varepsilon * u$, $f_\varepsilon = Lu_\varepsilon$, and $\varphi_{j\varepsilon} = B_j u_\varepsilon|_{t=0}$. Then the theorem is true for u_ε, f_ε, $\varphi_{j\varepsilon}$; but $[Pu_\varepsilon]_\alpha^q \to [Pu]$, $[f_\varepsilon]_{l-2m+\alpha}^n \to [f]_{l-2m+\alpha}^n$, and $[\varphi_{j\varepsilon}]_{lm_j+\alpha}^q \to [\varphi_j]_{l-m_j+\alpha}^q$, so it is true as stated.

5. Variable coefficients. The foregoing results concerning equations with constant coefficients in a half-space permit the derivation of certain similar results for more general domains and variable coefficients. The procedure we shall use is basically that in [2, § 7]; however, the arguments here will be more involved, and in the case $q < n$, the results are much less general.

Let $\mathscr{D}$ be a domain in $(n+1)$-dimensional space with boundary $\dot{\mathscr{D}}$, and consider the problem

$$\begin{aligned} L(x, D)u &= f(x)\,, & x &\in \mathscr{D}\,, \\ B_j(x, D)u &= \varphi_j(x)\,, & x &\in \dot{\mathscr{D}}\,. \end{aligned} \tag{5.1}$$

$L(x, D)$ is assumed to be uniformly elliptic in $\mathscr{D}$ with ellipticity constant E, and to satisfy the root condition of [2]. Also the B_j are to satisfy the complementing condition of [2] with "determinant constant" Δ.

As before let q be an integer, $0 \leqq q \leqq n$, and l an integer with $l \geqq \max[2m, m_j]$; but now we permit the $=$ sign in this latter inequality to hold *only* in the case $m_j < 2m$ for all j. Let $\mu_0 = \max[1, l-2m]$ and $\mu_j = \max[1, l-m_j]$. We assume the coefficients of L and B_j to belong to classes $\mathscr{C}_{\mu_0+\alpha}^{n+1}(\mathscr{D})$ and $\mathscr{C}_{\mu_j+\alpha}^{n+1}(\dot{\mathscr{D}})$ respectively, and to have $|\cdot|_{\mu_0+\alpha}^{n+1}$ and $|\cdot|_{\mu_j+\alpha}^{n+1}$ norms bounded by the constant H.

In addition to these assumptions on L and B_j, we shall require that coordinate tranformations may be introduced which, at least locally, flatten out the boundary $\dot{\mathscr{D}}$, and such that the operators L and B_j transform into operators of a special type. This special type is that in which the coefficients of all derivatives of order $2m$ in L and those of order m_j in B_j, which involve *only* differentiations with respect to "undistinguished" variables, be constant on the new flat boundary. As will be shown in § 6, this assumption involves no loss of generality when $q = n$ (this is the case when there is one "undistinguished" direction, and it is normal to $\dot{\mathscr{D}}$). However for $q < n$ it limits substantially the generality of the results. There is one exception however: the case when $m = 1$, $B_1 \equiv 1$, $q = 0$, and $n = 1$ or 2. In this case such transformations as required above are always possible; however in this case the same a priori estimates may be obtained much more easily by use of the known maximum principle for second order elliptic equations.

Theorem 5.1 treats the case when the domain $\mathscr{D}$ is the half-space $x_{n+1} > 0$, and L and B_j are of the special type. Theorem 5.4 indicates the same results to hold if L and B_j may be transformed locally to operators of the special type, $\dot{\mathscr{D}}$ at the same time being flattened locally. Theorem 5.2 treats the case when L and B_j are of special type throughout $\mathscr{D}$; then the full Hölder continuity of f is no longer required.

Constants appearing in this and the following section which depend only on E, Δ, H, m, m_j, α, and l will all be denoted by the letter C. Whenever an operator appears with a tilde ($\sim$) over it, it is to be understood that every term of the operator involves at least one differentiation with respect to a component of $\tilde{x}$ or in a "distinguished direction.". Symbols such as $|\cdot|_{\cdots}^{q,\mathscr{D}_1}$, where $\mathscr{D}_1$ is a subdomain of $\mathscr{D}$, simply mean the same as $|\cdot|_{\cdots}^{q}$, except that the function in brackets is considered to have only $\mathscr{D}_1$ as its domain of definition. We shall also use the symbol $|\tilde{u}|_{l+\alpha}^{n+1}$ as defined on page 526. An operator $Q(x, D)$ with variable coefficients is said to be normalized if the l.u.b. of all its coefficients for all x in its domain of definition is one.

THEOREM 5.1. *Let $L(x, D)$ and $B_j(x, D)$ satisfy the above conditions, and in addition assume L and B_j to be of the forms*

$$(5.2)\quad L(x, D) = L_0(D) + \tilde{L}(x, D) + L_1(x, D) + \textit{lower order terms},$$

$$(5.3)\quad B_j(x, D) = B_{j0}(D) + \tilde{B}_j(x, D) + \textit{lower order terms},^{1}$$

where L_0 and B_{j0} have constant coefficients, and L_1 has coefficients which vanish for $x_{n+1} = 0$. Let $u(x), f(x)$, and $\varphi_j(x)$ satisfy (5.1) *in the half-space $x_{n+1} > 0$, and have smoothness and boundedness properties which will guarantee the norms in* (5.5) *and in the proof of the theorem to exist. Let $P(x, D)$ be any homogeneous normalized operator of degree l with coefficients in $\mathscr{C}_{l+\alpha}^{n+1}$ whose $|\cdot|_{l+\alpha}^{n+1}$ norms are*

[1] Terms of the form $R(x, D)B_i(x, 0)$ $(i \neq j)$, R an operator of degree $m_j - m_i$, would also be permissible in the expression for B_j; but if they are present we may replace B_j by $B_j - RB_i$ and φ_j by $\varphi_j - R(x, D)\varphi_i$, obtaining an equivalent boundary-value problem in which they no longer appear.

bounded by H, and which may be represented in the form

(5.4) $P(x, D) = A(x, D)L_0(D) + \sum_j a_j(x, D)B_{j0}(D) + \tilde{P}(x, D) + P_1(x, D)$.

where P_1 vanishes for $x_{n+1} = 0$, ond the a_j involve no differentiations with respect to x_{n+1}. Then

$$(5.5a) \qquad | P(x, D)u |_\alpha^q \leqq C\Big\{| f |_{l-2m+\alpha}^{n+1} + \sum_{j=1}^m | \varphi_j |_{l-m_j+\alpha}^q + | u |_0^0\Big\} ,$$

$$(5.5b) \qquad | \tilde{u} |_{l+\alpha}^{n+1} \leqq C\Big\{| f |_{l-2m+\alpha}^{n+1} + \sum_{j=1}^m | \varphi_j |_{l-m_j+\alpha}^q + | u |_0^0\Big\} .$$

Proof. The proof will employ the following two lemmas, the first of which is contained in the results of [2].

LEMMA 5.1. *Let $u \in \mathscr{C}_{l-1+\alpha}^{n+1}, f \in \mathscr{C}_{l-2m+\alpha}^{n+1}, \varphi_j \in \mathscr{C}_{l-m_j}^0$ be solutions to* (5.1) *in an arbitrary domain $\mathscr{D}$ with smooth enough boundary. Then*

$$(5.6) \qquad | u |_{l-1+\alpha}^{n+1} \leqq C\{| f |_{l-2m+\alpha}^{n+1} + \sum_j | \varphi_j |_{l-m_j}^0 + | u |_0^0\} .$$

Proof. If $l > \max[2m, m_j]$ this follows directly from [2, Theorem 7.3]: there $\mathscr{A}$ is identified with $\mathscr{D}$, l is replaced by $l-1$, and the inequalities $| f |_{l-2m-1+\alpha}^{n+1} \leqq | f |_{l-2m-1+\alpha}^{n+1}$ and $| \varphi_j |_{l-m_j-1+\alpha}^{n+1\,\mathscr{D}} \leqq | \varphi_j |_{l-m_j}^{0\,\mathscr{D}}$ are employed. The other possibility is that $\max[m_j] < 2m$ and $l = 2m$. Let S_1 and S_2 be concentric balls with radii 1 and 2 respectively, and center in $\mathscr{D}$. Let a be some number such that the hyperplane $x_1 = a$ intersects S_2. Define $F_1(x) = \int_a^{x_1} f(\xi, x_2, \cdots, x_{n+1})d\xi$ (we may need to extend f outside $\mathscr{D}$ for this to be defined), and $F_\beta = 0$ for $\beta > 1$. Then Theorem 9.3 of [2] is applicable: set $p = 2m - 1$ and $\mathscr{A} = S_1$. The conclusion of that theorem is:

$$| u |_{l-1+\alpha}^{n+1\ S_1} \leqq C\{\sum_\beta | F_\beta |_\alpha^{n+1,S_2} + \sum_j | \varphi_j |_{2m-1-m_j+\alpha}^{n+1} + | u |_0^0\} .$$

But $| F_1 |_\alpha^{n+1,S_2} \leqq 4 | f |_\alpha^{n+1}$ and $F_\beta = 0, \beta > 1$, so (5.6) holds in this case also if a superscript S_1 is adjoined to the norm on the left. But it does not appear on the right and its center is arbitrary, so (5.6) is valid as written.

LEMMA 5.2. *Consider again the case when the domain $\mathscr{D}$ is the halfspace $x_{n+1} > 0$. Let $b(x)$ be a function in $\mathscr{C}_{l+\alpha}^{n+1}$ such that $b(x) = 0$ for $x_{n+1} = 0$, and $| b |_{l+\alpha}^{n+1} < H$. Then for every derivative of order l,*

$$(5.7) \qquad | b(x)D^l u |_\alpha^{n+1} \leqq C\{| f |_{l-2m+\alpha}^{n+1} + \sum_j | \varphi_j |_{l-m_j}^0 + | u |_0^0\} .$$

Proof. Let $v - b(x)u$. Then

$$L(x, D)v = bf + \sum_{0 \leqq k \leqq 2m-1} c_k(x)D^k u = F(x) ,$$

$$B_j v\,|_{x_{n+1}=0} = \sum_{k < m_j} e_{jk}(x)D^k u\,|_{x_{n+1}=0} = \Phi_j(x) .$$

It follows from Lemma 5.1 that

$$|\,F\,|_{l-2m+\alpha}^{n+1} \leqq C\{|\,f\,|_{l-2m+\alpha}^{n+1} + \sum_j |\,\varphi_j\,|_{l-m_j}^{0} + |\,u\,|_0^0\}$$

and

$$|\,\Phi_j\,|_{l-m_j+\alpha}^{n} \leqq C\{|\,f\,|_{l-2m}^{n+1} + \sum_j |\,\varphi_j\,|_{l-m_j}^{0} + |\,u\,|_0^0\} .$$

Hence from the main boundary estimate of [2] (Theorem 7.3),

$$|\,v\,|_{l+\alpha}^{n+1} \leqq C\{|\,f\,|_{l-2m+\alpha}^{n+1} + \sum_j |\,\varphi\,|_{l-m_j}^{0} + |\,u\,|_0^0 + |\,v\,|_0^0\} .$$

But since for every derivative D^l we have

$$bD^l u = D^l v + \text{lower order terms}$$

and since the lower order terms may be estimated by (5.6), and also since $|\,v\,|_0^0 \leqq C\,|\,u\,|_0^0$, (5.7) follows.

Now to proceed with the proof of Theorem 5.1, let $\mathscr{C}_0$ be the class of homogeneous operators of degree l which have a representation of the form (5.4), and whose coefficients

(1) are in $\mathscr{C}_\alpha^{n+1}$ and have $|\cdot|_\alpha^{n+1}$ norms bounded by H; and

(2) have first derivatives with respect to x_{n+1} in $\mathscr{C}_\alpha^{n+1}$ with norms $|\cdot|_\alpha^{n+1}$ bounded by H. Let $\mathscr{C}$ be the subclass consisting of those operators in $\mathscr{C}_0$ with coefficients in $\mathscr{C}_{1+\alpha}^{n+1}$ whose $|\cdot|_{1+\alpha}^{n+1}$ norms are bounded by H. Let δ be a fixed number, $0 < \delta < 1$, which will be defined later. We define the number M as

$$M = 4\delta^{-\alpha} \operatorname*{lub}_{\mathscr{C}_0} |\,\bar{Q}(x, D)u\,|_0^0 + \operatorname*{lub}_{\mathscr{C}} |\,\bar{Q}(x, D)u\,|_\alpha^q + [\tilde{u}]_{l+\alpha}^{n+1}$$

with the lub's taken over all operators $\bar{Q}(x, D) \in \mathscr{C}_0$ and $\mathscr{C}$ respectively. Then from the definition of $|\cdot|_\alpha^q$ there is a point y and an operator $Q(x, D) \in \mathscr{C}_0$ or in $\mathscr{C}$, or a derivative $\tilde{D}^l$, such that one of the following four quantities is $> \frac{1}{8}M$:

$$
\begin{aligned}
&U_1 = 4\delta^{-\alpha}\,|\,Q(y, D)u(y)\,| , \\
&U_2 = \frac{|\,Q(z, D)u(z) - Q(y, D)u(y)\,|}{|\,\tilde{z} - \tilde{y}\,|^\alpha} \quad \text{(for some } z \text{ with } \hat{z} = \hat{y}) , \\
(5.8)\qquad &U_3 = |\,\tilde{D}^l u(y)\,| , \\
&U_4 = \frac{|\,\tilde{D}^l u(z) - \tilde{D}^l u(y)\,|}{|\,z - y\,|^\alpha} \qquad \text{(for some } z) ,
\end{aligned}
$$

with U_2 and U_4 missing if $q = 0$. We shall carry out the proof first under the assumption that $U_2 > \frac{1}{8}M$. The proof for other cases will then require only slight additional arguments. Therefore we assume $q > 0, Q \in \mathscr{E}$, and $U_2 > \frac{1}{8}M$. It may be assumed that $|z - y| \leqq \delta$ since if not, the quotient U_2 will be $< 2\delta^{-\alpha} | Qu |_0^0$, and there will be a point y' such that

$$U_1' = 4\delta^{-\alpha} | Q(y', D)u(y') | > 2\delta^{-\alpha} | Qu |_0^0 > U_2 > \tfrac{1}{8}M .$$

The argument thus reduces to the case when the first of the four quantities in (5.8) is $> \frac{1}{8}M$, which case is treated separately.

Let $\zeta(t)$ be a $\mathscr{C}_\infty$ function of a single variable such that $\zeta(t) = 1$ for $|t| < 1$ and $\zeta(t) = 0$ for $|t| > 2$. Define

$$w(x) = \zeta\Big(\frac{x - y}{\delta}\Big)u(x) \tag{5.9a}$$

if both y any z are further than 2δ from $\dot{\mathscr{D}}$; i.e., both y and z have $(n + 1)$-st component $\geqq 2\delta$; and

$$w(x) = \zeta\Big(\frac{x - y_B}{3\delta}\Big)u(x) \tag{5.9b}$$

if either y or z is nearer than 2δ from $\dot{\mathscr{D}}$; here y_B is the projection of y onto $\dot{\mathscr{D}}$, so that if $y = (y_1 \cdots y_{n+1})$, $y_B = (y_1, \cdots, y_n, 0)$.

Let us assume the latter alternative (5.9b) to be the case; the proof for the former is similar. First, on the basis of (5.4), also considering (5.2) and (5.3), we may express $Q(y, D)$ as

$$Q(y, D) = Q_1(y, D) + Q_2(y, D) , \tag{5.10}$$

where Q_2 vanishes for y on $\dot{\mathscr{D}}$ and

$$Q_1(y, D) = A(y, D)L'(y_B, D) + \sum_j a_j(y, D)B_j'(y_B, D) + \tilde{Q}_3(y, D) ,$$

where L' and B_j' are those parts of L and B_j consisting of highest order terms only. Let us decompose the quotient U_2 as follows:

$$\frac{1}{8}M < \frac{| Q(z, D)u(z) - Q(y, D)u(y) |}{| \tilde{z} - \tilde{y} |} \leqq T_1 + T_2 + T_3 , \tag{5.11}$$

where

$$T_1 = \left| \frac{Q(z, D) - Q(y, D)}{| \tilde{z} - \tilde{y} |^\alpha} u(z) \right| ,$$

$$T_2 = \frac{Q_1(y, D)u(z) - Q_1(y, D)u(y) |}{| \tilde{z} - \tilde{y} |^\alpha} ,$$

$$T_3 = \frac{|Q_2(y, D)u(z) - Q_2(y, D)u(y)|}{|\tilde{z} - \tilde{y}|^\alpha} .$$

Owing to the smoothness of the coefficients of Q and to the definiton of $\mathscr{E}_0$, we have

$$T_1 \leqq C\delta^{1-\alpha} \operatorname*{lub}_{\bar{Q}\in\mathscr{E}} |\bar{Q}(x, D)u(x)|_0^0 \leqq C\delta M . \tag{5.12}$$

Theorem 4.4 (4.18), with Theorem 4.5 may be invoked to estimate T_2. In view of the definition of Q_1, that theorem tells us the following (where we have used $u(y) = w(y)$, $u(z) = w(z)$; notice also that condition (4.15) "neutralizes" $\tilde{Q}_3$):

$$\begin{aligned} T_2 &= \frac{|Q_1(y, D)w(z) - Q_1(y, D)w(y)|}{|\tilde{z} - \tilde{y}|} \\ &\leqq C\{[L'(y_B, D)w(x)]^n_{l-2m+\alpha} + \sum_j [B'_j(y_B, D)w]^q_{l-m_j+\alpha}\} . \end{aligned} \tag{5.13}$$

(y_B, of course, is to be considered a constant when the norms on the right are computed.) To further estimate the terms on the right, we introduce the symbol S_δ to denote the sphere of radius 6δ about y_B. First, for any derivative D^{l-2m},

$$D^{l-2m}L'\{(y_B, D)w = \zeta D^{l-2m}L'(y_B, D)u + \sum_{k\geqq 1} \gamma_k(x)(D^k\zeta)D^{l-k}u . \tag{5.14}$$

From (5.1), (5.2),

$$\begin{aligned} D_x^{l-2m}L'(y_B, D)u(x) &= D^{l-2m}L(x, D)u(x) + D_x^{l-2m}(\tilde{L}(y_B, D) \\ &\qquad - \tilde{L}(x, D))u(x) - D^{l-2m}L_1(x, D)u(x) + \text{l.o.t.} \\ &= D^{l-2m}f(x) + Q^*(x, D)u - L_1(x, D)D^{l-2m}u(x) \\ &\qquad + \text{l.o.t.} , \end{aligned} \tag{5.15}$$

where

$$Q^*(x, D) = (\tilde{L}(y_B, D) - \tilde{L}(x, D))D^{l-2m} .$$

Also

$$[\zeta D^{l-2m}f(x)]^n_\alpha \leqq [f]^n_{l-2m+\alpha} + \delta^{-\alpha}[f]^0_{l-2m} \leqq C(\delta)[f]^n_{l-2m+\alpha} . \tag{5.16}$$

Now since the coefficients of $\tilde{L}$ are in $\mathscr{C}^{n+1}_{1+\alpha}$, Q^* is a combination of derivatives of the form $\tilde{D}^l$ with coefficients bounded in $|\cdot|^{n+1}_\alpha$ norm by $6H\delta$, for $x \in S_\delta$. Since also $|\zeta|^{n+1}_\alpha < C\delta^{-\alpha}$, we have

$$[\zeta Q^*(x, D)u]^n_\alpha \leqq C(\delta[\tilde{u}]^n_{l+\alpha} + \delta^{1-\alpha}[\tilde{u}]^0_l) .$$

By a standard calculus lemma (see for example [2, § 5]),

$$[\tilde{u}]^0_l < \varepsilon[\tilde{u}]^n_{l+\alpha} + C(\varepsilon)|u|^0_0$$

for arbitrarily small ε. Choosing $\varepsilon = \delta^\alpha$, we have

(5.17) $$[\zeta Q^* u]_\alpha^n \leqq C\delta M + C(\delta) | u |_0^0 .$$

Lemma 5.1 easily yields

(5.18) $$[\zeta \cdot (\text{lower order terms})]_\alpha^n \leqq C\{| f |_{l-2m}^0 + \Sigma | \varphi_j |_{l-m_j}^0 + | u |_0^0\} .$$

For the same reasons

$$[\gamma_k(x)(D^k\zeta)(D^{l-k}u)]_\alpha^n \leqq C(\delta)\{| f |_{l-m_j+\alpha}^0 \sum_j | \varphi_j |_{l-2m+\alpha}^0 + | u |_0^0\} .$$

Combining this result with (5.14–18), we find

$$[L'(y_B, D)w]_{l-2m+\alpha}^n \leqq C\delta M + C(\delta)\{| f |_{l-2m+\alpha}^n + \Sigma | \varphi_j |_{l-m_j}^0 + | u |_0^0\} + [\zeta L_1(x, D)D^{l-2m}u]_\alpha^n .$$

In exactly the same way one obtains

$$[B_j'(y_B, D)w]_{l-m_j+\alpha}^q \leqq C\delta M + C(\delta)\{| f |_{l-2m}^0 + \Sigma | \varphi_j |_{l-m_j+\alpha}^q + | u |_0^0\} .$$

Combining these results with (5.13), (5.12), and (5.11), we have

(5.19) $$M \leqq C_2\delta M + C(\delta)\{| f |_{l-2m+\alpha}^n + \Sigma | \varphi_j |_{l-m_j+\alpha}^q + | u |_0^0\} + [\zeta L_1(x, D)D^{l-2m}u]_\alpha^n + T_3 ,$$

with C_2 independent of δ. The last two terms may be estimated with the use of Lemma 5.2. We shall illustrate the method by estimating T_3. By hypothesis the coefficients of Q_2 are in $\mathscr{C}_{1+\alpha}^{n+1}$ and vanish for $x_{n+1} = 0$; hence we may take out a factor x_{n+1} from each and have left a function in $\mathscr{C}_\alpha^{n+1}$. More specifically, define $b(x_{n+1})$ to be an infinitely differentiable function with $| b |_{l+\alpha}^{n+1} < H$ assuming the values

$$b(x_{n+1}) = \begin{cases} x_{n+1} & (0 \leqq x_{n+1} \leqq 1/2) , \\ 1 & (1 \leqq x_{n+1}) . \end{cases}$$

Then b may be factored out, and we have

$$Q_2(x, D) = b(x_{n+1})Q_4(x, D)$$

where the coefficients of Q_4 are in $\mathscr{C}_\alpha^{n+1}$ with $| \cdot |_\alpha^{n+1}$ norms bounded by H. Since b satisfies the hypotheses of Lemma 5.2,

$$| b(x_{n+1})Q_4(y, D)u(x) |_\alpha^{n+1} \leqq C\{| f |_{l-2m+\alpha}^{n+1} + \sum_j | \varphi_j |_{l-m_j}^0 + | u |_0^0\} .$$

But since $z_{n+1} = y_{n+1}$, we may write

(5.20) $$T_3 = \frac{| b(y_{n+1})Q_4(y, D)u(z) - b(y_{n+1})Q_4(y, D)u(y) |}{| \tilde{z} - \tilde{y} |^\alpha}$$

$$= \frac{|b(z_{n+1})Q_4(y, D)u(z) - b(y_{n+1})Q_4(y, D)u(y)|}{|\tilde{z} - \tilde{y}|^\alpha}$$
$$\leqq |b(x_{n+1})Q_4(y, D)u(x)|_\alpha^{n+1} \leqq C\{|f|_{l-2m+\alpha}^{n+1} + \Sigma |\varphi_j|_{l-m_j}^0 + |u|_0^0] .$$

The same estimate holds for the next to last term in (5.19), so that in all,

$$M \leqq C_2\delta M + C(\delta)\{|f|_{l-2m+\alpha}^{n+1} + \Sigma |\varphi_j|_{l-m_j+\alpha}^q + |u|_0^0\} . \tag{5.21}$$

If any one of the other three quantities U_1, U_3, or U_4 in (5.8) is assumed to be $> \frac{1}{8}M$, then an inequality similar to (5.21) with other constants C_1, C_3, C_4, all independent of δ, may be derived, In the case $U_1 > \frac{1}{8}M$, then we define w again according to (5.9) (forgetting about z). T_1 will be missing from (5.11) and T_2 and T_3 are no longer quotients, but rather $4\delta^{-\alpha}|Q_1(y, D)u(y)|$ and $4\delta^{-\alpha}|Q_2(y, D)u(y)|$ respectively. Theorem 4.5 again yields (5.13) except for an extra factor $\delta^{-\alpha}$ on the right. Repeating the argument from this point on, we obtain (5.21) with $C_2\delta M$ replaced by $C_1\delta^{1-\alpha}M$. U_3 and U_4 may be treated in similar manners.

Now the definition of δ is clear:

$$\delta = \min [(2C_1)^{-1/1-\alpha}, (2C_2)^{-1}, (2C_3)^{-1}, (2C_4)^{-1}] , \tag{5.22}$$

so that $C_1\delta^{1-\alpha} \leqq 1/2$, and $C_i\delta \leqq 1/2$ $(i = 2, 3, 4)$. Putting all terms in M on the left, (5.21) now implies (5.5a). Also since $|\tilde{u}|_{l+\alpha}^{n+1} = |\tilde{u}|_{l-1+\alpha}^{n+1} + [\tilde{u}]_{l+\alpha}^{n+1} \leqq M + |\tilde{u}|_{l-1+\alpha}^{n+1}$ and since the last term here may be estimated with Lemma 5.1, (5.5b) is deduced and the theorem proved.

The condition (5.2) imposed on L is really only a condition on L at the boundary x_{n+1}; consequently the full Hölder-continuity of f is needed for the estimate (5.5). The following theorem will only utilize f's Hölder continuity with respect to $\tilde{x}$; but as a price for it a condition on L analogous to (5.2) is imposed throughout the domain; and also the class of operators which are estimable is reduced.

THEOREM 5.2. *Let L, B_j, u, f, and φ_j satisfy the hypotheses of Theorem* 5.1, *except that f is required merely to be in $\mathscr{C}^q_{l-2m+\alpha}$, and $L(x, D)$ is of the form*

$$L(x, D) = L_0(D) + \tilde{L}(x, D) + \textit{lower order terms}. \tag{5.23}$$

Then

$$|\tilde{u}|_{l+\alpha}^{n+1} \leqq C\{|f|_{l-2m+\alpha}^q + \sum |\varphi_j|_{l-m_j+\alpha}^q + |u|_0^0\} . \tag{5.24}$$

Proof. The proof is the same in outline as that of Theorem 5.1; the following are the only differences. The pseudonorm $[\tilde{u}]_{l+\alpha}^{n+1}$ takes the place of M, so that U_1 and U_2 are missing from the list in (5.8).

Taking the case $U_4 \geqq 1/8[\tilde{u}]^{n+1}_{l+\alpha}$, Theorem 4.4 (4.18) is again invoked to yield (5.22). Again this takes the place of (5.13) and the argument is the same, except that the superscripts n in (5.16–19) are to be replaced by q, and last two terms in (5.19) are now missing. This proves the theorem.

Theorems 5.1 and 5.2 were based on the assumption that $\mathscr{D}$ is, or may be mapped onto, a half-space in such a manner that the transformed operators L and B_j satisfy certain properties. The following theorem serves to indicate that such a transformation property of $\mathscr{D}$, L, and B_j need only be local; i.e., we assume only that every point in $\mathscr{D}$ near the boundary has a neighborhood which may be mapped onto a hemisphere, L and B_j being transformed under this mapping in the desired manner.

Specifically, we assume that some portion Γ_1 of the boundary $\dot{\mathscr{D}}$ (it may happen that $\Gamma_1 = \dot{\mathscr{D}}$) is covered by a network of q families of "distinguished curves," each of class $\mathscr{C}^{n+1}_{l+\alpha}$, with no two curves of the same family intersecting each other, and no two curves from any two families tangent at any point. Then there will be q curves, one from each family, passing through each point in Γ_1. It is along these curves that we shall assume certain functions to be Hölder continuous. If $\Gamma_1 \neq \dot{\mathscr{D}}$, we speak of another portion Γ_2 with $\Gamma_1 \cup \Gamma_2 = \dot{\mathscr{D}}$, and Γ_2 overlapping Γ_1 so that the boundaries Γ_1 and Γ_2 are bounded away from each other by some number $d_1 > 0$. We also assume these q families may be extended in some manner to cover a subdomain $\mathscr{D}_1$ adjacent to Γ_1, $\mathscr{D}_1$ having the properties that $\dot{\mathscr{D}}_1 \cap \dot{\mathscr{D}} = \Gamma_1$, every point of $\mathscr{D}_1$ is nearer than $2d_1$ to Γ_1, and every point of $\mathscr{D}$-$\mathscr{D}_1$ is further than d_1 from Γ_1-Γ_2.

Our smoothness assumptions on $\dot{\mathscr{D}}$ will be very much the same as those made in [2, Theorem 7.3]. First of all, we assume Γ_2 to be of class $\mathscr{C}^{n+1}_{l+\alpha}$ and to satisfy the other requirements which are imposed in Theorem 7.3 of [2] on the boundary portion Γ spoken of there. Next, concerning Γ_1 and $\mathscr{D}_1$, we suppose there is some number $d \leqq d_1$ such that evey point $y \in \mathscr{D}_1$ has a neighborhood N_y whose boundary contains a portion of Γ_1 and which may be mapped by a one-to-one $\mathscr{C}^{n+1}_{l+\alpha}$ mapping $\mathscr{T}_y$ onto the hemisphere $\mathscr{H}$ ($|\bar{x}| = 1, \bar{x}_{n+1} > 0$) of radius 1 and center at origin in $(n+1)$-dimensional $\bar{x}$-space in such a manner that the following conditions are fulfilled:

(1) The image of y is closer than 1/3 to the origin.

(2) $\dot{N}_y \cap \dot{\mathscr{D}}$ is mapped onto the flat portion $\dot{\mathscr{H}}_0$ of the boundary of $\mathscr{H}$. Also, denoting the image of $N_y \cap \mathscr{D}_1$ by $\mathscr{H}_{1y}$, the distinguished curves in $N_y \cap \mathscr{D}_1$ are to be mapped onto line segments in $\mathscr{H}_{1y}$ which are parallel to the first q coordinate axes. In accordance with our usual practice, the first q coordinates of a point $\bar{x}$ in $\mathscr{H}_{1y}$

will be grouped together in $\tilde{\bar{x}}$, and the others in $\hat{\bar{x}}$ (these will not be defined outside $\mathscr{H}_{1y}$).

(3) L and B_j are transformed into new operators L_y and B_{jy} with the same smoothness, ellipticity, and complementing conditions as the original ones. We assume the same constant H will serve for the transformed operators independently of y.

(4) The transformed operators L_y and B_{jy} may be expressed, in $\mathscr{H}$, in the form:

$$L_y(\bar{x}, \bar{D}) = \lambda_y(\bar{x})L_{0y}(\bar{D}) + \tilde{L}_y(\bar{x}, \bar{D}) + L_{1y}(\bar{x}, \bar{D}) \\ + \text{lower order terms} \,, \tag{5.25}$$

where $\bar{D}$ denotes differentiation with respect to the $\bar{x}$, L_{0y} is an operator with constant coefficients, L_{1y} vanishes on $\dot{\mathscr{H}}_0$(i.e., for $\bar{x}_{n+1}=0$); and for $\bar{x} \in \mathscr{H}_{1y}$, each term of $\tilde{L}_y$ involves a differentiation with respect to a component of $\tilde{\bar{x}}$ (this with be true of all operators below with a "$\sim$"). Also for $\bar{x}_{n+1} = 0$,

$$B_{jy}(\bar{x}, \bar{D}) = \beta_y(\bar{x})B_{j0y}(\bar{D}) + \tilde{B}_{jy}(x, \bar{D}) + \text{l.o.t.} \tag{5.26}$$

Note that the ellipticity and complementing conditions guarantee λ_y and β_y to be bounded away from zero by a constant depending on H.

Referring back to the original coordinate system, let $\mathscr{E}$ be the class of operators $P(x, D)$ defined in $\mathscr{D}$ with coefficients in $\mathscr{C}^{n+1}_{1+\alpha}$, whose $|\cdot|^{n+1}_{1+\alpha}$ norms are bounded by H, with the property that when subjected to any transformation $\mathscr{T}_y$, P assumes the form

$$P_y(\bar{x}, \bar{D}) = A_y(\bar{x}, \bar{D})L_{0y}(\bar{D}) + \Sigma a_j(\bar{x}, \bar{D})B_{j0y}(\bar{D}) + \tilde{P}_y(\bar{x}, \bar{D}) + P_{1y}(\bar{x}, \bar{D}), \tag{5.27}$$

where $\tilde{P}_y$ and P_{1y} have the same properties as $\tilde{L}_y$ and L_{1y}. Let $\tilde{\mathscr{E}}$ be the subset of $\mathscr{E}$ consisting of those P which assume the form

$$P_y(\bar{x}, \bar{D}) = \tilde{P}_y(\bar{x}, \bar{D}) \tag{5.28}$$

with each transformation $\mathscr{T}_y$. Of course for points x in $\mathscr{D}$-$\mathscr{D}_1$ there are no distinguished directions and consequently there is no condition (except smoothness) imposed on $P(x, D)$ there.

The symbol $[\psi(x)]^{q,\mathscr{D}_1}_{l+\alpha}$ will be used below to denote

$$[\psi]^{q,\mathscr{D}_1}_{l+\alpha} = \text{lub}\,|D^l\psi(x)| + \text{lub}\,\frac{|D^l\psi(x_1) - D^l\psi(x_2)|}{|x_1 - x_2|^\alpha} \,, \tag{5.29}$$

where the first lub is over all points $x \in \mathscr{D}_1$ and derivatives of order l; the second, over all derivatives D^l and points $x_1, x_2 \in \mathscr{D}_1$ such that whenever $x_2 \in N_{x_1}$, the images $\bar{x}_1$ and $\bar{x}_2$ have the same components $\hat{\bar{x}}$ (i.e., x_1 and x_2 may be joined by a curve pieced together from portions of distinguished curves). A similar meaning is attached to $|\psi|^{q,\mathscr{D}_1}_{l+\alpha}$

and $|\psi|^{q,\Gamma_1}_{l+\alpha}$.

THEOREM 5.3. *Let L, B_j, and $\mathscr{D}$ satisfy the above conditions. Let u, f, and φ_j satisfy* (5.1) *and have the differentiability properties implied below. Then if $P(x, D) \in \mathscr{E}$,*

$$(5.31)\quad |P(x, D)u|^{q,\mathscr{D}_1}_{\alpha} \leqq C\{|f|^{n+1}_{l-2m+\alpha} + \Sigma |\varphi_j|^{q\,\Gamma_1-\Gamma_2}_{l-m_j+\alpha} + \Sigma |\varphi_j|^{n+1\,\Gamma_2}_{l-m_j+\alpha} + |u|^0_0\} .$$

Furthermore, if $\tilde{P} \in \tilde{\mathscr{E}}$, then

$$(5.32)\quad |\tilde{P}u|^{n+1}_{\alpha} \leqq C\{|f|^{n+1}_{l-2m+\alpha} + \Sigma |\varphi_j|^{q\,\Gamma_1-\Gamma_2}_{l-m_j+\alpha} + \Sigma |\varphi_j|^{n+1\,\Gamma_2}_{l-m_j+\alpha} + |u|^0_0\} .$$

Proof. We first define a third class $\mathscr{E}_0$ consisting of all operators $P(x, D)$ with the same properties a those in $\mathscr{E}$, except that the coefficients need not be differentiable in any tangential direction. Nevertheless, they are to be in $\mathscr{C}^{n+1}_{\alpha}$. In other words, when subjected to any $\mathscr{T}_y$, the coefficients are to be in $\mathscr{C}^{n+1}_{\alpha}$ and to have Hölder continuous derivatives with respect to $\bar{x}_{n+1}$. The corresponding norms are to be bounded by H. Let δ be a number to be defined later, and define

$$(5.33)\quad M = 4\delta^{-\alpha} \operatorname*{lub}_{\mathscr{E}_0} |P(x, D)u|^{0,\mathscr{D}_1}_0 + \operatorname*{lub}_{\mathscr{E}} |P(x, D)u|^{q,\mathscr{D}_1}_{\alpha} + \operatorname*{lub}_{\tilde{\mathscr{E}}} |\tilde{P}u|^{n+1}_{\alpha} .$$

Then there is a point $y \in \mathscr{D}$ and an operator $Q(x, D) \in \mathscr{E}$ (or $\mathscr{E}_0$) or an operator $\tilde{Q}(x, D) \in \tilde{\mathscr{E}}$ such that one of the following four quantities is $> \frac{1}{8}M$:

$$(5.34)\quad U_1 = 4\delta^{-\alpha} |Q(y, D)u(y)| ,$$
$$U_2 = \frac{|Q(z, D)u(z) - Q(y, D)u(y)|}{|z - y|^{\alpha}}$$

(for some $z \in \mathscr{D}_1$ with $\hat{\bar{z}} = \hat{\bar{y}}$ for every transformation $\mathscr{T}_x$),

$$U_3 = |\tilde{Q}(y, D)u(y)| ,$$
$$U_4 = \frac{|\tilde{Q}(z, D)u(z) - \tilde{Q}(y, D)u(y)|}{|z - y|^{\alpha}} \qquad \text{(for some } z \in \mathscr{D}) .$$

If $y \notin \mathscr{D}_1$, U_1 and U_2 are missing from the list, and if $q = 0$, U_2 and U_4 are missing.

First we assume $q > 0$, $y \in \mathscr{D}_1$, $Q \in \mathscr{E}$, and $U_2 > \frac{1}{8}M$. It may be assumed that $z \in N_y$ and that $|z - y| \leqq \delta$ (see the proof to Theorem 5.1). Subject the coordinates to the transformation $\mathscr{T}_y$. Then $|\bar{z} - \bar{y}| \leqq \delta' = (1/\kappa)\delta$, where κ is the minimum expansion coefficient for the transformations $\mathscr{T}_y$. We assume δ to be small enough so that $\delta' \leqq \frac{1}{9}$. Define

(5.35a) $$w(\bar{x}) = \zeta\Big(\frac{\bar{x}-\bar{y}}{\delta'}\Big)u(\bar{x}) ,$$

if both $\bar{y}$ and $\bar{z}$ are further than $2\delta'$ from $\dot{\mathscr{H}}_0$; and

(5.35b) $$w(\bar{x}) = \zeta\Big(\frac{\bar{x}-\bar{y}_B}{3\delta'}\Big)u(\bar{x})$$

if not (here $\bar{y}_B$ is the projection of $\bar{y}$ onto $\dot{\mathscr{H}}_0$). Since $|\bar{y}| \leqq \frac{1}{3}$ and $|\bar{y}-\bar{z}| \leqq \frac{1}{9}$ it follows that w always has support in $\mathscr{H}$.

We assume alternative (5.35b) as the proof for the other is similar. It is seen from (5.25), (5.27) that after transformation the operator Q may be written as

(5.36) $$\bar{Q}(\bar{y}, \bar{D}) = Q_1(\bar{y}, \bar{D}) + Q_2(\bar{y}, \bar{D}) ,$$

where

$$Q_1(\bar{y}, \bar{D}) = \frac{A(\bar{y}, \bar{D})}{\lambda(\bar{y}_B)} L'_y(\bar{y}_B, \bar{D}) + \sum_j \frac{a_j(\bar{y}, \bar{D})}{\beta_j(\bar{y}_B)} B'_{jy}(\bar{y}, \bar{D}) + \tilde{Q}_3(\bar{y}, \bar{D}) ,$$

where L'_y and B'_{jy} are those parts of L_y and B_{jy} consisting of highest order terms only, and Q_2 vanishes for $\bar{y}_{n+1} = 0$. Then

(5.37) $$\tfrac{1}{8}M \leqq T_1 + T_2 + T_3 ,$$

where

$$T_1 = \frac{|(Q(\bar{z}, \bar{D}) - Q(\bar{y}, \bar{D}))u(\bar{z})|}{|\bar{z}-\bar{y}|^{\alpha}} ,$$

$$T_2 = \frac{|Q_1(\bar{y}, \bar{D})u(\bar{z}) - Q_1(\bar{y}, \bar{D})u(\bar{y})|}{|\bar{z}-\bar{y}|^{\alpha}} ,$$

$$T_3 = \frac{|Q_2(\bar{y}, \bar{D})u(\bar{z}) - Q_2(\bar{y}, \bar{D})u(\bar{y})|}{|\bar{z}-\bar{y}|^{\alpha}} .$$

The smoothness of the coefficients of operators in $\mathscr{E}$ tells us

(5.38) $$T_1 \leqq CM .$$

Theorem 4.4 (4.18), with the aid of (5.36), may be applied to yield

(5.39) $$\begin{aligned} T_2 &= \frac{|Q_1(\bar{y}, \bar{D})w(\bar{z}) - Q_1(\bar{y}, \bar{D})w(\bar{y})|}{|\bar{z}-\bar{y}|^{\alpha}} \\ &\leqq C\{[L'_y(\bar{y}_B, \bar{D})w]^{n}_{l-2m+\alpha} + \sum_j [B'_{jy}(\bar{y}_B, \bar{D})w]^{q}_{l-m_j+\alpha}\} . \end{aligned}$$

Let $S_{\delta'}$ be the sphere of radius $6\delta'$ about y_B, so the support of w is in $S_{\delta'}$. As in (5.14), we have

$$\bar{D}^{l-2m}L'_y(\bar{y}_B, \bar{D})w = \zeta\bar{D}^{l-2m}L'_y(\bar{y}_B, \bar{D})u + \sum_{k\geqq 1} \gamma_k(\bar{x})(\bar{D}^k\zeta)\bar{D}^{l-k}u ,$$

and from (5.25),

$$\bar{D}_{\bar{x}}^{l-2m} L'_k(\bar{y}_B, \bar{D})u(\bar{x}) = \lambda_y(\bar{y}_B)(\lambda_y(\bar{x}))^{-1}\bar{D}_{\bar{x}}^{l-2m} L_y(\bar{x}, \bar{D})u + \tilde{Q}*(\bar{x}, \bar{D})u(\bar{x}) \tag{5.40}$$
$$+ \lambda_y(\bar{y}_B)(\lambda_y(\bar{x}))^{-1} L_{1y}(\bar{x}, \bar{D})\bar{D}^{l-2m}u(\bar{x})$$
$$+ \text{lower order terms ,}$$

where

$$\tilde{Q}^*(\bar{x}, \bar{D})u = \{\tilde{L}_y(\bar{y}_B, \bar{D})\bar{D}^{l-2m} - \lambda_y(x_B)(\lambda(\bar{x}))^{-1}\tilde{L}_y(\bar{x}, \bar{D})D^{l-2m}\}u \ .$$

Since $L_y(\bar{x}, \bar{D})u = f(\bar{x})$ and λ_y is estimable from above and below in terms of H, the first term on the right in (5.40) makes a contribution to $\bar{D}^{l-2m}L'_y(\bar{y}_B, \bar{D})w$ which is estimable in $[\cdot]_\alpha^n$ norm by $C(\delta)\,|f|_{l-2m+\alpha}^n$ (as in (5.16)). Now $\tilde{Q}^*(\bar{x}, \bar{D})$ is clearly the image under $\mathscr{T}_y$ of an operator in $\mathscr{E}$, and furthermore this operator has coefficients bounded in $|\cdot|_\alpha^{n+1}$ norm by $C\delta$ for some C depending only on H. Therefore, as in (5.17),

$$|\zeta\tilde{Q}^*(\bar{x}, \bar{D})u(\bar{x})]_\alpha^{n+1} \leqq C\delta M + C(\delta)\,|u|_0^0 \ .$$

Thus by continuing the reasoning we obtain

$$|L'_y(\bar{y}_B, \bar{D})w]_{l-2m+\alpha}^n \leqq C\delta M + C(\delta)\{|f|_{l-2m+\alpha}^n + \Sigma\,|\varphi_j|_{l-m_j}^0 + |u|_0^0\} \tag{5.41}$$
$$+ [\zeta\lambda_y(\bar{y}_B)(\lambda_y(\bar{x}))^{-1}L_{1y}(\bar{x}, \bar{D})\bar{D}^{l-2m}u]_\alpha^u \ .$$

In the same manner we obtain

$$[B'_{jy}(\bar{y}_B, \bar{D})w]_{l-m_j+\alpha}^q \leqq C\delta M + C(\delta)\{|f|_{l-2m}^0 + \Sigma\,|\varphi_j(\bar{x})|_{l-m_j+\alpha}^{q\ \dot{\mathscr{H}}_0} \tag{5.42}$$
$$+ |u|_0^0\} \ .$$

In this latter we use the fact that

$$|\varphi_j(\bar{x})|_{l-m_j+\alpha}^{q\ \dot{\mathscr{H}}_0} \leqq C(|\varphi_j|_{l-m_j+\alpha}^{q\ \Gamma_1-\Gamma_2} + |\varphi_j|_{l-m_j+\alpha}^{n+1\ \Gamma_2}) \ ,$$

then combine (5.41) and (5.42), estimate T_3 and the last term in (5.41) by Lemma 5.2, define δ to be small enough so that all terms involving M may be transposed to the left side of the inequality, and (5.31) is proved for this case. A similar proof goes through if the other alternative in (5.35) holds or if U_1, U_3, or U_4 is $> \frac{1}{8}M$ and $y \in \mathscr{D}_1$. Finally, if $y \notin \mathscr{D}_1$, the boundary estimates of Agmon, Douglis, and Nirenberg [2, Theorem 7.3] may be applied directly to estimate U_3 and U_4 in terms of $|f|_{l-m+\alpha}^{n+1\ \mathscr{D}_2}$ and $\Sigma\,|\varphi_j|_{l-m_j+\alpha}^{n+1\ \Gamma_2}$. This completes the proof.

6. The case $q = n$. In this section we shall see that somewhat more concerning equations with variable coefficients may be said when $q = n$ than when $q < n$. In fact, most of the properties of solutions of elliptic boundary value problems which are true under complete Hölder-continuity assumptions ($q = n + 1$) of the functions involved are also true (or analogs of them are true) under assumptions corresponding to the case $q = n$. Assuming $q = n$ we shall be able (1) to demonstrate improved versions of Theorems 5.1 and 5.3, and (2) to prove an existence theorem concerning problem (5.1).

First we consider the problem when $\mathscr{D}$ is the half-space $x_{n+1} > 0$. The assumption $q = n$ means that all functions concerned are Hölder continuous in all directions except possibly that of the x_{n+1}-axis.

THEOREM 6.1. *Let L and B_j satisfy the ellipticity, complementing, and smoothness conditions stated as hypotheses to Theorem* 5.1. *If $u(x) \in \mathscr{C}^n_{l+\alpha}$, $f(x) \in \mathscr{C}^n_{l-2m+\alpha}$, $\varphi_j(x) \in \mathscr{C}^n_{l-m_j+\alpha}$, satisfy* (5.1) *for $x_{n+1} > 0$, then*

$$(6.1) \qquad |u|^n_{l+\alpha} + |\tilde{u}|^{n+1}_{l+\alpha} + \leqq C\{|f|^n_{l-2m+\alpha} + \Sigma\,|\varphi_j|^n_{l-m_j+\alpha} + |u|^0_0\}\,.$$

Proof. We shall first show that without any further hypotheses, L and B_j may be put into the form (5.2), (5.3), and that the corresponding set $\mathscr{E}$ includes all derivatives D^l. Then an estimate of the form (6.1), with, however, $|f|^{n+1}_{l-2m+\alpha}$ replacing $|f|^n_{l-2m+\alpha}$ on the right, clearly follows immediately from (5.5). With no loss of generality we may assume the coefficient of $\partial^{2m}/\partial x^{2m}_{n+1}$ in $L(x, D)$ to be identically 1. Then setting $L_0(D) = \partial^{2m}/\partial x^{2m}_{n+1}$ and $L_1 = 0$, (5.2) is obtained. Since the complementing condition assures us that in each B_j there is a derivative $\partial^{m_j}/\partial x^{m_j}_{n+1}$ with non-vanishing coefficient, we do the same thing here ($B_{j0} = \partial^{m_j}/\partial x^{m_j}_{n+1}$). Also every derivative D^l is trivially of the form (5.4) with, in fact, $P_1 = 0$ and $a_j = 0$, so is contained in $\mathscr{E}$.

Lastly we must show that (6.1) is correct as it stands, rather than with $|f|^{n+1}_{l-2m+\alpha}$ on the right. To do this we refer to the proof of Theorem 5.1, in particular to (5.19). At that stage the proper superscript n appears on the right, but it is changed to $n + 1$ when the last two terms are estimated (by means of Lemma 5.2). In the present case, however, these last two terms are absent (we have mentioned that $L_1 = 0$, and T_3 is absent because $Q_2 = 0$ in (5.10)), so that the superscript n remains, and (6.1) is valid.

We now pass to the analog of Theorem 5.3. In that theorem q families of distinguished curves were assumed to cover $\mathscr{D}_1$, a portion of $\mathscr{D}$. It will be more convenient in the present case to speak of a one-parameter family of n-dimensional hypersurfaces covering $\mathscr{D}_1$, the boundary portion $\Gamma_1 = \dot{\mathscr{D}}_1 \cap \dot{\mathscr{D}}$ being one of this family. This amounts to the same thing, and the proof is unchanged; moreover this permits the inclusion of the important case when $\Gamma_1 = \dot{\mathscr{D}}$ but

$\dot{\mathscr{D}}$ may not be covered with an n-parameter family of curves with the required regularity properties holding everywhere (for example, when $\mathscr{D}$ is a sphere in 3-space). Along with this change in point of view, assumption (2) preceding Theorem 5.3 should be changed to require that these n-dimensional hypersurfaces be mapped onto hyperplanes $\bar{x}_{n+1} = \text{const.}$ We shall discard hypothesis (4) altogether. Lastly we define a new domain $\mathscr{D}_2$ with the properties that $\mathscr{D}_1 \cup \mathscr{D}_2 = \mathscr{D}$, $\dot{\mathscr{D}}_2 \cap \dot{\mathscr{D}} = \Gamma_2$. If $\Gamma_1 = \dot{\mathscr{D}}$, $\Gamma_2 = 0$ and $\mathscr{D}_2$ is an interior domain.

THEOREM 6.2. *Let $\mathscr{D}$, $\mathscr{D}_1$, $\mathscr{D}_2$, L, and B_j satisfy the hypotheses of Theorem* 5.3, *with the above modifications, and excluding* (5.25), (5.26). *Then if $u(x), f(x), \varphi_j$ satisfy* (5.1) *and have the required smoothness properties,*

$$(6.2) \quad |\tilde{u}|_{l+\alpha}^{n+1} + |u|_{l+\alpha}^{n,\mathscr{D}_1} \leqq C\{|f|_{l-2m+\alpha}^{n\ \mathscr{D}_1} + |f|_{l-2m+\alpha}^{n+1,\mathscr{D}_2} + \Sigma |\varphi_j|_{l-m_j+\alpha}^{n\ \dot{\mathscr{D}}} + |u|_0^0\} .$$

Proof. Since $q = n$, any operators L and B_j automatically satisfy (5.25) and (5.26); and in fact with $L_{1y} = 0$. Also clearly any derivative D^l is in $\mathscr{E}$, and any such directional derivative involving a differentiation in a direction tangent to a distinguished hypersurface is in $\tilde{\mathscr{E}}$. Hence (6.2) would immediately follow from Theorem 5.3 if the first two terms on the right were replaced by $|f|_{l-2m+\alpha}^{n+1}$. But as in Theorem 6.1, the fact that $L_{1y} = 0$ and that P_{1y} is not needed in (5.27) results in our not having to require $D^{l-2m}f$ to be Hölder continuous in the one undistinguished direction, for points in $\mathscr{D}_1$-$\mathscr{D}_2$. This completes the proof. This theorem is analogous to Theorem 7.3 of [2].

The domain $\mathscr{D}_2$ was introduced not only for greater generality, but also because in general such a domain would be needed for topological reasons; it is not always possible to cover the entire domain $\mathscr{D}$ with a regular family of hypersurfaces, one of which is $\dot{\mathscr{D}}$. It is therefore important that such a covering be resticted to $\mathscr{D}_1$. However the theorem may be improved to the extent that f still need not be fully Hölder continuous in $\mathscr{D}_2$ If there is a second family of distinguished hypersurfaces covering $\mathscr{D}_2$ in a regular manner, and not necessarily fitting in with the first family in $\mathscr{D}_1 \cap \mathscr{D}_2$, then the second term on the right of (6.2) may be replaced by $|f|_{l-2m+\alpha}^{n\ \mathscr{D}_2}$, which is of course to be understood as defined with reference to the second family. The proof offers no difficulties but we shall not give it.

Our final task will be to prove that a solution to the basic problem (5.1) may be expected to exist under the smoothness hypotheses corresponding to $q = n$, provided one exists under the stronger hypotheses corresponding to $q = n + 1$. But first we consider questions of uniqueness. It is clear from the remark on page 517 that if $f \in \mathscr{C}^{n+1}_{l-2m+\alpha}(\mathscr{D})$ (and $\varphi_j \in \mathscr{C}^{n}_{l-m_j+\alpha}(\dot{\mathscr{D}})$) then any solution $u \in \mathscr{C}^{n}_{l+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D}_2)$ to (5.1) is in $\mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D})$. For any lth order directional derivative $\tilde{D}^l u$ written in terms of local coordinate system, which involves a differentiation in a distinguished direction is completely Hölder continuous; but then the only derivative D^l not involving such a direction is also completely Hölder continuous, for it may be expressed by means of the differential equation in terms of f and derivative $\tilde{D}^l$. Hence under the hypotheses of Theorem 6.2, if problem (5.1) has at most one solution in $\mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D})$ for every $f \in \mathscr{C}^{n+1}_{l-2m+\alpha}(\mathscr{D})$, then it has at most one solution $u \in \mathscr{C}^{n}_{l+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D}_2)$ for every $f \in \mathscr{C}^{n}_{l-2m+\alpha} \cap \mathscr{C}^{n+1}_{l-2m+\alpha}(\mathscr{D}_2)$.

THEOREM 6.3. *If uniqueness holds in problem* (5.1) *with* $q = n$, *then the term* $|u|^0_0$ *may be omitted from* (6.2).

Proof. If this were not true, there would be a sequence u^ν of functions in $\mathscr{C}^{n}_{l+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D}_2)$ with $|Lu^\nu|^{n+1\,\mathscr{D}_2}_{l-2m+\alpha}$, $|Lu\,|^{n\,\mathscr{D}_1}_{l-2m+\alpha}$, and $|B_j u^\nu|^{n\,\dot{\mathscr{D}}}_{l-m_j+\alpha}$ bounded, but $|u^\nu|^0_0$ and $|\tilde{u}\,|^{n+1}_{l+\alpha}$ or $|u\,|^{n\,\mathscr{D}_1}_{l+\alpha} \to \infty$. Define the new sequence $\bar{u}^\nu = u^\nu / |u^\nu|^0_0$. Then

$$|L\bar{u}^\nu|^{n\,\mathscr{D}_1}_{l-2m+\alpha} + |L\bar{u}^\nu|^{n+1\,\mathscr{D}_2}_{l-2m+\alpha} + \Sigma\,|B_j\bar{u}^\nu|^{n\,\dot{\mathscr{D}}}_{l-m_j+\alpha} \to 0\,,$$

but $|\bar{u}^\nu|^0_0 = 1$, and according to (6.2), $|\tilde{\bar{u}}^\nu|^{n+1}_{l+\alpha}$ and $|\bar{u}^\nu|^{n\,\mathscr{D}_1}_{l+\alpha}$ are bounded. From this last fact we know the derivatives of the form $\tilde{D}^l\bar{u}^\nu$ to be equicontinuous, and there is a subsequence $\bar{u}^k \to \bar{u}$ with $\tilde{D}^l\bar{u}^k \to \tilde{D}^l\bar{u}$, and $D^\lambda\bar{u}^l \to D^\lambda\bar{u}$, $\lambda \leqq l-1$, all these convergence processes being uniform. Write L in terms of local coordinates $\bar{x}_j$, where $\tilde{\bar{x}} = (\bar{x}_1, \cdots, \bar{x}_n)$. Then if $a(\bar{x})$ is the coefficient of $\partial^{2m}/\partial\bar{x}^{2m}_{n+1}$ in this expression, we have $\partial^l\bar{u}^k/\partial\bar{x}^l_{n+1} = \Sigma$ (coeffs.) $\tilde{D}^l\bar{u}^k$ + lower order terms + $(a(\bar{x}))^{-1}(\partial^{l-2m}/\partial\bar{x}^{l-2m}_{n+1})L\bar{u}^k$. Since the last term approaches 0 as $k \to \infty$, $\partial^l\bar{u}^k/\partial x_{n+1}$ converges uniformly to a function, which will therefore be $\partial^l\bar{u}/\partial\bar{x}^l_{n+1}$, and $\bar{u}$ will satisfy

$$L\bar{u} = 0 \text{ in } \mathscr{D}\,,$$
$$B_j\bar{u} = 0 \text{ on } \dot{\mathscr{D}}\,.$$

But $\bar{u} \not\equiv 0$, which contradicts the uniqueness assumption, and the theorem is proved.

THEOREM 6.4. *Let* $\mathscr{D}$, $\mathscr{D}_1$, $\mathscr{D}_2$, L, *and* B_j *satisfy the hypotheses of Theorem* 6.2. *Suppose the problem* (5.1) *has a unique solution*

$u \in \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D})$ for every $f \in \mathscr{C}^{n+1}_{l-2m+\alpha}(\mathscr{D})$ and $\varphi_j \in \mathscr{C}^{n}_{l-m_j+\alpha}(\dot{\mathscr{D}})$. Then it has a unique solution $u \in \mathscr{C}^{n}_{l+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D}_2)$ for every $f \in \mathscr{C}^{n}_{l-2m+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l-2m\alpha}(\mathscr{D}_2)$ and $\varphi_j \in \mathscr{C}^{n}_{l-m_j+\alpha}(\dot{\mathscr{D}})$.

Proof. Given any $f \in \mathscr{C}^{n}_{l-2m+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l-2m+\alpha}(\mathscr{D}_2)$, let the family $f_\varepsilon(x)$ be made up of functions in $\mathscr{C}^{n+1}_{l-2m+\alpha}(\mathscr{D})$ such that as $\varepsilon \to 0$, $D^\lambda f_\varepsilon \to D^\lambda f$, $\lambda \leqq l - 2m$, for every point in $\mathscr{D}$ at which the latter derivatives are continuous; and $|f_\varepsilon|^{n\,\mathscr{D}_1}_{l-2m+\alpha} \to |f|^{n\,\mathscr{D}_1}_{l-2m+\alpha}$. For example, we could set $f_\varepsilon(x) = j_\varepsilon * f$ with j_ε as defined in the proof to Theorem 4.6. By assumption and Theorems 6.2, 6.3, for each ε there is a unique $u_\varepsilon(x) \in \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D})$ such that

$$Lu_\varepsilon = f_\varepsilon \text{ in } \mathscr{D}\,,$$
$$B_j u_\varepsilon = \varphi_j \text{ on } \dot{\mathscr{D}}\,,$$

and

$$|\tilde{u}_\varepsilon|^{n+1,\mathscr{D}}_{l+\alpha} + |u_\varepsilon|^{n\,\mathscr{D}_1}_{l+\alpha} \leqq C\{|f_\varepsilon|^{n\,\mathscr{D}_1}_{l-2m+\alpha} + |f_\varepsilon|^{n+1\,\mathscr{D}_2}_{l-2m+\alpha} + \Sigma\,|\varphi_j|^{n}_{l-m_j+\alpha}\}.$$

Hence the norms $|\tilde{u}_\varepsilon|^{n+1,\mathscr{D}}_{l+\alpha}$ and $|u_\varepsilon|^{n\,\mathscr{D}_1}_{l+\alpha}$ form bounded sets. We shall show that the set of functions u_ε is compact in $\mathscr{C}^{n}_{l+\alpha}(\mathscr{D}_1)$. The boundedness of the norms $|\tilde{u}_\varepsilon|^{n+1}_{l+\alpha}$, shows the set of derivatives $\tilde{D}^l u_\varepsilon$ to be equicontinuous. Denoting by $\mathscr{D}_\delta$ the portion of $\mathscr{D}_1$ that remains after a δ-neighborhood of every point of discontinuity of $D^{l-2m}f$ has been deleted, it is clear that the $D^{l-2m}f_\varepsilon$ will be equicontinuous in $\mathscr{D}_\delta$. Solving the differential equation for $\partial^l u_\varepsilon/(\partial\bar{x}_{n+1})^l$, the only lth order derivative not of the form $\tilde{D}^l u_\varepsilon$, we see that it, hence all $D^l u_\varepsilon$, are equicontinuous in $\mathscr{D}_\delta$, and a subsequence of the u_ε converges to a function u_δ which satisfies the differential equation $Lu_\delta = f$ in $\mathscr{D}_\delta$, and the boundary conditions $B_j u_\delta = \varphi_j$ on $\dot{\mathscr{D}}$. Now taking a sequence of positive numbers $\delta_\nu \to 0$ and a diagonal subsequence of the u_ε, we find that the latter converges to a function $u \in \mathscr{C}^{n}_{l+\alpha}(\mathscr{D}_1) \cap \mathscr{C}^{n+1}_{l+\alpha}(\mathscr{D}_2)$ which satisfies (5.1).

APPENDIX A. *Proof of Lemma* 4.3. What we shall show specifically is that if the contour γ and function $F(\xi, \tau)$ are as in Lemma 4.1, $k > q$, and F is differentiable with respect to ξ for $\tau \in \gamma$, $|\xi| = 1$, then

$$\text{(A.1)} \quad \int_{\tilde{x}\text{-space}} d\tilde{x} \int_{|\xi|=1} d\omega_\xi \int_\gamma F(\tilde{\xi}, \hat{\xi}; \tau)(x\cdot + \xi + t\tau)^{-k} d\tau$$
$$= \frac{(2\pi i)_q (k-q-1)!}{(k-1)!} \int_{|\hat{\xi}|=1} d\omega_{\hat{\xi}} \int_\gamma F(0, \hat{\xi}; \tau)(\hat{x}\cdot\hat{\xi} + t\tau)^{-k+q} d\tau\,.$$

With this established, (4.9) will follow as a special case, in view of

(4.6) (see the definition of $b_{j,s}$ in [2]). The lemma tells us nothing new in case $q = 0$. First we prove it for the case $q = 1, \tilde{x} = x_1$; an obvious iteration process will then yield the result for the general case. First we recognize that

$$\int d\omega_\xi \int_\gamma F(\xi, \tau)(x\cdot\xi + t\tau)^{-k} d\tau$$
$$= \frac{1}{1-k}\frac{\partial}{\partial x_1}\int_{\substack{|\xi|=1 \\ PV}} d\omega_\xi \int_\gamma \frac{1}{\xi_1} F(\xi, \tau)(x\cdot\xi + t\tau)^{-k+1} d\tau \ ,$$

where "PV" indicates that the integral is taken in the principal value sense. This is easily checked by forming the derivative as the limit of difference quotients. Letting I denote the left side of (A.1), it follows that

$$I = \frac{1}{1-k}\left\{\int_{\substack{|\xi'|=1 \\ PV}} d\omega_\xi \int_\gamma \frac{1}{\xi_1} F(\xi_1, \hat{\xi}; \tau)(x\cdot\xi + t\tau)^{1-k} d\tau\right\}_{x_1=-\infty}^{x_1=+\infty} . \tag{A.2}$$

Now let $\zeta_\delta(\xi_1)$ be an infinitely differentiable function depending on a small parameter $\delta > 0$, such that

$$\zeta_\delta(\xi_1) = 0 \ \text{ for } \ |\xi_1| > 1 - \delta$$
$$\zeta_\delta(\xi_1) = 1 \ \text{ for } \ |\xi_1| < 1 - 2\delta \ .$$

The principal value integral (A.2) may be converted into an ordinary integral by subtracting from the coefficient of ξ_1^{-1} in the integrand any even smooth function of ξ_1 which takes on the same value as the original coefficient when $\xi_1 = 0$. For this function we choose

$$F(0, \hat{\xi}(1-\xi_1^2)^{-1/2}; \tau)\zeta_\delta(\xi_1)(1-\xi_1^2)^{(-n+3)/2}(\hat{x}\cdot\hat{\xi}(1-\xi_1^2)^{-1/2} + t\tau)^{1-k} .$$

Carrying out the subtraction in three parts, we obtain $I = I_1 + I_2 + I_3$, where

$$I_1 = \frac{1}{1-k}\left\{\int_{|\xi|=1} d\omega_\xi \int_\gamma \frac{F(\xi_1, \hat{\xi}; \tau) - F(0, \hat{\xi}(1-\xi_1^2)^{-1/2}; \tau)\zeta_\delta(\xi_1)(1-\xi_1^2)^{(-n+3)/2}}{\xi_1} \cdot \right.$$
$$\left. \cdot (x\cdot\xi + t\tau)^{1-k} d\tau\right\}_{x_1=-\infty}^{x_1=+\infty} ,$$

$$I_2 = \frac{1}{1-k}\left\{\int_{|\xi|=1} d\omega_\xi \int_\gamma F(0, \hat{\xi}(1-\xi_1^2)^{-2/2}; \tau)\zeta_\delta(\xi_1)(1-\xi_1^2)^{(-n+3)/2} \cdot \right.$$
$$\left. \cdot \frac{(x\cdot\xi + t\tau)^{1-k} - (x_1\xi_1 + \hat{x}\cdot\hat{\xi}(1-\xi_1^2)^{-1/2} + \tau t)^{1-k}}{\xi_1} d\tau\right\}_{-\infty}^{+\infty} ,$$

$$I_3 = \frac{1}{1-k}\left\{\int_{|\xi|=1} d\omega_\xi \int_\gamma F(0, \hat{\xi}(1-\xi_1^2)^{-1/2}; \tau)\zeta_\delta(\xi_1)(1-\xi_1^2)^{(-n+3)/2} \cdot \right.$$
$$\left. \cdot \frac{(x_1\xi_1 + \hat{x}\cdot\hat{\xi}(1-\xi_1^2)^{-1/2} + \tau t)^{1-k} - (\hat{x}\cdot\hat{\xi}(1-\xi_1^2)^{-1/2} + t\tau)^{1-k}}{\xi_1} d\tau\right\}_{-\infty}^{+\infty} .$$

The coefficient of $(\cdots)^{1+k}$ in the integrand of I_1 is a continuous and bounded function of ξ on $|\xi| = 1$; hence by Lemma 4.1 the integral is bounded by $C(|x|^2 + t^2)^{1-k}$ (C possibly depending on δ), which approaches 0 as $x_1 \to \pm\infty$. Hence $I_1 = 0$.

Next we use the mean value theorem to write the fraction in the integrand of I_2 as

$$\frac{(1-\xi_1^2)^{-1/2} - 1}{\xi_1} \hat{x}\cdot\hat{\xi}(k-1)(x_1\xi_1 + \hat{x}\cdot\hat{\xi}(1-\xi^2)^{-1/2} + t\tau)^{-k},$$

and observe that

$$\frac{(1-\xi_1^2)^{-1/2} - 1}{\xi_1} < \xi_1 \qquad \text{for } \xi_1 < 1/2 .$$

It is then easy to see that for x_1 large enough and $\xi_1 \leqq |x_1|^{-1/2}$, the integrand of I_2 is bounded by $C(\hat{x}, t)|x_1|^{-1/2}$ independently of $\hat{\xi}$ and τ. Also for $\xi_1 \geqq |x_1|^{-1/2}$ and x_1 large enough the integrand is bounded by $C(\delta, \hat{x}, t)|x_1\xi_1|^{-k} \leqq C(\delta, \hat{x}, t)|x_1|^{-1}$ (since $k \geqq 2$). Both of these bounds approach zero as $x_1 \to \pm\infty$, so we conclude that $I_2 = 0$.

To analyze I_3 we use the fact that

$$\int_{|\xi|=1} G(\xi_1, \hat{\xi})d\omega_\xi = \int_{-1}^{1} d\xi_1 \int_{|\hat{\xi}|=1} G(\xi_1, \hat{\xi}(1-\xi_1^2)^{1/2})(1-\xi_1^2)^{(n-3)/2} d\omega_{\hat{\xi}} ,$$

and obtain, after rearranging terms,

$$\text{(A-3)} \quad I = I_3 = \frac{1}{1-k}\int_{|\hat{\xi}|=1} d\omega_{\hat{\xi}} \int_\gamma F(0, \xi; \tau) \cdot \sum_{r=1}^{k-1} (\hat{x}\cdot\hat{\xi} + t\tau)^{-r} \left\{ \int_{-1}^{1} x_1(x\cdot\xi + t\tau)^{r-k}\zeta_\delta(\xi_1)d\xi_1 \right\}_{x_1=-\infty}^{x_1=+\infty} d\tau .$$

Now changing to a new integration variable $u = x_1\xi_1$,

$$\int_{-1}^{1} x_1(x\cdot\xi + t\tau)^{r-k}\zeta_\delta(\xi_1)d\xi_1 = \int_{-x_1}^{x_1} (u + \hat{x}\cdot\hat{\xi} + t\tau)^{r-k}du - x_1\int_{|\xi_1|>1-2\delta} [1 - \zeta_\delta(\xi_1)](x_1\xi_1 + \hat{x}\cdot\hat{\xi} + t\tau)^{r-k} d\xi_1 .$$

First we estimate the last integral. For x_1 large enough, $|\xi_1| > 1 - 2\delta$, $|x_1\xi_1 + \hat{x}\cdot\hat{\xi} + t\tau| > x_1/2$, so that the integral is less in absolute value than $2\delta x_1(x_1/2)^{r-k} = C\delta x_1^{r+1-k}$. For $r < k-1$ this vanishes as $x_1 \to \pm\infty$, and for $r = k-1$ it remains bounded by $C\delta$. As for the first integral,

$$\lim_{x_1\to\pm\infty}\int_{-x_1}^{x_1} (u + \hat{x}\cdot\hat{\xi} + t\tau)^{r-k}du = \lim \frac{1}{r+1-k}(u + \hat{x}\cdot\hat{\xi} + t\tau)^{r+1-k}\Big]_{-x_1}^{x_1} = 0$$

for $r < k-1$. For $r = k-1$ the same integral

$$= \lim_{x_1 \to \pm\infty} \log \frac{x_1 + \hat{x}\cdot\hat{\xi} + t\tau}{-x_1 + \hat{x}\cdot\hat{\xi} + t\tau} = \mp\pi i \ .$$

Hence all terms of the summation in (A.3) vanish except the one with $r = k - 1$, and

$$I = \frac{2\pi_i}{k-1}\int_{|\hat{\xi}|=1} d\omega_{\hat{\xi}} \int_\gamma F(0, \hat{\xi}; \tau)(\hat{x}, \hat{\xi} + t\tau)^{1-k} d\tau + 0(\delta) \ .$$

But since δ may be arbitrarily small, the term here of order δ is really zero, and we have proven (A.1) for the case $q = 1$. But if $q > 1$ the above procedure may be iterated by integrating successively with respect to $x_1, \cdots, x_q$. Thus Lemma 4.3 is proved.

APPENDIX B. *Proof of Theorem* 4.3. We shall prove that a necessary condition for (4.16) to hold is that in every representation (4.14), the $a_j(\xi)$ satisfy

$$a_j(0, -\hat{\xi}) = (-1)^{l-m_j} a_j(0, \hat{\xi}) \ . \tag{B.1}$$

That this implies the condition stated may be demonstrated by setting $a_{0j}(\hat{\xi}) = a_j(0, \hat{\xi})$ and showing that there is a polynomial $A_0(\hat{\xi}, \tau)$ such that

$$a(0, \hat{\xi}, \tau)M^+(0, \hat{\xi}, \tau) = A_0(\hat{\xi}, \tau)L(0, \hat{\xi}, \tau) \ .$$

Since

$$P(-\xi, -\tau) = (-1)^l P(\xi, \tau),\ M^+(-\xi, -\tau) = (-1)^m M^-(\xi, \tau)$$

where $M^+(\xi, \tau)M^-(\xi, \tau) = L(\xi, \tau)$, and $B_j(-\xi, -\tau) = (-1)^{m_j} B_j(\xi, \tau)$, we have from (4.14) and (B.1)

$$\begin{aligned} &a(0, \hat{\xi}, \tau)M^+(0, \hat{\xi}, \tau) + \sum_j a_j(0, \hat{\xi})B_j(0, \hat{\xi}, \tau) \\ &\quad = P(0, \hat{\xi}, \tau) = (-1)^l P(0, -\hat{\xi}, -\tau) \\ &\quad = (-1)^{l+m} a(0, -\hat{\xi}, -\tau)M^-(0, \hat{\xi}, \tau) + \Sigma\, a_j(0, \hat{\xi})B_j(0, \hat{\xi}, \tau) \ . \end{aligned}$$

Hence

$$(-1)^{l+m} a(0, -\hat{\xi}, -\tau)M^-(0, \hat{\xi}, \tau) = a(0, \hat{\xi}, \tau)M^+(0, \hat{\xi}, \tau) \ .$$

But for $\hat{\xi} \neq 0$, $M^+(0, \hat{\xi}, \tau)$ and $M^-(0, \hat{\xi}, \tau)$, as polynomials in τ, have no factors in common; hence $M^-(0, \hat{\xi}, \tau)$ must divide $a(0, \hat{\xi}, \tau)$:

$$a(0, \hat{\xi}, \tau)M^+(0, \hat{\xi}, \tau) = A_0(\hat{\xi}, \tau)M^-M^+ = A_0(\tilde{\xi}, \tau)L(0, \hat{\xi}, \tau) \ .$$

Now we proceed to show that (B.1) is necessary. Assume, on the contrary, that there is a value $\hat{\xi} = \hat{\xi}_0$ such that $a_j(0, -\hat{\xi}_0) \neq (-1)^{l-m_j} a(0, \hat{\xi}_0)$. We shall construct a family of functions $\varphi_j^\varepsilon(x)$ such that $[\varphi_j^\varepsilon]_{l-m_j+\alpha}^q$ are bounded as $\varepsilon \to 0$, whereas $[Pw^\varepsilon]_\alpha^q$ are not (here w^ε is defined in terms of φ_j^ε by (4.3)). We may rotate the coordinate system so that $\hat{\xi}_0$ is directed along the x_n-axis; also since the a_j are homogeneous of degree $l - m_j$, it may be assumed that $|\hat{\xi}_0| = 1$: $\hat{\xi}_0 = (0, \cdots, 0, 1)$.

Define

$$\begin{aligned} \eta^\varepsilon(s) &= 0 \qquad \text{for } s \leqq -\varepsilon\,, \\ &= s^{l-m_j} \text{ for } s \geqq \varepsilon\,, \end{aligned}$$

and smooth it off in the range $-\varepsilon \leqq s \leqq \varepsilon$ so that it is an infinitely differentiable function whose derivatives of orders $\leqq l - m_j$ are monotonically increasing in $-\varepsilon \leqq s \leqq \varepsilon$. Our sequence φ_j^ε will be

$$\varphi_j^\varepsilon(x) = \eta^\varepsilon(x_n)\zeta(|x|)\,,$$

where $\zeta(s)$ is a $\mathscr{C}_\infty$ function with $\zeta = 0$ for $|s| > 1$ and $\zeta = 1$ for $|s| < 1/2$. It is easily checked that $(\partial^{l-m_j}/\partial x_n^{l-m_j})\varphi_j^\varepsilon(x)$ is bounded independently of ε; hence so is $[\varphi_j^\varepsilon]_{l-m_j+\alpha}^\varepsilon$. We shall show that $P(D)w^\varepsilon$ may be made arbitrarily large by choosing ε and t small. We assume $l - m_j$ to be even; the proof for odd case is similar. At this point we apply representation (2.7) to (4.3) after integrating by parts as before; and for this purpose we redefine $\tilde{x} = (x_1 \cdots, x_{n-1})$, $\hat{x} = x_n$. Hence

$$\begin{aligned} P(D)w^\varepsilon = {} & P\Delta^{(1/2)(n+s-l+m_j)}K_{j,s}*[\Delta^{(l-m_j)/2}\varphi_j^\varepsilon(y) - \Delta^{(l-m_j)/2}\varphi_j^\varepsilon(x_1, \cdots, x_{n-1}, y_n)] \\ & + \int_{-\infty}^{\infty} \Delta^{(1/2)(l-m_j)}\varphi_j^\varepsilon(x_1 \cdots x_{n-1}, y_n)K^*(x_n - y_n; t)dy_n \end{aligned}$$

where

$$K^*(x_n; t) = \iint_{-\infty}^{\infty} P\Delta^{(1/2)(n+s-l+m_j)}K_{j,s}(x, t)dx_1 \cdots dx_{n-1}\,.$$

Since the behavior of φ_j^ε with respect to the variables $x_1, \cdots, x_{n-1}$ is essentially independent of ε, the bracketed expression in the first term on the right is certainly bounded by const. $|x - y|^\alpha$, where the constant is independent of ε. Hence by the methods of Theorem 2.1, this first term and its Hölder difference quotients are bounded by a constant independent of ε. Also Lemma 4.3 with (4.6) and (4.14) tell us

$$\begin{aligned} K^*(x_n; t) &= \text{const.} \sum_{\varepsilon_n=\pm 1}\int_\gamma \frac{P(0, \xi_n, \tau)N_j(0, \xi_n, \tau)}{M^+(0, \xi_n, \tau)[x_n\xi_n + t\tau]}d\tau \\ &= \text{const.} \sum_{\xi_n=\pm 1}\int_\gamma \frac{P(0, \xi_n, \tau)N_j(0, \xi_n, \tau)}{M^+(0, \xi_n, \tau)}[(x_n\xi_n + t\tau)^{-1} \\ &\quad - (x_n\xi_n + t\tau_0)^{-1}]d\tau + \end{aligned}$$

$$+ \text{const.} \sum_{\xi_n=\pm 1} (x_n\xi_n + t\tau_0)^{-1} \int a(0, \xi_n, \tau) N_j(0, \xi_n, \tau) + \frac{\Sigma a_i(0, \xi_n) B_i(0, \xi_n, \tau) N_j(0, \xi_n, \tau)}{M^+(0, \xi_n, \tau)} d\tau ,$$

where τ_0 is a point on γ and on the imaginary axis: $\tau_0 = i\,|\tau_0|$. The first term on the right may be written as

$$\text{const.} \sum_{\xi_n=\pm 1} t \int_\gamma d\tau \int_{\tau_0}^{\tau} \frac{PN_j}{M^+} (x_n\xi_n + t\sigma)^{-2} d\sigma ,$$

hence estimated (Lemma 4.1) by const. $t(x_n^2 + t^2)^{-1}$. By virtue of the properties of N_j, the final term may be expressed as

$$\begin{aligned} \sum_{\xi_n=\pm 1} (x_n\xi_n + t\tau_0)^{-1} a_j(0, \xi_n) &= a_j(0, 1)\{(x_n + t\tau_0)^{-1} + (-x_n + t\tau_0)^{-1} \\ &\quad - (-x_n + t\tau_0)^{-1}(a_j(0, 1) - a_j(0, -1)) \\ &= - \frac{2a_j(0, 1)\tau_0 t}{x_n^2 + |\tau_0|^2 t^2} + \frac{\Delta a_j}{x_n - \tau_0 t} , \end{aligned}$$

where $\Delta a_j = a_j(0, 1) - a_j(0, -1)$. Thus

$$K^*(x_n; t) = K_1^*(x_n; t) + \text{const.} \frac{\Delta a_j}{x_n - \tau_0 t}$$

where $|K_1^*| < \text{const.}\, t(x_n^2 + t^2)^{-1}$. Therefore, using the fact that $\int_{-\infty}^{\infty} [t/((x_n - y_n)^2 + t^2)] dy_n$ is bounded independently of t and x_n, and $\Delta^{(1/2)(l-m_j)} \varphi_j^\varepsilon(x)$ is bounded independently of ε, we have (setting $\tilde{x} = 0$)

$$\begin{aligned} &Pw^\varepsilon(0, x_n) \\ &= W^\varepsilon + \text{const.}\, (\Delta a_j) \int_{-\infty}^{\infty} \zeta(|y_n|) \frac{d^{l-m_j}\eta(y_n)}{dy_n^{l-m_j}} [(x_n - y_n) - t\tau_0]^{-1} dy_n , \end{aligned}$$

where $|W^\varepsilon| <$ const. (independent of ε). Now this last integral may be written as $I_1 + I_2$, the two parts corresponding to the ranges $-\varepsilon < y_n < 1/2$ and $1/2 < y_n < 1$ (the integrand vanishes for $y_n < -\varepsilon$ and $y_n > 1$). At this point we set $x_n = 0$. Then since $(d^{l-m_j}/dy^{l-m_j})\eta^\varepsilon(y_n) = (l - m_j)!$ for $y_n > \varepsilon$ and $|y_n - t\tau_0|^{-1} < 4$ for $y_n > 1/2$, I_2 is easily estimated as

$$|I_2(0, t)| < \text{const.}\, (l - m_j)!$$

For I_1 we obtain

$$\begin{aligned} \text{const.}\, I_1(0, t) &= (\Delta a_j) \int_{-\varepsilon}^{1/2} D^{l-m_j} \eta^\varepsilon [-y_n - t\tau_0]^{-1} dy_n \\ &= -(\Delta a_j)(D^{l-m_j}\eta(y_n)) \log(-y_n - t\tau_0)]_{-\varepsilon}^{1/2} + \end{aligned}$$

$$
\begin{aligned}
&+ (\Delta a_j)\int_{-\varepsilon}^{1/2} D^{l-m_j+1}\eta^{\varepsilon} \log(-y_n - t\tau_0)dy_n \\
&= -(\Delta a_j)(l - m_j)! \log(-1/2 - t\tau_0) \\
&\quad + (\Delta a_j)\int_{-\varepsilon}^{\varepsilon} D^{l-m_j+1}\eta^{\varepsilon}(y_n) \log(-y_n - t\tau_0)dy_n \,.
\end{aligned}
$$

The first term in the last expression is independent of ε and t. By construction, $D^{l-m_j+1}\eta^{\varepsilon}(y_n)$ is a nonnegative function vanishing for $|y_n| > \varepsilon$ such that $\int_{-\varepsilon}^{\varepsilon} D^{l-m_j+1}\eta^{\varepsilon} dy_n = (l - m_j)!$. Since τ_0 is imaginary,

$$
-Re \log(-y_n - t\tau_0) = \tfrac{1}{2}|\log(y_n^2 + |t\tau_0|^2)| \geqq \tfrac{1}{2}|\log(\varepsilon^2 + t^2|\tau_0|^2)|
$$

for t and ε small enough and $|y_n| < \varepsilon$. Therefore the last integral in the expression for $I_1(0, t)$ is unbounded as ε and t approach 0, and the theorem is proved.

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POWERS OF A CONTRACTION IN HILBERT SPACE

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Introduction. Let H be a Hilbert space and P an operator with $\|P\| = 1$. Our main problem is to find the weak limits of $P^n x$ as $n \to \infty$. This is applied to Markov Processes and to Measure Preserving Transformations.

Markov Processes. Let (Ω, Σ, μ) be a measure space. Let x_n be a sequence of real valued measurable functions on Ω and:

1. $\mu(x_{n+\alpha} \in A \cap x_{m+\alpha} \in B) = \mu(x_n \in A \cap x_m \in B)$.
2. *Conditional probability that $x_k \in A$ given x_i and x_j, $i < j < k$, is equal to conditional probability that $x_k \in A$ given x_j.*

Let $I(\sigma)$ denote the characteristic function of σ. Define $P(n)$ by linear extension of:

$P(n)\, I(x_0 \in A) =$ *Conditional probability that $x_n \in A$ given x_0.*

Then:

1′. $\|P(1)\| = 1$

2′. $P(n) = P(1)^n$.

For details see [1] and [2].

We will study limits of

$$(P(1)^n\, I(x_0 \in A), I(x_0 \in B)) = \mu(x_n \in A \cap x_0 \in B)\ .$$

Many of the results here appear in particular cases in [1,] [2] and [3].

1. Reduction to unitary operators. For every $x \in H$

a. $\|P^{*k}P^kP^nx - P^nx\|^2 \leqq 2\|P^nx\|^2 - 2\,Re(P^{*k}P^kP^nxP^nx)$
$= 2(\|P^nx\|^2 - \|P^{n+k}x\|^2) \underset{n\to\infty}{\longrightarrow} 0$

b. $\|P^kP^{*k}P^nx - P^nx\|^2 \leqq \|P^{*k}P^kP^{n-k}x - P^{n-k}x\|^2 \underset{n\to\infty}{\longrightarrow} 0.$

Therefore:

If weak lim $P^{n_i}x = y$ then $P^{*k}P^ky = P^kP^{*k}y = y$ (here and elsewhere n_i or m_i will denote a subsequence of the integers). This means $\|y\| = \|P^ky\| = \|P^{*k}y\|$. Notice that if $P^*Px = x$ then $\|Px\|^2 = (P^*Px, x) = \|x\|^2$. On the other hand

$$\|Px\|^2 = (P^*Px, x) \leqq \|P^*Px\|\,\|x\| \leqq \|x\|^2 \text{ since } \|P\| = 1.$$

Hence if $\|Px\| = \|x\|$ then $(P^*Px, x) = \|P^*Px\|\,\|x\|$ and thus $P^*Px = x$.

THEOREM 1.1. *Let* $K = \{x \mid \|P^kx\| = \|P^{*k}x\| = \|x\|\ k = 1, 2, \cdots\}$

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then K is a subspace of H, invariant under P and P^*. On K the operator P is unitary. If $x \perp K$ then

$$\text{weak} \lim_{n\to\infty} P^n x = \text{weak} \lim_{n\to\infty} P^{*n} x = 0 .$$

Proof. It is only necessary to prove the last part. If $x \perp K$ and $y = \text{weak} \lim P^{n_i} x$ then by the preceding remark $y \in K$ hence $y = 0$. Now from the weakly sequentially compactness follows: weak lim $P^n x = 0$.

This theorem is a consequence of Theorem 2 of [9] and was reproduced here only because of the elementary proof.

If F is the selfadjoint projection on K and H is finite dimensional, then F is the spectral measure of the circumference of the unit circle in the sence of Dunford's spectral theory, with respect to P. This is no longer true when H is infinite dimensional and P a spectral operator (even a scalar type operator) in the sense of Dunford. These remarks are proved in [4].

LEMMA 2.1. *Let $y = \text{weak} \lim P^{n_i} x$. Then $\|y\|^2 \leqq \limsup |(P^n x, x)|$.*

Proof. Let $x = u + v$ where $u \in K$ and $v \perp K$. Then $y = \text{weak} \lim P^{n_i} u$, $\limsup |(P^n x, x)| = \limsup |(P^n u, u)|$. Now

$$|(y, P^k u)| = \lim_{i\to\infty} |(P^{n_i} u, P^k u)| = \lim |(P^{n_i - k} u, u)|$$

since $u \in K$. Thus

$$\|y\|^2 = \lim |(y, P^{n_i} u)| \leqq \limsup |(P^n u, u)| .$$

This could also be written in the form

$$\limsup |(P^n x, z)| \leqq \|z\| \limsup |(P^n x, x)|^{1/2} .$$

DEFINITION A. Let $H_0 = \{x \mid \lim (P^n x, x) = 0\}$.

THEOREM 3.1. *$x \in H_0$ if and only if weak $\lim P^n x = 0$, if and only if weak $\lim P^{*n} x = 0$. The set H_0 is a closed subspace of H containing $K^\perp$. If T commutes with P or with P^* and $x \in H_0$ then $Tx \in H_0$.*

Proof. The first parts of the theorem follow from Lemma 2.1 and Theorem 1.1. Now if $TP = PT$ and $P^n x \xrightarrow{w} 0$ then $P^n T x = TP^n x \xrightarrow{w} 0$.

Applications.

1. Markov processes.

a. If $\lim_{n\to\infty} \mu(x_n \in A \cap x_0 \in A) = 0$ then $\lim_{n\to\infty} \mu(x_n \in A \cap x_0 \in B) = 0$ and $\lim_{n\to\infty} \mu(x_0 \in A \cap x_n \in B) = 0$ for every set B.

b. Let $\lim \mu(x_n \in A \cap x_0 \in A) = \mu(x_0 \in A)^2$. Put $x = I(x_0 \in A) - \mu(x_0 \in A)$. (Provided that $\mu(\Omega) < \infty$ so that $1 \in L_2$).
Then

$$\begin{aligned}(P(1)^n x, x) &= (I(x_n \in A) - \mu(x_0 \in A), I(x_0 \in A) - \mu(x_0 \in A)) \\ &= \mu(x_n \in A \cap x_0 \in A) - \mu(x_0 \in A)^2 \to 0 \;.\end{aligned}$$

Thus for every Borel set B:

$$\lim\,(I(x_n \in A) - \mu(x_0 \in A), I(x_0\, B)) = 0$$

or

$$\mu(x_n \in A \cap x_0 \in B) \to \mu(x_0 \in A)\,\mu(x_0 \in B) \;.$$

Similarly

$$\mu(x_0 \in A \cap x_n \in B) \to \mu(x_0 \in A)\,\mu(x_0 \in B) \;.$$

2. Measure preserving transformations. Let φ be a M.P.T. on (Ω, Σ, μ). If $\mu(\varphi^{-n}(A) \cap A) \to 0$ then

$$\lim_{n\to\infty} \mu(\varphi^{-n}(A) \cap B) = \lim_{n\to\infty} \mu(A \cap \varphi^{-n}(B)) = 0 \;.$$

if $\lim \mu(\varphi^{-n}(A) \cap A) = \mu(A)^2$ and $\mu\,(\Omega) < \infty$ then

$$\begin{aligned}&\mu(\varphi^{-n}(A) \cap B) \to \mu(A)\mu(B) \\ &\mu(A \cap \varphi^{-n}(B)) \to \mu(A)\mu(B) \;.\end{aligned}$$

3. Measure theory. Let μ be a positive finite measure on Borel subsets of $(0, 2\pi)$. Define the operator P by $Pf(\vartheta) = e^{i\vartheta}f(\vartheta)$. Then H_0 is the set of all functions f such that

$$\int_0^{2\pi} e^{in\vartheta}\,|f(\theta)\,|^2\mu(d\vartheta) \to 0 \;.$$

Let $f \in H_0$ and $A_\varepsilon = \{\vartheta\,|\,|f(\vartheta)\,| \geqq \varepsilon\}$. Define $g_\varepsilon = 1/f$ on A_ε and zero elsewhere. Finally let

$$T_\varepsilon h(\vartheta) = g_\varepsilon(\vartheta)h(\vartheta) \;.$$

Then T_ε commutes with P and by Theorem 3.1

$$\int_A e^{in\vartheta}\mu(d\vartheta) \to 0$$

where $A = \cup A_\varepsilon$.

By taking unions of such sets one can prove:
There exists a set B such that for every h whose support is contained in B a.e.

$$\int e^{in\vartheta} |h(\vartheta)|^2 \mu(d\vartheta) \to 0$$

and this holds only for such functions.

2. **Positive contractions.** In this section we assume that H is the real Hilbert space $L_2(\Omega, \Sigma, \mu)$ where $\mu \geqq 0$ and $\mu(\Omega) = 1$. An operator S will be called positive if:

a. If $f \geqq 0$ a.e. than $Sf \geqq 0$ a.e.
b. $S1 = 1$.
c. $||S|| = 1$.

We will assume that P is positive. It is easily seen that so are P^*, $P^n P^{*n}$ and $P^{*n} P^n$.

LEMMA 1.2. *Let S be a positive operator on $L_2(\Omega, \Sigma, \mu)$. The space*

$$L = \{f \mid Sf = f\}$$

is generate by characteristic functions of a σ subfield, Σ', of Σ: $f \in L$ if and only if f is Σ' measurable.

Proof. Let Σ' contain all $\sigma \in \Sigma$ such that $SI(\sigma) = I(\sigma)$. If $Sf = f$ then

$$||f||^2 \geqq (S|f|, |f|) \geqq |(Sf, f)| = ||f||^2$$

hence $S|f| = |f|$ therefore if $f, g \in L$ so do $\max(f, g)$ and $\min(f, g)$. This shows in particular that Σ' is a field and since L is closed it is a σ field.

Now if $f \in L$ so does $f - c$ for any constant, thus it is enough to show that

$$\{\omega \mid f(\omega) > 0\} \in \Sigma' :$$

Let f_+ be the positive part of f, $2f_+ = |f| + f \in L$. Thus $\varepsilon^{-1} \min(\varepsilon, f^+) \in L$ but as $\varepsilon \to 0$ this converges to $I\{\omega \mid f(\omega) > 0\}$.

This Lemma was proved in [8].

THEOREM 2.2. *The space K is generated by characteristic functions of a σ subfield Σ_1 of Σ. If $\sigma \in \Sigma_1$ then $PI(\sigma) = I(\tau)$ where $\tau \in \Sigma_1$, similarly for P^*.*

Proof. The space K is the intersection of the space

$$\{f \mid \|P^n f\| = \|f\|\}, \quad \{f \mid \|P^{*n} f\| = \|f\|\} \quad n = 1, 2, \cdots$$

By Lemma 1 each of this is generated by a σ subfield of Σ. Thus K is generated by the intersection of these subfields.

Now if $\sigma \in \Sigma_1$ then $\sigma' = \Omega - \sigma \in \Sigma_1$ too. The functions $P(I(\sigma))$ and $P(I(\sigma'))$ are positive, bounded by 1 and $(P(I(\sigma)), P(I(\sigma'))) = (P^*P(I(\sigma)), I(\sigma')) = (I(\sigma), I(\sigma')) = 0$. Moreover $P(I(\sigma)) + P(I(\sigma')) = 1$, therefore, both functions are characteristic functions. As K is invariant under P these are characteristic functions of sets in Σ_1.

Let $I(A)$ and $I(B)$ belong to K. Then

$$P(I(A) \cdot I(B)) \leqq \min\{P(I(A)), P(I(B))\} = P(I(A)) \cdot P(I(B)) .$$

On the other hand

$$P^*[(P(I(A)) \cdot P(I(B))] \leqq I(A) \cdot I(B)$$

or

$$P(I(A)) \cdot P(I(B)) \leqq P(I(A) \cdot I(B)) .$$

Therefore

$$P(I(A) \cdot I(B)) = P(I(A)) \cdot P(I(B)) .$$

It could be shown that if $f, g \in K$ and $f \cdot g \in L_2$ then $P(fg) = Pf \cdot Pg$.

Thus if $Pf = \alpha f$ and $Pg = \beta g$ where $|\alpha| = |\beta| = 1$ then $f, g \in K$ and if $f \cdot g\, L_2$ then $P(fg) = \alpha\beta fg$.

If $Pf = \alpha f$ where $|\alpha| = 1$ let $f = |f|\, h$ then:

$$\|f\|^2 \geqq (P|f|, |f|) \geqq |(Pf, f)| = \|f\|^2 .$$

Therefore, $P|f| = |f|$ necessarily $Ph = \alpha h$. It follows that

$$P(|f|\, h^2) = \alpha^2 |f|\, h^2 .$$

This is a Theorem of [8].

Following [1] let us define:

Doeblin's Condition. There exists a positive finite measure ν define on Σ, and a positive ε such that: If $\nu(\sigma) < \varepsilon$ then for some n either

$$\|P^n(I(\sigma))\| < \mu(\sigma)^{1/2}$$

or

$$\|P^{*n}(I(\sigma))\| < \mu(\sigma)^{1/2} .$$

Using the same arguments as in Theorem 3.11 and its corollaries of [1] we conclude.

THEOREM 3.2. *If Doeblin's condition holds then* $\Sigma_1 = \{\sigma_1, \cdots, \sigma_n\}$ *where* σ_i *are disjoint sets such that*

1. $\bigcup_{i=1}^n \sigma_i = \Omega$
2. $P^n(I(\sigma_i)) = I(\sigma_i) = P^{*n}(I(\sigma_i))$.
3. *The operator* $P(P^*)$ *acts as a permutation on the* σ_i *sets.*
4. *For each* $f, g, \in L_2$

$$\lim_{k\to\infty} (P^{nk+d}f, g) = \sum_{i=1}^n \mu(\sigma_i)^{-1} \int_{\sigma_i} f(\omega)\mu(d\omega) \int_{P^d\sigma_i} g(\omega)\mu(d\omega)$$

where $P^d\sigma_i$ *denotes the set whose characteristic function is* $P^d(I(\sigma_i))$.

Thus if x_n is a Markov process and $\mu(\Omega) = 1$ then

$$\lim \mu(x_{kn+d} \in A \cap x_0 \in B) = \sum_{i=1}^n \mu(\sigma_i)^{-1}\mu(x_0 \in A \cap \sigma_i)\mu(x_0 \in B \cap P^d\sigma_i) .$$

For detailed proves of these results and treatment of the case $\mu(\Omega) = \infty$ in the case of Markov processes see [1] and [3].

Measure Preserving Transformations. Let φ be a measure preserving transformation on (Ω, Σ, μ). The operator P is defined on $L_2(\Omega, \Sigma, \mu)$ by $Pf = g$ where $g(\omega) = f(\varphi(\omega))$. It is a positive contraction. Thus the space K is generated by all characteristic functions f that satisfy $\|P^{*n}f\| = \|f\|$, for P is an isometry. Let the restriction of P to K be denoted by U and let Σ_1 be the Boolean algebra that generates K. On Σ_1 φ acts like a measure preserving invertable transformation. (It maps Σ_1 onto itself).

We will use here the terminology of [5]

THEOREM 4.2. *The transformation* φ *on* Σ *is ergodic, weakly mixing or strongly mixing, if and only if,* φ *on* Σ_1 *is ergodic, weakly mixing or strongly mixing, respectively.*

Proof. It is clear that if P satisfies any of the requirements so does U. Conversely:

a. Let U be ergodic. If P was not then for some nonconstant function f, $Pf = f$. But then $P^nf = P^{*n}f = f$ and $f \in K$, so U is not ergodic.

b. Let U be weakly mixing. Given $f = f_1 + f_2$ where $f_1 \in K f_2 \perp K$ then for every g

$$\frac{1}{n}\sum_{j=0}^{n-1} |(P^jf, g) - (f, 1)(1, g)| \leqq \frac{1}{n}\sum_{j=0}^{n-1} |(P^jf_1, g) - (f_1, 1)(1, g)|$$
$$+ \frac{1}{n}\sum_{j=0}^{n-1} |(P^jf_2, g) - (f_2, 1)(1, g)| .$$

The first term tends to zero because U is weakly mixing and g can be replaced by the projection of g on K. The second term is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1}|(P^j f_2, g)|$$

for $(f_2, 1) = 0$. Thus it tends to zero with $(P^n f_2, g)$.

c. Let U be strongly mixing. Put again $f = f_1 + f_2$ $P^n f_1$ tends weakly to $(f_1, 1)1 = (f, 1)1$ and $P^n f_2$ tends weakly to zero.

COROLLARY. *The transformation φ is weakly mixing, if and only if, P has on the unit circle no eigenvalue except for* 1 *which is a simple eigenvalue.*

This generalizes the 'Mixing Theorem' in [5] page 39.

Proof. The operator U satisfies the same condition and by the 'Mixing Theorem' is weakly mixing. By the previous theorem so is P.

3. The space H_c.

DEFINITION. $H_c = \{x \mid x \in K$ *and the set* $P^n x$ $n = 1, 2, \cdots$ *is conditionally compact*$\}$.

The set H_c is a subspace of H, invariant under P and P^*. $P^{n_i}x$ converges for $x \in K$ iff $(P^{n_i}x, P^{n_j}x) \longrightarrow_{n_i, n_j \to \infty} ||x||^2$. This is equivalent to $(P^{*n_i}x, P^{*n_j}x) \longrightarrow ||x||^2$ because P is unitary. Thus P could be replaced by P^* in the definition.

THEOREM 1.3. *The following conditions are equivalent*:
a. $x \in K$ *and* $P^n x$ *contains a convergent subsequence.*
b. *There exists a subsequence* m_i *such that* $x = \lim P^{m_i}x$.
c. $\limsup |(P^n x, x)| = ||x||^2$.

Proof.
$a \Rightarrow b$: Let $P^{n_i}x \longrightarrow y$ then

$$||x||^2 = ||y||^2 = \lim (P^{n_i}x, P^{n_{i-1}}x) = \lim (P^{n_i - n_{i-1}}x, x)$$

because $x \in K$.

Hence $||x - P^{n_i - n_{i-1}}x|| \longrightarrow 0$.

$b \Rightarrow c$: obvious.

$c \Rightarrow a$: Let $\lim |(P^{n_i}x, x)| = ||x||^2$ and weak $\lim P^{n_i}x = y$. Then $|(y, x)| = ||x||^2$ while $||y|| \leqq ||x||$ hence $y = \alpha x$ where $|\alpha| = 1$. From [7] page 79 $P^{n_i}x$ converges strongly to αx. Finall if $Z \in H_0$ then:

$$(Z, x) = \lim \alpha^{-1}(Z, P^{n_i}x) = \lim \alpha^{-1}(P^{*n_i}Z, x) = 0 .$$

It is clear that if $x \in H_c$ then condition (a) is satisfied hence the other conditions. In particular $H_c \perp H_0$.

THEOREM 2.3. *If $x \in H_c$ and $y = \lim_{i\to\infty} P^{n_i}x$ then there exists a subsequence k_i so that*

$$x = \lim P^{k_i}y .$$

Proof Let k_i be chosen so that

$$x = \lim P^{n_i+k_i}x .$$

Then

$$\lim \|x - P^{k_i}y\| = \lim \|P^{n_i}x - y\| = 0 .$$

4. Finitely many limits. Let x be such that the sequence $(P^n x, x)$ has finitely many limits. Let these be $c_1, c_2, \cdots, c_r$ where $|c_i| \leqq |c_{i+1}|$.

DEFINITION C. $L = \{z \mid P^n z = z$ for some $n\}$. If $z \in L$ then $az \in L$. If $z \in L$ and $y \in L$ then:

$$P^n z = z , \quad P^m y = y \Rightarrow P^{nm}(z + y) = z + y .$$

Thus L is a linear manifold, also $\bar{L} \subset H_c$.

If $z \in H$ let $\{z\}^0$ be the set consisting of z alone and $\{z\}^n$ be the set of all weak limits of $P^m y$ where $y \in \{z\}^{n-1}$.

Let $x = x_0 + x_1$ where $x_0 \in H_0$ $x_1 \perp H_0$. Then

$$(P^n x, x) = (P^n x_0, x_0) + (P^n x_1, x_1), \lim (P^n x_0, x_0) = 0 .$$

Thus we will assume that $x \perp H_0$.

LEMMA 1.4. *For some k $\{x\}^k \cap L \neq 0$.*

Proof. Let $0 \neq y \in \{x\}^1$ then for every n $(y, P^n x)$ is equal to one of the values c_i and:

a. For every $n \geqq 0$ $(P^n y, y)$ can assume only the values c_i $1 \leqq i \leqq r$.

Let $(y, y) = |c_i|$. If for some k $|(P^k y, y)| = (y, y)$ then $P^k y = \lambda y$ with $|\lambda| = 1$. Thus λ must be a root of one for $(P^{nk}y, y) = \lambda^n(y, y)$ assumes finitely many values. Therefore in this case $y \in L$.

If $|(P^n y, y)| < (y, y)$ for every n then

$$\limsup_{n\to\infty} |(P^n y, y)| < (y, y) .$$

Also $\limsup (P^n y, y,) \neq 0$ for $y \perp H_0$. Thus we may choose a subsequence n_i so that $P^{n_i}y$ will converge weakly to $z \neq 0$. Now z satisfies a and $||z|| < ||y||$ by Lemma 2.1.

This procedure cannot be continued more than r times thus at some stage we must get an element of L.

LEMMA 2.4. *If u is the projection of x on $\bar{L}$ then $u \in L$.*

Proof. Let $0 \neq y \in \{x\}^k \cap L$. Then $y \in \{u\}^k + \{x - u\}^k$. Now $y \in L$ and $x - u \perp L$. Also L is invariant under P and P^* hence $\{x - u\}^k \perp L$ and $y \in \{u\}^k$. By Theorem 2.3 $u \in \{\overline{P^n y}\}$ which is a finite set in L.

THEOREM 3.4. *If the sequence $(P^n x, x)$ has finitely many limits then $x = x_0 + x_1$ where $x_0 \in H_0$ and $x_1 \in L$.*

Proof. Let $x_1 = u + v$ where $u \in L$ (by Lemma 2.4.) and $v \perp L$. Now $(P^n v, v) = (P^n x_1, x_1) - (P^n u, u)$ has finitely many limits and by Lemma 1.4 cannot be orthogonal to L unless it is zero.

If limit $(P^n x, x)$ exists then $Px_1 = x_1$.

If L is one dimensional (for instance ergodic transformations) then the conditions of Theorem 3.4 imply that $Px_1 = x_1$.

THEOREM 4.4. *Let $A = \{x$ the sequence $(P^n x, x)$ has finitely many limits$\}$. If linear combinations of elements of A are dense in H, then the eigenvalues of P on the circumference of the unit circle, are roots of 1.*

Proof. Let $Px = \lambda x$ where $|\lambda| = 1$. Let $x_i \in A$ and $y = \Sigma a_i x_i$ where $||x - y|| < 1/2 ||x||$.
Since $x \perp H_0$ we may assume that for some integers k_i $P^{k_i} x_i = x_i$. Hence for $k = k_1 k_2 \cdots k_n$ we have $P^k y = y$. Thus

$$\lambda^{km} x = P^{km} x = y + P^{km}(x - y) .$$

Therefore

$$|\lambda^{km} - 1| \, ||x|| \leqq ||\lambda^{km} x - y|| + ||y - x|| < ||x|| .$$

This equation cannot be satisfied for all values of m unless λ^k is a root of 1.

5. Semi groups of contractions. Let $P(t)$ be a strongly continuous semi group of contractions $0 \leqq t$. For every $\delta > 0$ $P(\delta)$ defines the subspace $K(\delta)$ as in Theorem 1.1.

LEMMA 1.5. *$x \in K(\delta)$ if and only if*

$$||P(t)x|| = ||P(t)^*x|| = ||x|| \quad 0 \leqq t < \infty \ .$$

Proof. Trivially the condition is sufficient. If $x \in K(\delta)$ and $t \leqq n\delta$ then

$$||x|| = ||P(n\delta)x|| = ||P(n\delta - t)\, P(t)x|| \leqq ||P(t)x|| \leqq ||x|| \ .$$

Thus $||P(t)x|| = ||x||$ and similarly $||P(t)^*x|| = ||x||$.

Thus all the spaces $K(\delta)$ are the same and will be denoted by K.

THEOREM 2.5. *The space K is invariant under $P(t)$ and $P(t)^*$ for all t. On K $P(t)$ is unitary. If $x \perp K$ then*

$$\text{weak} \lim_{t\to\infty} P(t)x = 0$$

and by symmetry

$$\text{weak} \lim_{t\to\infty} P(t)^*x = 0 \ .$$

Proof It was shown that $K = K(t)$ hence by Theorem 1.1 K is invariant under $P(t)$ and $P(t)^*$ and $P(t)$ is unitary on K.

Let $x \perp K$ and let $y \in H$ and $\varepsilon > 0$ be given. Choose η so that

$$||P(s)x - x|| < \varepsilon. \quad \text{if } s \leqq \eta \ .$$

Choose n_0 so that

$$|(P(n\eta)x, y)| < \varepsilon \quad \text{if } n \geqq n_0 \ .$$

This is possible by Theorem 1.1. If

$$(n+1)\eta \geqq t \geqq n\eta > n_0\eta$$

then

$$|(P(t)x, y)| \leqq |(P(n\eta)x, y)| + |(P(t)x - P(n\eta)x, y)| \ .$$

The first term is less than ε because $n > n_0$. The second term is bounded by

$$\begin{aligned} ||y||\,||P(t)x - P(n\eta)x|| &= ||y||\,||P(n\eta)\,(P(t-n\eta)x - x)|| \\ &\leqq ||y||\,||P(t-n\eta)x - x|| \leqq ||y||\varepsilon \end{aligned}$$

for $0 \leqq t - n\eta \leqq \eta$.

This is proved also in [9] Theorem 4.

Let us assume in this section:

(*) *For some $t_0 > 0$ the operator $P(t_0)\,P(t_0)^*$ is the sum of a compact operator and an operator of norm less then one.*

This is equivalent to:

(**) *For some* $0 < t_0$ *the point* 1 *is isolated in the spectrum of* $P(t_0)\,P(t_0)^*$ *and the space of eigenvectors corresponding to it is finite.*

It is clear that (**) implies (*). Now if 1 is not an isolated point of the spectrum, with finite eigenvectors space, there is a sequence of orthonormal vectors x_n such that

$$\|P(t_0)\,P(t_0)^* x_n - x_n\| \to 0 .$$

(We use here the fact that $P(t_0)\,P(t_0)^*$ is self adjoint). Let

$$P(t_0)\,P(t_0)^* = A + B$$

where B is compact and $\|A\| < 1$. Then

$$\|Ax_n + Bx_n - x_n\| \to 0 .$$

But B is compact hence $Bx_n \to 0$ hence

$$\|Ax_n - x_n\| \to 0$$

and 1 is the spectrum of A contrary to assumption.

It is easily seen that $P(t)\,P(t)^*$ satisfy, also, the condition if $t > t_0$: $P(t)\,P(t)^* = P(t-t_0)P(t_0)P(t_0)^*P(t-t_0)^*$. Let

$$K(t) = \{x \mid \|P(t)^*x\| = \|x\|\} = \{x \mid P(t)P(t)^*x = x\} .$$

Then $K(t_1) \subset K(t_2)$ if $t_1 > t_2$ and $K(t)$ is finite dimensional when $t \geqq t_0$.

For some $s > 0$ dim $K(s)$ is minimal hence $K(s) = K(s+h)$ for all $h \geqq 0$. Let us denote $K(s)$ by K.

LEMMA 3.5. *The space* K *is invariant under* $P(h)^*$ *and* $P(h)$ *for all* $h > 0$.

Proof. If $x \in K$ then $x \in K(s+h)$ hence

$$\|P(s+h)^*x\| = \|x\|$$

hence

$$\|x\| = \|P(s)^*P(h)^*x\| \leqq \|P(h)^*x\| \leqq \|x\|$$

or $P(h)^*x \in K$.

Now on the finite dimensional space K, the operator $P(h)^*$ is norm preserving and therefore onto.

If $x \in K$ then for some $y \in K$ $P(h)^*y = x$ and $\|x\| = \|y\|$. Thus $P(h)x = y \in K$.

We may assume that $s \geqq t_0$.

The subspace $K^\perp$ is also invariant under $P(t)$ and $P(t)^*$. Now

$P(s)\, P^*(s)$ is quasi compact on K and

$$(P(s)\, P^*(s)x, x) < 1 \quad x \in K^{\perp} .$$

Hence on $K^{\perp}$ $\|P(s)\| = c < 1$:

The operator $P(s)$ is quasi compact on H (in the sense of (*).

Let A be the infinitesimal generator of $P(t)$ then:

1. On K the operator $(1/i)A$ is self adjoint.
2. On $K^{\perp}$

$$\sigma(\mathrm{A}) \subset \{\lambda \mid Re\, \lambda \leqq \omega_0\}$$

where

$$\omega_0 = \lim_{t \to \infty} t^{-1} \log \|P(t)\| .$$

See [6] corollary to Theorem 11.5.1

Now

$$\omega_0 = \lim_{n \to \infty} (ns)^{-1} \log \|P(ns)\| \leqq \lim_{n \to \infty} (ns)^{-1} \log \|P(s)\|^n \leqq s^{-1} \log c < 0 .$$

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THE STRUCTURE OF THE ORBITS AND THEIR LIMIT SETS IN CONTINUOUS FLOWS

N. E. FOLAND

1. Introduction. If f is a mapping of a product space $X \times Y$ into a space Z, then the image of $(x, y) \in X \times Y$ under f is denoted by xy. A continuous flow $\mathscr{F}$ on a metric space X is a continuous mapping f of the product space $X \times R$, where R is the space of real numbers, onto X such that (1) for each $r \in R$, xr is a homeomorphism of X onto X and (2) for each $x \in X$ and $r, s \in R$, $(xr)s = x(r+s)$.

For each $x \in X$ the sets $O(x) = \{xr \mid r \in R\}$, $O_+(x) = \{xr \mid r \geqq 0\}$, $O_-(x) = \{xr \mid r \leqq 0\}$ are called the orbit, positive semi-orbit and negative semi-orbit of x under $\mathscr{F}$, respectively. The orbit $O(x)$ is either (1) a point, (2) a simple closed curve, or (3) a one-to-one and continuous image of R. In general one can not replace (3) by (3′) a homeomorphic image of R.

Bebutoff [1] has given necessary and sufficient conditions that the entire collection of orbits of a continuous flow be homeomorphic to a family of parallel lines in Hilbert space. In the second section of this paper we solve the simpler problem of describing those points of an arbitrary metric space with orbits homeomorphic to R. These will be the points which are neither positively nor negatively recurrent.

In the last section we discuss the structure of the orbit family of continuous flows on a 2-cell, with special attention being given to the α and ω limit sets of an orbit [5; 6; 7]. The author wishes to acknowledge the referee's assistance in condensing the original paper.

2. The topological nature of the orbits under a continuous flow. Consider a metric space $\{X, \rho\}$ and a continuous flow $\mathscr{F}$ on X. The following definitions are well-known in Topological Dynamics:

DEFINITION 1. A point $x \in X$ is said to be a *rest* point under $\mathscr{F}$ if

$$xr = x$$

for each $r \in R$.

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DEFINITION 2. A point $x \in X$ is said to be *periodic* under $\mathscr{F}$ and $\mathscr{F}$ is said to be periodic at x if there is a $t \in R, t \neq 0$, for which $xt = x$. If $\mathscr{F}$ is periodic at a non-rest point x, then the smallest positive number $w \in R$ for which $xw = x$ is called the *primitive period* of x.

DEFINITION 3. A point $x \in X$ is said to be positively (negatively) *recurrent* under $\mathscr{F}$ if for each $\varepsilon > 0$ there exists a strictly increasing (decreasing) sequence $\{r_i\}$ of points of R such that $\lim_{i\to\infty} r_i = +\infty(-\infty)$ and

$$\rho(xr_i, x) < \varepsilon$$

for all i.

THEOREM 1. *The point x is neither positively nor negatively recurrent if and only if ϕ: $R \to 0(x)$ defined by $\phi(t) = xt, t \in R$, is a homeomorphism.*

Proof. Since the mapping $f: X \times R \to X$ is continuous, it follows that ϕ is continuous. Assume that x is neither positively nor negatively recurrent. It follows that x is not periodic and thus ϕ is a one-to-one map of R onto $O(x)$. Let $\{xt_i \mid i = 1, 2, \cdots\}$ be a sequence of points of $O(x)$ converging to xt_0. To prove that ϕ^{-1} is continuous it is sufficient to prove that $\lim_{i\to\infty} t_i = t_0$. If this is not the case, either the sequence $\{t_i\}$ contains a subsequence which is unbounded and x is either positively or negatively recurrent or the sequence $\{t_i\}$ contains a subsequence converging to $s \neq t_0$, and x is a periodic point. We conclude that ϕ^{-1} is continuous and ϕ is a homeomorphism.

Now suppose ϕ is a homeomorphism and suppose x is positively recurrent. Then there exists a sequence $\{t_i \mid t_i \in R, i = 1, 2, \cdots\}$ with $\lim_{i\to\infty} t_i = +\infty$ and such that $\lim_{i\to\infty} xt_i = x$. But then, since ϕ^{-1} is continuous,

$$\infty = \lim_{i\to\infty} t_i = \lim_{i\to\infty} \phi^{-1}(xt_i) = \phi^{-1}(x) = 0 .$$

Thus x is not positively recurrent. Similarly, x is not negatively recurrent.

The proof is completed.

THEOREM 2. *Let $x \in X$ and let $O(x)$ be homeomorphic to R. Then x is neither positively nor negatively recurrent.*

Proof. By assumption there exists a homeomorphism h of R onto $O(x)$. Then x is not a periodic point. For if x is a periodic point, $O(x)$ is either a point or a simple closed curve which is homeomorphic to a circle. It follows that $xt_1 = xt_2$ implies $t_1 = t_2$.

Let $r \in R$ and let $h(r) = xt$. Then t is uniquely determined. Let $\psi\colon R \to R$ be defined by $\psi(r) = t$. Since h is an onto homeomorphism, ψ is an onto map and ψ is one-to-one. Let $\phi\colon R \to O(x)$ be defined by $\phi(t) = xt$, $t \in R$. Then ϕ is continuous, onto and one-to-one, and $\psi^{-1} = h^{-1}\phi$ and thus ψ^{-1} is continuous. Now ψ^{-1} is a continuous, one-to-one, onto map of R onto R and hence is a homeomorphism. Since $\phi = h\psi^{-1}$, it follows that ϕ is a homeomorphism and from Theorem 1 we infer that x is neither positively nor negatively recurrent.

3. The stucture of the α-and ω-limit sets in a continuous flow on a 2-cell. Let X be an open or closed 2-cell, that is, a homeomorphic image of the interior of the unit circle or of the unit disk. Let $\mathscr{F}$ be a continuous flow on X and let $A \subset X$ be the set of rest points under $\mathscr{F}$.

We recall the following definition due to Whitney [9] (cf., also, [8]).

DEFINITION 4. A closed set $S \subset X$ is a *local section* of $\mathscr{F}$ if there exists a $\tau \in R$, $\tau > 0$, such that for each $x \in S$

$$\{xt \mid |t| \leqq \tau\} \cap S = x .$$

If $x \in S$, then S is called a *local section through* x.

Whitney [9] (cf., also, [5]) proved, for the spaces under discussion, that for each $x \in X - A$ there is an arc $S \subset X$ such that S forms a local section of $\mathscr{F}$ through x. Using this Whitney [9] (cf., also, [5]) proved the following:

LEMMA 1. *If $x \in X - A$, then there exists a local section S of $\mathscr{F}$ through x such that the set*

$$E = \{yt \mid y \in S, \mid t \mid \leqq \tau\}$$

can be mapped homeomorphically onto the closed rectangle $|u| \leqq 1$, $|v| \leqq 1$ in such a way that the arcs $\{yt \mid |t| \leqq \tau\}$, for $y \in S$, become the lines $v =$ constant of the rectangle while S has image $u = 0$, $|v| \leqq 1$.

The local section S of Lemma 1 divides the interior of the set E into two disjoint subregions.

DEFINITION 5. Let x, S, and E be as in Lemma 1. That one of the two regions into which S separates E, which the orbit $O(x)$ of x enters under increasing values of r, will be termed (after Bendixson [3]) the *positive side* S^+ of S. The other region will be termed the *negative side* S^- of S.

LEMMA 2. *Let x, S, and E be as in Definition 5. Then each orbit which enters E crosses S from S^- to S^+ under increasing values of r.*

Proof. Suppose the contrary. Then there exists a sequence $\{y_i\}$ of distinct points of S converging to $y \in S$ such that the orbit $O(y_i)$ of each y_i enters one of the two regions S^+ or S^- under increasing values of r, while the orbit $O(y)$ of y enters the other region under increasing values of r. Thus for any t such that $0 < t < \tau$ the points $y_i t$ and yt lie in disjoint subregions of E. This is impossible since $\lim_{i\to\infty} y_i t = yt$.

Let S be any local section of $\mathscr{F}$ and let $y \in S$. Let S^- and S^+ be as in Definition 5, it follows from Lemma 2 that S^- and S^+ are independent of y. Thus if $O(x)$ is any orbit such that $O(x) \cap S \neq 0$, then each crossing of S by $O(x)$ is from S^- to S^+ under increasing values of r. Let the orbit $O(x)$ meet S in successive points x' and x'' in the positive direction on $O(x)$, then x is an interior point of X and $(x'x'') \cup S_1$, where $(x'x'')$ and S_1 denote the subarcs joining x' and x'' of $O(x)$ and S, respectively, is a simple closed curve lying in the interior of X. Let $C = (x'x'') \cup S_1$, it follows that $X - C$ consists of exactly two components. Denote by C^+ that component of $X - C$ which lies on the positive side S^+ of S along S_1 and by C^- the other component of $X - C$. Any simple closed curve C determined in this manner will be termed a harbor [7].

LEMMA 3. *If C is a harbor, then the positive semi-orbit $O_+(y)$ of each $y \in C^+$ lies in C^+, and the negative semi-orbit $O_-(y)$ of each $y \in C^-$ lies in C^-.*

Proof. If $y \in C^+$ and $O_+(y) \cap C^- \neq 0$, then $O_+(y)$ must first cross S on S_1 and hence cross S from S^+ to S^- under increasing values of r which is impossible. If $y \in C^-$ and $O_-(y) \cap C^+ \neq 0$, then $O_-(y)$ must first enter C^+ on S_1 crossing from S^- to S^+ under decreasing values of r which is also impossible.

Using Lemma 3 one can construct a very short proof of the following result proved by Bohr and Fenchel ([4], Vol. II, C38).

If x is a positively or negatively recurrent point of X under $\mathscr{F}$, then x is periodic under $\mathscr{F}$.

Since the only points of X with orbits not homeomorphic to R are those which are either positively or negatively recurrent, it follows that (3) may be replaced by (3′) for continuous flows on 2-cell. Thus if $x \in X$, then the orbit $O(x)$ is either a point, a simple closed curve, or a homeomorphic image of R.

DEFINITION 6. A point $y \in X$ is said to be an *ω-limit (α-limit) point* of an orbit $O(x) \subset X$ if there exists a strictly increasing (decreasing) sequence $\{r_i\}$ of points of R such that $\lim_{i\to\infty} r_i = +\infty(-\infty)$ and $\lim_{i\to\infty} xr_i = y$. The set of all ω-limit (α-limit) points of an orbit $O(x)$ will be denoted by $w(x)\,(\alpha(x))$.

THEOREM 3. *If x is a nonperiodic point of X under $\mathscr{F}$, then $\omega(x) \cap \alpha(x) \subset A$.*

Proof. Suppose there exists a point y in the set $\omega(x) \cap \alpha(x) - A$. Choose a local section S of $\mathscr{F}$ through y. Then, since $y \in \omega(x) \cap \alpha(x)$, $O_+(x)$ and $O_-(x)$ must both cross S an infinite number of times near y. Thus an arc $(x'x'')$ of $O(x)$ and a subarc S_1 of S form a harbor C. Let p and q denote the end-points of S and assume the labeling so that the order p, x', x'', q holds on S. Then the half-open subarc $(px') - x'$ of S lies in C^- while the half-open subarc $(x''q) - x''$ of S lies in C^+. Now $y \notin S_1$ since $O(x)$ can not cross S on S_1. If $y \in (px')$, then $y \notin \omega(x)$ since the positive semi-orbit $O_+(x'')$ from x'' on lies in C^+. Thus $y \notin (px')$. If $y \in (x''q)$, then $y \notin \alpha(x)$ since the negative semi-orbit $O_-(x')$ from x' on lies in C^-. Thus $y \notin (x''q)$. This is a contradiction of the fact that $y \in S$ and $y \in \omega(x) \cap \alpha(x)$. Hence the theorem is proved.

Throughout the remainder of this section X shall denote a closed 2-cell. Then X contains at least one point a such that a is a rest point under the continuous flow $\mathscr{F}$ [2]. Thus $A \neq 0$. Let F denote the family of orbits $\{O(x) \mid x \in X - A\}$. Then each member of F is either an open arc or a simple closed curve. Since X is compact each of the sets $\omega(x)$, $\alpha(x)$ for any $x \in X$ is a non-null closed and connected subset of X and is the union of points of A and curves of F [8]. It follows from a theorem due to Kaplan [5] that $\omega(x)(\alpha(x))$ is identical with any nondegenerate periodic orbit contained in $\omega(x)$ $(\alpha(x))$. Thus the set $\omega(x)(\alpha(x))$ is either the union of points of A and open arcs of F or a simple closed curve of F.

THEOREM 4. *Let A be a totally disconnected set. If z is a nonperiodic point in $\overline{O(x)} - O(x)$, then $O(z)$ is an open arc whose closure, $\overline{O(z)}$, is either a closed arc with end-points in A or a simple closed curve consisting of the orbit $O(z)$ together with a point of A.*

Proof. The theorem will be proved when it is shown that $\omega(z) \subset A$ and $\alpha(z) \subset A$, since each of the sets $\omega(z)$, $\alpha(z)$ is connected and A is totally disconnected.

Thus suppose $y \in \omega(z) - A$ and let S be a local section of $\mathscr{F}$ through y. Then $O_+(z)$ must cross S an infinite number of times near y. Thus there exists successive points z', z'' in the positive direction on $O_+(z)$ such that $z \in O_-(z') - z'$ and $z', z'' \in S$. Let C be the harbor formed by the arc $(z'z'')$ of $O(z)$ and the subarc S_1 of S between z' and z''. As in the proof of Theorem 3, y and $O_+(z'') - z''$ lie together in C^+ while $O_-(z) \subset C^-$. Since $\overline{O(x)} - O(x) \neq 0$, x is a non-periodic point. Thus, by Theorem 3, z is in exactly one of the sets $\omega(x), \alpha(x)$. $z \in \omega(x)$ implies $\overline{O(z)} \subset \omega(x)$ and $z \in \alpha(x)$ implies $\overline{O(z)} \subset \alpha(x)$. If $z \in \omega(x)$, then $O_+(x)$ must cross S entering C^+ under increasing values of r. By Lemma 3, $O_+(x)$, from where it enters C^+ on, lies in C^+. Then $\omega(x) \subset C^+$ which is impossible since $\overline{O(z)} \subset \omega(x)$, $O_-(z) \subset C^-$ and $\alpha(z) \neq 0$. Thus $z \in \alpha(x)$. Let $U(z)$ be a neighborhood of z such that $U(z) \subset C^-$, and let x' be a point on $O(x)$ in $U(z)$. Then, by Lemma 3, $O_-(x') \subset C^-$. This together with $y \in C^+$ implies $y \notin \alpha(x)$ which is a contradiction of $\overline{O(z)} \subset \alpha(x)$. Hence $z \notin \alpha(x)$. But z is in one or the other of the sets $\omega(x), \alpha(x)$. Thus the assumption that $\omega(z) - A \neq 0$ is false and $\omega(z) \subset A$. In a like manner $\alpha(z) \subset A$. It follows that $\lim_{t\to+\infty} zt = \omega(z) \in A$ and $\lim_{t\to-\infty} zt = \alpha(z) \in A$. Thus $\overline{O(z)} = O(z) \cup \omega(z) \cup \alpha(z)$ is a closed arc with end-points in A or a simple closed curve consisting of $O(z)$ and a point of A according as $\omega(z) \neq \alpha(z)$ or $\omega(z) = \alpha(z)$.

The proof of the theorem is completed.

THEOREM 5. *Let A be a totally disconnected set and let $x \in X - A$ be such that $\overline{O(x)} \cap A \neq 0$. Let $a \in \omega(x) \cap A$ $(\alpha(x) \cap A)$, and suppose $\omega(x) \neq a$ $(\alpha(x) \neq a)$. Let $G(a) = \{O(z) \mid \overline{O(z)} = O(z) \cup a, z \in \overline{O(x)} - O(x), z \neq a\}$. Then $G(a)$ is an at most countable set of open arcs, and if $G(a) = \bigcup_{n=1}^{\infty} D_n$ is infinite, and if $\{y_n\}$ is any sequence of points with $y_n \in D_n$, then $\lim_{n\to\infty} y_n = a$.*

Proof. Let $D \in G(a)$. It follows that D is the orbit of a point $z \in \overline{O(x)} - O(x) - A$, and $\bar{D} = D \cup a$ is a simple closed curve. For each $D \in G(a)$, let D^i denote the interior of $\bar{D}$. Then if D_j and D_k are distinct members of $G(a)$, the sets D_j^i and D_k^i are either disjoint or one is a proper subset of the other. If $D_j^i \subset D_k^i$, then D_k^i must contain $G(a) - D_k$, since then $O(x) \subset D_k^i$. Such a member of $G(a)$ will be termed a boundary arc of $G(a)$. Clearly $G(a)$ can contain at most one boundary arc. Also, if D_j and D_k are distinct members of $G(a)$, neither of which is a boundary arc of $G(a)$, then D_j^i and D_k^i are disjoint. It follows that $G(a)$ consists of an at most countable set of open arcs.

Suppcse that $G(a) = \bigcup_{n=1}^{\infty} D_n$ is an infinite set of open arcs. Let $\{y_n\}$ be any sequence of points with $y_n \in D_n$, and let $\lim_{n\to\infty} y_n = z$. The proof of the theorem is completed by showing $z = a$. In order to show $z = a$, it is first shown that $z \in A$. If $z \notin A$, let S be a local section of $\mathscr{F}$ through z. If D_k is not a boundary arc of $G(a)$, then $z \notin D_k^i$. Since $G(a)$ has at most one boundary arc, the removal of a boundary are will not alter $\lim_{n\to\infty} y_n$. Thus suppose $G(a)$ has no boundary arc. Then $z \in X - H$, where $H = \bigcup_{n=1}^{\infty} D_n^i$. No D_n can cross S more than once, since D_n is an orbit and $\bar{D}_n$ is a simple closed curve. For each n, let D_n^e denote the exterior of $\bar{D}_n$. Then if S crosses D_n, S must pass from D_n^i to D_n^e. But then S can cross at most two D_ns. Hence $z \in A$, for $\lim_{n\to\infty} y_n = z$ implies that an infinite number of $O(y_n) = D_n$ intersect S. That $z = a$ is shown next. Consider the subarcs of D_n joining y_n and a. From this sequence of arcs we can choose a converging snbfequence converging to a set B. It follows that both z and a are in B. If $b \in B, a \neq b \neq z$, then b is the limit of a sequence $\{y'_{n_k}\}$ with $y'_{n_k} \in D_{n_k}$. Thus, by the same argument used to show $z \in A$, it can be shown that $b \in A$. But the set B is connected [10] and A is totally disconnected, hence $z = a = B$.

This theorem is a generalization of a theorem due to Kaplan [7].

THEOREM 6. *Let A be a totally disconnected set and let $x \in X - A$ be such that $\omega(x) \cap A = \bigcup_{n=1}^{k} a_n$ $(\alpha(x) \cap A = \bigcup_{n=1}^{k} a_n)$. Then $\omega(x)$ $(\alpha(x))$ consists of a finite number of open arcs, each of which is an orbit joining distinct elements of $\bigcup_{n=1}^{k} a_n$ together with $\bigcup_{n=1}^{k} G(a_n)$.*

Proof. Consider the sets $G(a_n)$, $n = 1, 2, \cdots, k$. Let $G^*(a_n)$ denote the point set union of all open arcs in $G(a_n)$. One can easily show that the point set closure of $G^*(a_n)$ is $G^*(a_n) \cup a_n$ and that $G^*(a_i) \cap G^*(a_j) = 0$ for $a_i \neq a_j$. The set $w(x)$ $(\alpha(x))$ is connected. By Theorem 4, the orbit of any point in $\omega(x) - \bigcup_{n=1}^{k} G(a_n)$ $(\alpha(x) - \bigcup_{n=1}^{k} G(a_n))$ is an open arc terminating at distinct points of $\bigcup_{n=1}^{k} a_n$. Thus each a_i must be joined to some a_j $(i \neq j)$ by the orbit of some point in $\omega(x)$ $(\alpha(x))$. Clearly, no two a_n's are connected by more than two such arcs. Hence, $w(x)$ $(\alpha(x))$ contains only a finite number of arcs joining distinct a_n's. Thus the theorem is proved.

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ANALYTIC MEASURES

FRANK FORELLI

1. In this note we consider from a measure-theoretic point of view the Helson-Lowdenslager generalization to compact abelian groups of the F. and M. Riesz theorem on analytic measures [3]. Our contribution to this matter is the *proof* of Theorem 1. From Theorem 1 the Helson-Lowdenslager generalization readily follows. That which is new here is the proof of Theorem 1. For the most part, the statement of Theorem 1 can be obtained from the generalization in [3] of the F. and M. Riesz theorem.

We have a second theorem (Theorem 2) which is about analytic measures (Theorem 1 is not) and which adds to the information about analytic measures given in [3]. Although Theorem 2 does not appear in [3] it can be obtained from the generalization in [3] of the F. and M. Riesz theorem, and we will indicate how this may be done at the end of the proof of Theorem 2. In recent work (completed before our work was undertaken) de Leeuw and Glicksberg have found a generalization of the F. and M. Riesz theorem which includes Theorem 2 and much more. Nevertheless, it is hoped that the proof of Theorem 2 given here will be of interest.

Although the proof of Theorem 1 is given in the language of harmonic analysis, we wish to point out that the argument is valid in the more general context of Dirichlet algebras. This however is not true of Theorem 2.

2. Throughout G will denote a compact abelian group with Haar measure σ and with dual group Γ. Following [3], a subset S of Γ is said to be a half-space if S is closed under multiplication and if for each χ in Γ one and only one of the following occurs:

$$\chi = 1$$
$$\chi \in S$$
$$\bar{\chi} \in S .$$

We will assume that Γ contains half-spaces and in all that follows S will denote a fixed half-space in Γ.

$M(G)$ is the space of all regular Borel measures on G. ν in $M(G)$ is said to be analytic (more accurately analytic with respect to S) if the Fourier transform $\hat{\nu}$ vanishes on $\bar{S}$:

$$\hat{\nu}(\chi) = \int \bar{\chi} d\nu = 0$$

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for χ in $\bar{S}$.

$\boldsymbol{A}$ is the algebra of continuous analytic functions on $\boldsymbol{G}$: f belongs to $\boldsymbol{A}$ if and only if f belongs to $\boldsymbol{C}(G)$ and

$$\hat{f}(\chi) = \int \bar{\chi} f d\sigma = 0$$

for χ in $\bar{S}$. A nonnegative measure μ in $\boldsymbol{M}(G)$ with total mass one and such that

$$\int fgd\mu = \left(\int fd\mu\right)\left(\int gd\mu\right)$$

for all f and g in $\boldsymbol{A}$ is called a representing measure for $\boldsymbol{A}$. Among the representing measures is the measure σ.

It is important that the linear space of analytic measures is an $\boldsymbol{A}$-module: if ν is analytic and f is in $\boldsymbol{A}$, then $f\nu$ is also analytic.

The classical example of this abstract situation is the case in which $\boldsymbol{G}$ is the circle group, Γ the integer group, and $\boldsymbol{S}$ the positive integers. $\boldsymbol{A}$ is then the algebra of continuous functions on the circle whose Fourier coefficients vanish for negative indices, and the representing measures (other than normalized Lebesgue measure and the unit point masses) are the Poisson kernels μ_r $(0 < r < 1)$ and their translates:

$$\mu_r = P_r\sigma$$
$$P_r(x) = \Sigma r^{|n|}e^{inx} .$$

The celebrated theorem of F. and M. Riesz [4] states: *An analytic measure on the circle is absolutely continuous with respect to Lebesgue measure.*

As usual, $\|f\|_\infty$ is the supremum norm of f for f in $\boldsymbol{C}(G)$, $|\nu|$ is the total variation of ν and $\|\nu\|$ the total variation norm of ν for ν in $\boldsymbol{M}(G)$, and $*$ is convolution.

3. THEOREM 1. *Let μ be a representing measure for $\boldsymbol{A}$ and let ν be any measure in $\boldsymbol{M}(G)$. Then there is a sequence f_n in $\boldsymbol{A}$ such that*

$$\|f_n\|_\infty \leqq 1 \tag{1}$$

$$f_n \rightarrow 1 \quad \text{a.e. } \mu \tag{2}$$

$$\|f_n\nu - \nu_a\| \rightarrow 0 \tag{3}$$

where

$$\nu = \nu_a + \nu_s$$

is the Lebesgue decomposition of ν with respect to μ:

$$\nu_a \ll \mu , \qquad \nu_s \perp \mu .$$

Both the statement and proof of Theorem 1 should be compared with earlier work done by Helson in [2] for the circle group.

Because ν_s is a regular measure singular with respect to μ, we may choose a sequence E_n of compact sets such that

$$|\nu_s|(G \sim E_n) \leqq 1/n \tag{4}$$

$$\mu E_n = 0 . \tag{5}$$

Now choose a second sequence F_n of compact sets such that E_n and F_n are disjoint and

$$\mu(G \sim F_n) \leqq 1/n^4 . \tag{6}$$

Let v_n be a real continuous function on G such that

$$-2n \leqq v_n \leqq 0 \qquad \text{on } G \tag{7}$$

$$v_n = 0 \qquad \text{on } F_n \tag{8}$$

$$v_n = -2n \qquad \text{on } E_n \tag{9}$$

and let g_n be a real trigonometric polynomial such that

$$-2n \leqq g_n \leqq 0 \qquad \text{on } G \tag{10}$$

$$\| g_n - v_n \|_\infty \leqq 1/n \tag{11}$$

(g_n may be obtained by convolution of v_n with an approximate identity consisting of trigonometric polynomials).

Denote by $\tilde{g}_n$ the trigonometric polynomial conjugate to g_n. Here we mean conjugacy relative to the half-space S: conjugate to the trigonometric polynomial

$$\Sigma a(\chi)\chi$$

is the trigonometric polynomial

$$\Sigma - i\varepsilon(\chi)a(\chi)\chi$$

where

$$\varepsilon(\chi) = \begin{array}{l} 1 \text{ if } \chi \in S \\ 0 \text{ if } \chi = 1 \\ -1 \text{ if } \chi \in \bar{S} . \end{array}$$

Now let

$$k_n = g_n + i\left(\tilde{g}_n - \int \tilde{g}_n d\mu\right) .$$

Then k_n belongs to A, the real part of k_n is g_n, and

(12) $$\int k_n d\mu = \int g_n d\mu \,.$$

Finally let

$$f_n = e^{k_n} \,.$$

Then f_n also belongs to A and, because of (10), satisfies (1).

(9) and (11) give

$$|f_n| \leqq e^{-n} \qquad \text{on } E_n$$

and thus

$$\int_{E_n} |f_n|\, d|\nu_s| \leqq e^{-n} ||\nu_s|| \,.$$

From (1) and (4)

$$\int_{G\sim E_n} |f_n|\, d|\nu_s| \leqq 1/n \,,$$

and combining this estimate with the previous estimate leads to

(13) $$||f_n\nu_s|| \to 0.$$

From (8) and (11)

$$\int_{F_n} |g_n|^2 d\mu \leqq 1/n^2 \,,$$

and from (6) and (10)

$$\int_{G\sim F_n} |g_n|^2 d\mu \leqq 4/n^2 \,.$$

Combining these two estimates gives

(14) $$\int |g_n|^2 d\mu \leqq 5/n^2 \,.$$

Now, since

$$2g_n = k_n + \bar{k}_n \,,$$

we have

(15) $$4\int |g_n|^2 d\mu = 2\int |k_n|^2 d\mu + \int k_n^2 d\mu + \int \bar{k}_n^2 d\mu \,.$$

Moreover, since μ is a representing measure for A and because of (12),

(16) $$\int k_n^2 d\mu = \left(\int g_n d\mu\right)^2 \geqq 0 \,.$$

(15) and (16) combine to give

$$\int |k_n|^2 d\mu \leqq 2\int |g_n|^2 d\mu\ ,$$[1]

and this inequality together with (14) shows that the sequence k_n converges to zero in the norm of $L^2(\mu)$. Therefore, passing to a subsequence if necessary, we may assume the sequence k_n converges to zero almost everywhere with respect to μ, and now the sequence f_n satisfies (2).

The conditions (1) and (2) and the dominated convergence theorem imply

$$\|f_n\nu_a - \nu_a\| \to 0\ . \tag{17}$$

Finally, (13) and (17) give (3).

Because the space of analytic measures is an A-module, statement (3) of Theorem 1 gives the Helson-Lowdenslager theorem on analytic measures [3, Theorem 7]:

COROLLARY 1. *If ν is analytic, then so are ν_a and ν_s.*

Helson and Lowdenslager found more than just the statement of the corollary. They showed [3, Lemma 3]: *If ν is an analytic measure singular with respect to Haar measure, then ν has mean value zero.* This too follows from Theorem 1, but more is true.

THEOREM 2. *Let ν be an analytic measure which is singular with respect to Haar measure. Then $\nu * \mu$ is singular with respect to Haar measure for every representing measure μ.*

Since ν is singular with respect to σ, Theorem 1 (with σ in place of μ) provides a sequence f_n belonging to A such that

$$\|f_n\|_\infty \leqq 1 \tag{18}$$

$$f_n \to 1 \qquad \text{a.e. } \sigma \tag{19}$$

$$\|f_n\nu\| \to 0\ . \tag{20}$$

Now because ν is analytic and μ is a representing measure for A,

$$(f_n\nu) * \mu = (f_n * \mu)(\nu * \mu)\ . \tag{21}$$

(21) surely holds if ν is replaced by a member of A. But since ν is analytic, ν is in the weak-star closure of $A\sigma$, and since convolution is continuous in the weak-star topology for $M(G)$, (21) continues to hold for ν.

[1] This inequality and its proof are of course not new.

(20) implies

$$\|(f_n\nu)*\mu\| \to 0$$

and so because of (21)

$$\|(f_n*\mu)(\nu*\mu)\| \to 0$$

and this implies, passing to a subsequence if necessary,

$$f_n*\mu \to 0 \qquad \text{a.e. } \nu*\mu \,. \tag{22}$$

On the other hand, (18) and (19) imply, using dominated convergence,

$$\|f_n - 1\| = \int |f_n - 1|\, d\sigma \to 0 \tag{23}$$

and therefore

$$\|(f_n*\mu) - 1\| = \|(f_n - 1)*\mu\| \to 0 \,. \tag{24}$$

Because of (24) we may assume, again passing to a subsequence if necessary,

$$f_n*\mu \to 1 \qquad \text{a.e. } \sigma \,. \tag{25}$$

(22) and (25) show that $\nu*\mu$ and σ are carried on disjoint sets, and so they are mutually singular.

We mentioned in the introduction that Theorem 2 can be obtained from the generalization in [3] of the F. and M. Riesz theorem. Indeed, all that is required in our proof of Theorem 2 is a sequence belonging to $\boldsymbol{A}$ and satisfying (20) and (23), and the existence of such a sequence is implied (by using a standard argument) by Lemma 3 and Theorem 7 of [3].

4. Corollary 1 and Theorem 2 applied to the circle group give the F. and M. Riesz theorem. For if ν is an analytic measure on the circle, the singular part with respect to Lebesgue measure, ν_s, is also analytic. But $\nu_s*\mu_r$ is absolutely continuous with respect to Lebesgue measure. Therefore $\nu_s*\mu_r$ is the zero measure, and this implies, as $\hat{\mu}_r$ does not vanish at any point of the integer group, that ν_s is the zero measure.

There is also an F. and M. Riesz theorem for finite Borel measures ν on the real line $\boldsymbol{R}$, which is sometimes proved by mapping a half-plane conformally on the unit disk and using the F. and M. Riesz theorem for the circle. We wish to show that Theorem 1 applied to the Bohr compactification $\boldsymbol{B}$ of the line leads to an easy and, we believe, natural proof of the Riesz theorem for the line.

ν in $\boldsymbol{M}(\boldsymbol{R})$ is said to be analytic if

$$\hat{\nu}(t) = \int_{-\infty}^{\infty} e^{-ist} d\nu(s) = 0$$

for $t < 0$. $\boldsymbol{B}$ is the compact abelian group dual to $\boldsymbol{R}$ when $\boldsymbol{R}$ is given the discrete topology. The mapping π of $\boldsymbol{R}$ into $\boldsymbol{B}$ defined by

$$(\pi s, t) = e^{ist}$$

is a continuous isomorphism of $\boldsymbol{R}$ into $\boldsymbol{B}$, and the image $\boldsymbol{B}_0$ of $\boldsymbol{R}$ is a dense subgroup of $\boldsymbol{B}$. Using the transformation on measures which carries ν in $\boldsymbol{M}(\boldsymbol{R})$ into $\nu\pi^{-1}$ in $\boldsymbol{M}(\boldsymbol{B})$ we may identify $\boldsymbol{M}(\boldsymbol{R})$ with those measures in $\boldsymbol{M}(\boldsymbol{B})$ which are carried on $\boldsymbol{B}_0$. Moreover the Fourier transform of ν in $\boldsymbol{M}(\boldsymbol{R})$ is the same whether we consider ν as an element of $\boldsymbol{M}(\boldsymbol{R})$ or as an element of $\boldsymbol{M}(\boldsymbol{B})$. For $0 < r < 1$, the Cauchy measure μ_r is the measure carried on $\boldsymbol{B}_0$ defined by

$$\hat{\mu}_r(t) = r^{|t|} .$$

Each Cauchy measure is a representing measure for the algebra $\boldsymbol{A}$ of continuous analytic functions on $\boldsymbol{B}$ (here S is the set of positive real numbers), and the Cauchy measures and Lebesgue measure are mutually absolutely continuous.

With this brief description of $\boldsymbol{B}$ it is now easy to show: *An analytic measure on the line is absolutely continuous with respect to Lebesgue measure.*

Assume ν is an analytic measure carried on $\boldsymbol{B}_0$, and denote by σ_0 Lebesgue measure (transferred to $\boldsymbol{B}_0$). Since the Cauchy measures and Lebesgue measure are mutually absolutely continuous, Theorem 1 provides a sequence f_n belonging to $\boldsymbol{A}$ such that

$$\|f_n\|_\infty \leqq 1 \tag{26}$$

$$f_n \to 1 \qquad \text{a.e. } \sigma_0 \tag{27}$$

$$\|f_n\nu - \nu_a\| \to 0 \tag{28}$$

where ν_a is the absolutely continuous component of ν with respect to σ_0.

Consider a Cauchy measure μ_r. Because of (28)

$$\|(f_n\nu)*\mu_r - (\nu_a*\mu_r)\| \to 0 . \tag{29}$$

Also, since ν is analytic,

$$(f_n\nu)*\mu_r = (f_n*\mu_r)(\nu*\mu_r) . \tag{30}$$

Now $f_n*\mu_r$ converges pointwise to 1 on $\boldsymbol{B}_0$, and this is important. This is because of (26) and (27) and because a null set of μ_r remains a null set when translated by an element of $\boldsymbol{B}_0$.

Since ν is carried on $\boldsymbol{B}_0$, $\nu*\mu_r$ is also carried on $\boldsymbol{B}_0$ (and indeed

$\nu * \mu_r \ll \sigma_0$) and therefore, because $f_n * \mu_r$ converges boundedly to 1 on $\boldsymbol{B}_0$,

$$\| (f_n * \mu_r)(\nu * \mu_r) - (\nu * \mu_r) \| \to 0 \ . \tag{31}$$

From (29), (30), and (31) we obtain

$$\nu * \mu_r = \nu_a * \mu_r$$

which implies

$$\nu = \nu_a$$

since $\hat{\mu}_r$ does not vanish at any point of $\boldsymbol{R}$.

5. Corollary 1 and Theorem 2 when applied to the Bohr group give: *If ν is an analytic measure on $\boldsymbol{B}$ and $\nu * \mu_r$ is absolutely continuous with respect to Haar measure (for some $0 < r < 1$), then ν is absolutely continuous.* This is due to Bochner [1].

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ON A CLASSICAL THEOREM OF NOETHER IN IDEAL THEORY

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A classical result in the ideal theory of commutative rings is that an integral domain D with unit is a Dedekind domain if and only if D is noetherian, of dimension less than two, and integrally closed. [8; 275]. The statement of this theorem is due essentially to Noether [6; 53], though the present statement is a refined version of Noether's theorem. (See Cohen [1; 32] for the historical development of the theorem above.) Noether did not, in fact, require that the domain D contain a unit element. By imposing greater restrictions on the prime ideal factorization of each ideal, she showed that D must contain a unit element.

This paper considers an integral domain J with Property C: Every ideal of J may be expressed as a product of prime ideals.

In particular, it is shown that an integral domain J with property C need not contain a unit element. However, factorization of an ideal as a product of prime ideals is unique and J is noetherian, of dimension less than two, and integrally closed.[1] A domain without unit having these three properties need not have property C. If J does not contain a unit element, J is the maximal ideal of a discrete valuation ring V of rank one such that V is generated over J by the unit element e, and conversely. The structure of all such valuation rings V is known. [4; 62].

If J is an integral domain with quotient field k, then J^* will denote the subring of k generated by J and the unit element e of k. We will assume that all domains considered contain more than one element.

If D is an integral domain, not necessarily containing a unit, and if k is the quotient field of D, the definitions of fractionary ideals of D, of sums, products and quotients of fractionary ideals, and of the fractionary ideal $(u_1, u_2, \cdots, u_t)$ of D generated by finitely many elements $u_1, u_2, \cdots, u_t$ of k, are generalized in the obvious ways. In particular, D^* is a fractionary ideal of D and if $\mathscr{S}$ is the collection of all nonzero fractionary ideals of D, $\mathscr{S}$ is an abelian semigroup under multiplication with unit element D^*. A fractionary ideal F of

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[1] A domain D with quotient field k is integrally closed if D contains every element x of k with the following property: There exist elements $d_0, d_1, \cdots, d_n$ of D such that $x^{n+1} + d_nx^n + \cdots + d_1x + d_0 = 0$.

D is said to be invertible if F has an inverse when considered as an element of $\mathscr{S}$. A nonzero principal fractionary ideal is invertible and $(d)^{-1} = (1/d)$. A product of fractionary ideals is invertible if and only if each of the factors is invertible. [8; 271].

The following two lemmas may be proved by making minor changes in the usual proofs given in the case of a domain with unit. [8; 272-273]. While the proof of Theorem 1 is definitely a modification of the usual proof for a domain with unit, the author feels enough difficulties arise to prove Theorem 1 here.

LEMMA 1. *If A is an invertible fractional ideal of the integral domain D, then $A^{-1} = D^*: A$. Further, A has a finite module basis over D.*

LEMMA 2. *Suppose A is a proper ideal of the domain D such that A may be expressed as a product of invertible prime ideals of D. This representation is unique if $D \subset D^*$, or unique to within factors of D if $D = D^*$.*

Henceforth in this paper, J will denote an integral domain without unit such that J has property C.

THEOREM 1. *Every nonzero proper prime ideal of J is invertible and maximal.*

Suppose first that there exists a nonzero proper invertible prime ideal P of J such that P is not maximal. We chose a such that $P \subset P + (a) \subset J$. We express $P + (a)$ and $P + (a^2)$ as products of prime ideals: $P + (a) = J^k P_1 \cdots P_r$, $P + (a^2) = J^t Q_1 \cdots Q_s$ where each P_i and each Q_j is a proper ideal of J. In $\bar{J} = J/P$ we have: $(\bar{a}) = \bar{J}^k \bar{P}_1 \cdots \bar{P}_r$, $(\bar{a})^2 = \bar{J}^t \bar{Q}_1 \cdots \bar{Q}_s$. By Lemma 2, $s = 2r$ and by proper labeling $P_i = Q_{2i-1} = Q_{2i}$. If $\bar{J}$ does not contain a unit element, then Lemma 2 implies also that $t = 2k$ so that $P + (a^2) = [P + (a)]^2$. If $\bar{J}$ contains a unit, then $(\bar{a}) = \bar{J}^k \bar{P}_1 \cdots \bar{P}_r$ so that r is positive and $(\bar{a}) = \overline{P_1 \cdots P_r}$. Similarly, $(\bar{a})^2 = \overline{Q_1 \cdots Q_s}$. Therefore $[P + (a)]^2 = P_1^2 \cdots P_r^2 = P + (a^2)$. For either case, therefore, $P + (a^2) = [P + (a)]^2$. The remainder of the proof of the theorem is the same as the proof appearing in [8; 273].

THEOREM 2. *J is a noetherian domain.*

We first show that J is finitely generated. Thus if J contains a proper nonzero prime ideal P, then $P = (p_1, \cdots, p_s)$ is maximal and finitely generated by Theorem 1 and Lemma 1. Therefore if $d \in J$, $d \notin P$, then $J = (p_1, \cdots, p_s, d)$. If (0) is the only proper prime ideal of J, then given $d \in J$, $d \neq 0$, $(d) = J^k$ for some integer $k \geqq 1$. Then J is invertible, and hence finitely generated.

It follows that every prime ideal of J is finitely generated. Since J has property C, every ideal of J is finitely generated.

THEOREM 3. *Every nonzero ideal of J is a power of J and, in fact, J is a principal ideal domain.*

Since J is noetherian and $J \subset J^*$, $J^2 \subset J$. [5; 172–73]. We choose $x \in J$, $x \notin J^2$. Because J has property C, (x) is prime. We shall show that $(x) = J$. We suppose that $(x) \subset J$. Because (x) is invertible and $J \subset J^*$, $(x) \supset (x)J \supset (x^2)$. If A is any ideal such that $(x) \supset A \supset (x^2)$ and if P is a prime factor of A, then $P \supseteqq (x)$ so that $P = (x)$ or $P = J$. Because $(x) \supset A \supset (x^2)$, $A = (x)J^k$ for some $k \geqq 1$. But $x \notin J^2$ so that $x^2 \notin (x)J^k$ for $k \geqq 2$. Therefore $k = 1$ and $(x)J$ is the unique ideal properly between (x) and (x^2).

We next show that (x^2) is a primary ideal. Thus if $a, b \in J$, $ab \in (x^2)$, and $a \notin (x)$, then $b \in (x)$. Hence $(x^2) \subseteqq (x^2, b) \subseteqq (x)$. Now (x) is maximal and prime in J so that $J/(x)$ contains a unit element $\bar{u}$. Because $a \notin (x)$, $ua \notin (x)$ so that $uax \notin (x^2)$ and therefore $ux \notin (x^2, b)$. This means $(x^2, b) \not\supseteqq (x)J$ so that $(x^2, b) = (x^2)$ by the preceding paragraph. Hence $b \in (x^2)$ and (x^2) is primary.

Now $ua - a \in (x)$ so that $(ua - a)^3 \in (x^2)$. If $z \in J$, then $z(ua - a)^3 = a^3(tz - z) \in (x^2)$ where t is a fixed element of J independent of z. Since $a^3 \notin (x)$ and (x^2) is primary, $tz - z \in (x^2)$ for each $z \in J$—i.e., $J/(x^2)$ contains a unit element. This means, however, that $V = (x)/(x^2)$ is a vector space over the field $J/(x)$. There is a one-to-one correspondence between subspaces of V and ideals of J between (x) and (x^2). Hence V has exactly one nonzero proper subspace, which is impossible. Therefore $J = (x)$ as asserted.

If P is a proper prime ideal of J, the argument above shows that $P \subseteqq J^2 = (x^2)$. This means for some ideal A of J, $P = A(x)$. Since P is prime, $P = A$. Now $(x) = J \subset J^*$ so that P is not invertible and thus $P = (0)$. Hence J is the only nonzero prime ideal of J. Therefore if A is a nonzero ideal of J, $A = J^k = (x^k)$ for some positive integer k.

A ring R with at most two prime ideals is called a *primary ring*. Theorem 3 shows that J is a primary domain. The author has investigated primary rings in [3].

THEOREM 4. *J^* is a discrete valuation ring of rank one. Conversely if D is a discrete valuation ring of rank one with maximal* *ideal M and if $D = M^*$, then M is a domain without unit having property C.*

The proof will use the following.

LEMMA 3. *Suppose S is a ring with unit e and that R is a subring of S such that S is generated by R and e. A subset of R is an ideal of S if and only if it is an ideal of R. S is noetherian if and only if R is noetherian.*

For the proof of the lemma, see [3].

To prove the theorem, we let $\xi \in k$, the quotient field of J^*. For some elements a and b of J, $\xi = a/b$. By Theorem 3 the ideals (a) and (b) of J compare—i.e. $a/b \in J^*$ or $b/a \in J^*$. Therefore, J^* is a valuation ring. Because J^* is noetherian, J^* is discrete and of rank one. [9; 41].

If M is the maximal ideal of J^* then $J = M^r$ for some r. Then $M^{r+1} \subset J$ implies $M^{r+1} = (M^r)^s$ for some integer s so that $r + 1 = rs$ and $r = 1$ – i.e., $J = M$. Hence J^*/J is a field. Because J^* is generated over J by e, $J^*/J = Z/(p)$ for some prime integer p.

The proof of the converse is an immediate consequence of Lemma 3 and of the fact that a discrete valuation ring of rank one is a Dedekind domain. [8; 278].

It is possible to classify all discrete valuation rings V of rank one such that $V = M^*$ where M is the maximal ideal of V, for if V has this property, so does the completion $\bar{V}$ of V. [2; 60]. If now p is a fixed prime, if Π denotes the prime field with p elements, x an indeterminate over π, if $V_1 = Z_{(p)}$ and $V_2 = (\Pi[x])_{(x)}$ then V_1 and V_2 are discrete valuation rings of rank one and with residue field Π. Further V_1 and V_2 are regular and *unramified* in Cohen's sense. [2; 88]. Thus $\bar{V}_1$ and $\bar{V}_2$ are so-called *p-adic* rings. [2; 59–60, 89]. Now $\bar{V}_1$ has characteristic zero (unequal characteristic case for $\bar{V}_1$ and its residue field) and $\bar{V}_2$ has characteristic p (equal characteristic case). The within isomorphism, $\bar{V}_1$ and $\bar{V}_2$ are the only two p-adic rings of dimension one having residue class field Π. [2; 89]. Now $\bar{V}_1$ is simply the domain of Hensel's p-adic integers and $\bar{V}_2$ is the domain of formal power series in one indeterminate over the field Π. [7; 242–243]. Finally, $\bar{V}$ is an Eisenstein extension of $\bar{V}_1$ or $\bar{V}_2$, and in case $\bar{V}$ has characteristic p, $\bar{V} \cong \bar{V}_2$. In short we have: If V has characteristic p, then to within isomorphism V is a ring between V_2 and $\bar{V}_2$. If V is unramified of characteristic 0, then $V_1 \subseteqq V \subseteqq \bar{V}_1$. If V is ramified of characteristic zero, then V is isomorphic to a valuation ring contained in an Eisenstein extension of $\bar{V}_1$. Conversely, if V is a ring having any of the three properties just described, V is a discrete valuation ring of rank one having residue field Π. [2; 59–60].

We add the following remarks:

In the last paragraph of the proof of Theorem 2, it is not necessary to use the fact that J has property C to conclude J is noetherian. That J is noetherian follows from a theorem of Cohen [1; 29] if all prime ideals of J are finitely generated.

In the proof of Theorem 3, it is not true in general that if $D/(x)$ is a field, that the ring $D/(x^2)$ contains a unit element, and hence that $(x)/(x^2)$ is a vector space over $D/(x)$. One can take D to be the ring of even integers and $x = 6$.

Theorem 3 implies that J is noetherian and of dimension less than two. Using Theorem 4, it is easily seen that J is integrally closed. That these three conditions do not imply that a domain D has property C may be seen by taking D to be the domain of even integers. Theorems 3 and 4 imply a bit more than the above. They even imply that J is a noetherian integrally closed primary domain. It can be shown that a noetherian integrally closed primary domain D without unit is the Jacobson radical of D^*, which is a semi-local ring, and that further, $D^*/D \cong Z/(p_1p_2 \cdots p_k)$ for some distinct primes $p_1, \cdots, p_k$. [3]. However, D need not have property C as can be seen by choosing D as the Jacobson radical of Z_M where M consists of all integers relatively prime to 6. An analog to the classical Noether theorem cited earlier in the case of a domain without unit, while obtainable, now seems not as desirable to the author as Theorem 4.

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PARTIAL ISOMETRIES

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0. Introduction. For normal operators on a Hilbert space the problem of unitary equivalence is solved, in principle; the theory of spectral multiplicity offers a complete set of unitary invariants. The purpose of this paper is to study a special class of not necessarily normal operators (partial isometries) from the point of view of unitary equivalence.

Partial isometries form an attractive and important class of operators. The definition is simple: a partial isometry is an operator whose restriction to the orthogonal complement of its null-space is an isometry. Partial isometries play a vital role in operator theory; they enter, for instance, in the theory of the polar decomposition of arbitrary operators, and they form the cornerstone of the dimension theory of von Neumann algebras. There are many familiar examples of partial isometries: every isometry is one, every unitary operator is one, and every projection is one. Our first result serves perhaps to emphasize their importance even more; the assertion is that the problem of unitary equivalence for completely arbitrary operators is equivalent to the problem for partial isometries. Next we study the spectrum of a partial isometry and show that it can be almost anything; in the finite-dimensional case even the multiplicities can be prescribed arbitrarily. In a special (finite) case, we solve the unitary equivalence problem for partial isometries. After that we ask how far a partial isometry can be from the set of normal operators and obtain a very curious answer. Generalizing and simplifying a result of Nagy, we show also that if two partial isometries are sufficiently near, then some natural cardinal numbers (dimensions) associated with them are the same. This result yields a partitioning of the metric space of all partial isometries into open-closed sets, and we conclude by proving that these sets are exactly the components.

For any operator A with null-space $\mathfrak{N}$ we write $\nu(A) = \dim \mathfrak{N}$ and we call $\nu(A)$ the *nullity* of A. If A is a partial isometry with range $\mathfrak{R}$, we write $\rho(A) = \dim \mathfrak{R}$ and $\rho'(A) = \dim \mathfrak{R}^{\perp}$; the cardinal numbers $\rho(A)$ and $\rho'(A)$ are the *rank* and the *co-rank* of A. The subspace $\mathfrak{N}^{\perp}$ is the *initial space* of A; the range $\mathfrak{R}$ (which is the same as the image $A\mathfrak{N}^{\perp}$) is the *final space* of A. If A is a partial isometry then so is A^*; the initial space of A^* is the final space of A, and vice versa. It follows that $\nu(A^*) = \rho'(A)$ and

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$\rho'(A^*) = \nu(A)$.

It is natural to define a partial order for partial isometries as follows: $A \leqq B$ in case B agrees with A on the initial space of A. (This implies that the initial space of A is included in the initial space of B.) A partial isometry is maximal with respect to this order if and only if either its initial space or its final space is the entire underlying Hilbert space. It follows that every partial isometry can be enlarged to either an isometry or a co-isometry (the adjoint of an isometry). A necessary and sufficient condition that a partial isometry possess a unitary enlargement (i.e., that there exist a unitary operator that dominates it) is that its nullity be equal to its co-rank. If the underlying Hilbert space is finite-dimensional, this condition is always satisfied; in the infinite-dimensional case it may not be.

1. Reduction. If A is a construction (i.e., if $\|A\| \leqq 1$) on a Hilbert space $\mathfrak{H}$, then $1 - AA^*$ is positive, and, consequently, $1 - AA^*$ has a unique positive square root A'. Assertion: if $M = M(A)$ is the operator matrix $\begin{pmatrix} A & A' \\ 0 & 0 \end{pmatrix}$, interpreted as an operator on $\mathfrak{H} \oplus \mathfrak{H}$, then M is a partial isometry. One quick proof is to compute MM^* and observe that it is a projection; this can happen if and only if M is a partial isometry. Consequence: every contraction on a Hilbert space can be extended to a larger Hilbert space so as to become a partial isometry.

THEOREM 1. *If A and B are unitarily equivalent contractions, then $M(A)$ and $M(B)$ are unitarily equivalent; if, conversely, A and B are invertible contractions such that $M(A)$ and $M(B)$ are unitarily equivalent, then A and B are unitarily equivalent.*

Proof. If U is a unitary operator that transforms A onto B, then U transforms A^* onto B^*, and therefore U transforms A' onto B'; it follows that $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ transforms $M(A)$ onto $M(B)$.

Suppose next that A and B are invertible and that $M(A)$ and $M(B)$ are unitarily equivalent. The range of $M(A)$ consists of all column vectors of the form $\begin{pmatrix} Af + A'g \\ 0 \end{pmatrix}$. This set is included in the set of all column vectors with vanishing second coordinate; the invertibility of A implies that the range of $M(A)$ consists exactly of all column vectors with vanishing second coordinate. Since the same is true for $M(B)$, it follows that every unitary operator matrix that transforms $M(A)$ onto $M(B)$ maps the subspace of all vectors of the form $\begin{pmatrix} f \\ 0 \end{pmatrix}$ onto itself. This implies that that subspace reduces every such unitary operator matrix, and hence that every such unitary operator matrix is diagonal. Since the diagonal entries of a diagonal unitary matrix are unitary operators, it follows that A and B are unitarily equivalent, as asserted.

The theorem implies that the problem of unitary equivalence for partial isometries is equivalent to the problem for invertible contractions. The latter problem, in turn, is equivalent to the problem for arbitrary operators. The reason is that by a translation ($A \to A + \alpha$) and a change of scale ($A \to \beta A$) every operator becomes an invertible contraction, and translations and changes of scale do not affect unitary equivalence.

Here is a comment on the technique used in the proof. There are many ways that a possibly "bad" operator A can be used to manufacture a "good" one $\left(\text{e.g., } A + A^* \text{ and } \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\right)$. None of these ways has ever yielded sufficiently many usable unitary invariants for A. It is usually easy to prove that if A and B are unitarily equivalent, then so are the various constructs in which they appear. It is, however, usually false that if the constructs are unitarily equivalent, then so are A and B. In the case treated by Theorem 1 this converse is true, and its proof is the less trivial part of the argument.

2. **Spectrum.** What can the spectrum of a partial isometry be? Since a partial isometry is a contraction, its spectrum is included in the closed unit disc. If the partial isometry is invertible (i.e., if 0 is not in the spectrum), then it is unitary, and therefore the spectrum is a non-empty compact subset of the unit circle; well known constructions prove that every such set is the spectrum of some unitary operator. If the partial isometry is not invertible, then its spectrum contains 0; what else can be said about it? The answer is, nothing else. This answer was pointed out to us by Arlen Brown; its precise formulation is as follows.

THEOREM 2. *If a compact subset of the closed unit disc contains the origin, then it is the spectrum of some partial isometry.*

Proof. It is sufficient to prove that if A is a contraction, then the spectrum of $M(A)$ is the union of the spectrum of A and the singleton $\{0\}$. (This is sufficient because every non-empty compact subset of the closed unit disc is the spectrum of some contraction.) It is easy enough to see that 0 always belongs to the spectrum of $M(A)$; indeed every vector of the form $\begin{pmatrix} 0 \\ f \end{pmatrix}$ is in the null-space of $M(A)^*$. It remains to prove that if $\lambda \neq 0$, then a necessary and sufficient condition that $\begin{pmatrix} A-\lambda & A' \\ 0 & -\lambda \end{pmatrix}$ be invertible is that $A-\lambda$ be invertible. This assertion belongs to the theory of formal determinants of operator matrices. Here is a sample theorem from that theory: if C and D commute and if D is invertible, then a necessary and sufficient condition that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be invertible is that $AD-BC$ be invertible. For our present purpose it is sufficient to consider the special case $C=0$, in which case the commutativity hypothesis is automatically satisfied; we proceed to give the proof for that case. If A is invertible, then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ can be proved to be invertible by exhibiting its inverse: it is $\begin{pmatrix} A^{-1} & A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$. (Recall that the invertibility hypothesis on D is in force throughout.) If, conversely, $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is invertible, with inverse $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ say, then

$$\begin{pmatrix} AP+BR & AQ+BS \\ DR & DS \end{pmatrix} = \begin{pmatrix} PA & PB+QD \\ RA & RB+SD \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $DR=0$; since D is invertible, this implies that $R=0$, and hence that $AP=PA=1$. The proof is complete.

3. **Multiplicity.** For finite sets what the preceding argument proves is this: if $\lambda_1, \cdots, \lambda_n$ are distinct complex numbers with $|\lambda_i| \leqq 1$ for all i, and if $\lambda_i = 0$ for at least one i, then there exists a partial isometry whose spectrum is the set $\{\lambda_1, \cdots, \lambda_n\}$. The partial isometry that the proof yields acts on a space of dimension $2n$ and has a large irrelevant null-space. There is an alternative proof that yields much more for finite sets.

THEOREM 3. *If $\lambda_1, \cdots, \lambda_n$ are complex numbers (not necessarily distinct) with $|\lambda_i| \leqq 1$ for all i, and if $\lambda_i = 0$ for at least one i, then there exists a partial isometry on a space of dimension n, whose characteristic roots are exactly the λ's, each with the algebraic multiplicity equal to the number of times it occurs in the list.*

Proof. The proof can be given by induction on n. For $n=1$, the operator 0 on a space of dimension 1 satisfies all the conditions. The induction step is implied by the following assertion: if an $n \times n$ matrix U with 0 in its spectrum is a partial isometry, and if $|\lambda| \leqq 1$, then there exists a column vector f with n coordinates such that $\begin{pmatrix} U & f \\ 0 & \lambda \end{pmatrix}$ is a partial isometry. To prove this, observe that, since 0 is in the spectrum of U, the column-rank of U is less than n. This makes it possible to find a non-zero vector f orthogonal to all the columns of U; to finish the construction, normalize f so that $\|f\|^2 = 1 - |\lambda|^2$.

4. Equivalence. In at least one case, a very special case, the unitary equivalence problem for partial isometries has a simple and satisfying solution.

THEOREM 4. *If two partial isometries on a finite-dimensional space are such that 0 is a simple root of each of their characteristic equations, then a necessary and sufficient condition that they be unitarily equivalent is that they have the same characteristic equation (i.e., that they have the same characteristic roots with the same algebraic multiplicities).*

REMARK. The principal hypothesis is that 0 is a root of multiplicity 1 of the characteristic equation. If this were replaced by the hypothesis that 0 is not a root of the characteristic equation at all (i.e., is a root of multiplicity 0), then the statement would become the classical solution of the unitary equivalence problem for normal operators on a finite-dimensional space.

Proof. The necessity of the condition is trivial. Sufficiency can be proved by induction on the dimension. If the dimension is 1, the assertion is trivial. For the induction step, if the dimension is $n+1$, represent the given partial isometries by triangular matrices with 0 in the northwest corner, and write the results in the form

$$U = \begin{pmatrix} U_0 & f \\ 0 & \lambda \end{pmatrix}, \qquad V = \begin{pmatrix} V_0 & g \\ 0 & \lambda \end{pmatrix},$$

where U_0 and V_0 are $n \times n$ matrices, and f and g are n-rowed column vectors. Since both U and V are partial isometries with first column 0 and rank n, it follows that, in both cases, the remaining n columns constitute an orthonormal set, and hence, in particular, that f is orthogonal to the columns of U_0 and g is orthogonal to the columns of V_0. The thing to prove is that if U_0 and V_0 are unitarily equivalent, then so also are U and V. Suppose therefore that W_0 is unitary and $W_0 U_0 W_0^* = V_0$. Assertion: there exists a complex number θ of modulus 1 such that $W = \begin{pmatrix} W_0 & 0 \\ 0 & \theta \end{pmatrix}$ transforms U onto V. Indeed, if $|\theta| = 1$, then

$$WUW^* = \begin{pmatrix} V_0 & \bar{\theta} W_0 f \\ 0 & \lambda \end{pmatrix},$$

Since this matrix is a partial isometry with first column 0 and rank n, it follows that $\bar{\theta} W_0 f$ is orthogonal to the span (of dimension $n-1$) of the columns of V_0. Since g also is orthogonal to the columuns of V_0, it follows that θ can indeed be chosen so that $\bar{\theta} W_0 f = g$. The only case that gives a moment's pause is the one in which $W_0 f = 0$. In that case $f = 0$, and therefore $|\lambda| = 1$; this implies that $g = 0$, and all is well.

5. Distance. Since the unitary equivalence problem is solved for normal operators, it is reasonable to approach its solution in the general case by asking how far any particular operator is from normality. The figurative "how far" can be interpreted literally, and its literal interpretation yields a curious unitary invariant. Let $\boldsymbol{N}$ be the set of all normal operators, and for each (not necessarily normal) operator A consider the distance $d(A, \boldsymbol{N})$ from A to $\boldsymbol{N}$. The distance here is meant in the usual sense appropriate to subsets of metric spaces: $d(A, \boldsymbol{N}) = \inf \{\| A - N \| : N \in \boldsymbol{N}\}$. The definition makes sense for all operators, and, in particular, for partial isometries. We proceed to study one of the simplest questions that the definition suggests: as U varies over the set $\boldsymbol{P}$ of partial isometries, what possible values can $d(U, \boldsymbol{N})$ attain? The answer we obtain is rather peculiar.

THEOREM 5. *The set of all possible values of $d(U, \boldsymbol{N})$, for U in $\boldsymbol{P}$, is the closed interval* $[0, 1/2]$ *together with the single number* 1.

Proof. We begin with the assertion that if a partial isometry U has a unitary enlargement, then $d(U, \boldsymbol{N}) \leqq 1/2$. The proof consists in verifying that if W is a unitary enlargement of U, then

$$\left\| U - \frac{1}{2} W \right\| = \frac{1}{2} .$$

Indeed, if $\mathfrak{N}$ is the null-space of U, then U is equal to 0 on $\mathfrak{N}$ and to W on $\mathfrak{N}^\perp$; it follows that $U - \frac{1}{2} W$ is equal to $-\frac{1}{2} W$ on $\mathfrak{N}$ and to $\frac{1}{2} W$ on $\mathfrak{N}^\perp$. This implies that $U - \frac{1}{2} W$ is 1/2 times a unitary operator and hence that its norm is 1/2.

It is easy to exhibit a partial isometry U such that $d(U, \boldsymbol{N}) = 1/2$; in fact this can be done on a two-dimensional Hilbert space. A simple example is the operator U_0 given by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. That U_0 is a partial isometry can be verified at a glance. (Its matrix is obtained from a unitary matrix by "erasing" a column.) The preceding paragraph implies that $d(U_0, \boldsymbol{N}) \leqq 1/2$; it remains to prove that if N is normal, then $\| U_0 - N \| \geqq 1/2$. For this purpose, let f be an arbitrary unit vector and note that

$$\big|\, \|U_0 f\| - \|U_0^* f\| \,\big| \leqq \big|\, \|U_0 f\| - \|Nf\| \,\big| + \big|\, \|N^* f\| - \|U_0^* f\| \,\big|$$
$$\leqq 2\|U_0 - N\| \, .$$

(Recall that, by normality, $\|Nf\| = \|N^*f\|$.) If f is the column vector $\begin{pmatrix}0\\1\end{pmatrix}$, then $\|U_0 f\| = 1$ and $\|U_0^* f\| = 0$; the proof is complete.

For each number t in the interval $[0, 1]$ write $t' = \sqrt{1-t^2}$. The mapping $t \rightarrow U_t = \begin{pmatrix} t & t' \\ 0 & 0 \end{pmatrix}$ is a continuous path in the metric space $\boldsymbol{P}$, which joins the partial isometry U_0 to the projection $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (a normal partial isometry). Conclusion: as U varies over all partial isometries, $d(U, \boldsymbol{N})$ can take (at least) all values between 0 and 1/2 inclusive. $\Big($The technique of the preceding paragraph can be used to show that $d(U_t, \boldsymbol{N}) = \frac{1}{2}t'.\Big)$

For the next step we need the following lemma: if P is a projection, and if A is an operator such that $P + A$ is one-to-one, then $\nu(A) \leqq \rho(P)$. To prove this, observe that the null-spaces of P and A have only 0 in common, so that the restriction of P to the null-space $\mathfrak{N}$ of A is one-to-one. It follows that the dimension of $\mathfrak{N}$ is less than or equal to the dimension of the entire range of P, which is the desired conclusion. (We use here the assertion that one-to-one bounded linear transformations do not lower dimension; cf. [2, Lemma 3].)

Suppose now that U is a partial isometry such that $\nu(U) < \rho'(U)$. Assertion: no operator at a distance less than 1 from U can be invertible. Suppose, indeed, that $\|U - A\| < 1$, so that $\|U^*U - U^*A\| < 1$. Write $P = 1 - U^*U$; since U is a partial isometry, P is the projection onto the null-space of U. Since $U^*U - U^*A = 1 - (P + U^*A)$, it follows that $P + U^*A$ is invertible, and hence, from the lemma of the preceding paragraph, that $\nu(U^*A) \leqq \rho(P) = \nu(U)$. If A were invertible, then U^*A and U^* would have the same nullity, and it would follow that $\nu(U^*) \leqq \nu(U)$. This contradicts the assumption on U, and it follows that A cannot be invertible.

Since the closure of the set of all invertible operators includes $\boldsymbol{N}$, it follows from the preceding paragraph that if U is a partial isometry with $\nu(U) < \rho'(U)$, then $d(U, \boldsymbol{N}) \geqq 1$. This result quickly implies some minor improvements of itself. To begin with, the hypothesis $\nu(U) < \rho'(U)$ can be replaced by $\nu(U) \neq \rho'(U)$. (If $\nu(U) > \rho'(U)$, then $\rho'(U^*) > \nu(U^*)$, and the original formulation is applicable to U^*.) Next, the conclusion $d(U, \boldsymbol{N}) \geqq 1$ can be replaced by $d(U, \boldsymbol{N}) = 1$. (Since 0 is normal, no partial isometry is at a distance greater than 1 from $\boldsymbol{N}$.) Finally, the result implies the principal assertion: if U is a partial isometry such that $d(U, \boldsymbol{N}) > 1/2$, then $d(U, \boldsymbol{N}) = 1$. Indeed, if $d(U, \boldsymbol{N}) > 1/2$, then $\nu(U) \neq \rho'(U)$, for otherwise U would have a unitary enlargement, and therefore, by the first paragraph of this proof, U would be at a distance not more than 1/2 from $\boldsymbol{N}$. The proof of Theorem 5 is complete.

6. Continuity. Associated with each partial isometry U there are three cardinal numbers: the rank $\rho(U)$, the nullity $\nu(U)$, and the co-rank $\rho'(U)$. Our next purpose is to prove that the three functions ρ, ν, and ρ' are continuous. For the space $\boldsymbol{P}$ of partial isometries we use the topology induced by the norm; for cardinal numbers we use the discrete topology. With this explanation the meaning of the continuity assertion becomes unambiguous: if U is sufficiently near to V, then U and V have the same rank, the same nullity, and the same co-rank. The following assertion is a precise quantitative formulation of the same result.

THEOREM 6. *If U and V are partial isometries such that $\| U - V \| < 1$, then $\rho(U) = \rho(V)$, $\nu(U) = \nu(V)$, and $\rho'(U) = \rho'(V)$.*

Proof. The null-space of U and the initial space of V can have only 0 in common. Indeed, if f is a nonzero vector such that $Uf = 0$ and $\| Vf \| = \| f \|$, then $\| Uf - Vf \| = \| f \|$, and this contradicts the hypothesis $\| U - V \| < 1$. It follows that the restriction of U to the initial space of V is one-to-one, and hence (see [2] again) that the dimension of the initial space of V is less than or equal to the dimension of the entire range of U. In other words, the result is that $\rho(V) \leqq \rho(U)$; the assertion about ranks follows by symmetry. This part of the theorem generalizes (from projections to arbitrary partial isometries) a theorem of Nagy (see [4, § 105]), and, at the time, considerably shortens its proof. The original proof is, in a sense, more constructive; it not only proves that two subspaces have the same dimension, but it exhibits a partial isometry for which the first is the initial space and the second the final space.

The assertion about ν can be phrased this way: if $\nu(U) \neq \nu(V)$, then $\| U - V \| \geqq 1$. Indeed, if $\nu(U) \neq \nu(V)$, say, for definiteness, $\nu(U) < \nu(V)$, then there exists at least one unit vector f in the null-space of V that is orthogonal to the null-space of U. To say that f is orthogonal to the null-space of U is the same as to say that f belongs to the initial space of U. It follows that $1 = \| f \| = \| Uf \| = \| Uf - Vf \| \leqq \| U - V \|$, and the proof of the assertion about nullities is complete.

The assertion about co-ranks is an easy corollary: if $\| U - V \| < 1$, then $\| U^* - V^* \| < 1$, and therefore $\rho'(N) = \nu(U^*) = \nu(V^*) = \rho'(V)$.

If the dimension of the underlying Hilbert space is δ, then the rank, nullity, and co-rank of each partial isometry are cardinal numbers

ρ, ν, and ρ' such that $\rho + \nu = \rho + \rho' = \delta$. If, conversely, ρ, ν, and ρ' are any three cardinal numbers satisfying these equations, then there exist partial isometries with rank ρ, nullity ν, and co-rank ρ'. Let $\boldsymbol{P}(\rho, \nu, \rho')$ be the set of all such partial isometries. Clearly the sets of the form $\boldsymbol{P}(\rho, \nu, \rho')$ constitute a partition of the space $\boldsymbol{P}$ of all partial isometries; it is a consequence of Theorem 6 that each set $\boldsymbol{P}(\rho, \nu, \rho')$ is both open and closed.

7. **Connectivity.** We proved in § 5 that there is a continuous path in the space $\boldsymbol{P}$ joining a normal partial isometry (in fact a projection) to one whose distance from $\boldsymbol{N}$ is 1/2. On the other hand, § 6 shows that $\boldsymbol{P}$ is not connected, and this suggests the question of just how disconnected $\boldsymbol{P}$ is. The following assertion is the answer.

THEOREM 7. *For each ρ, ν, and ρ', the set $\boldsymbol{P}(\rho, \nu, \rho')$ of all partial isometries of rank ρ, nullity ν, and co-rank ρ' is arcwise connected.*

Proof. The principal tool is the theorem that the set $\boldsymbol{P}(\rho, 0, 0)$ of all unitary operators is arcwise connected. This is a consequence of the functional calculus. Indeed, if U is unitary, then there exists a Hermitian operator A such that $U = e^{iA}$, If $U_t = e^{itA}$, $0 \leqq t \leqq 1$, then $t \rightarrow U_t$ is a continuous path of unitary operators joining 1 $(= U_0)$ to $U (= U_1)$. Since each unitary operator can be joined to 1, it follows that any two can be joined to each other. This settles the case $\boldsymbol{P}(\rho, 0, 0)$. A useful consequence is that if two partial isometries are unitarily equivalent, then they can be joined by a continuous path. Indeed if U_0 and U_1 are partial isometries, and if V is a unitary operator such that $V^* U_0 V = U_1$, then let $t \rightarrow V_t$ be a continuous path joining 1 to V, and observe that $t \rightarrow V_t^* U_0 V_t$ is a continuous path joining U_0 to U_1.

For the next step we need to recall the basic facts about shifts (see [1] or [3]). A simple shift (more precisely, a simple unilateral shift) is an isometry V for which there exists a unit vector f such that the vectors f, Vf, V^2f, $\cdots$ form an orthonormal basis for the space. A shift (not necessarily simple) is, by definition, the direct sum of simple ones. It is easy to see that every shift is an isometry whose co-rank is the number of simple direct summands. Two shifts are unitarily equivalent if and only if they have the same co-rank. The fundamental theorem about shifts is that every element of $\boldsymbol{P}(\rho, 0, \rho')$ (i.e., every isometry of co-rank ρ') is either unitary (in which case $\rho' = 0$), or a shift of co-rank ρ', or the direct sum of a unitary operator and a shift of co-rank ρ'.

Suppose now that U_0 and U_1 are in $\boldsymbol{P}(\rho, 0, \rho')$, with $\rho' \neq 0$. If both U_0 and U_1 are shifts, then (since they have the same co-rank) they are unitarily equivalent, and, therefore, they can be joined by a continuous path.

Suppose next that U_0 is a shift (of co-rank ρ') and that $U_1 = V_1 \oplus W_1$, where V_1 is a shift (of co-rank ρ') and W_1 is unitary. Since the dimension of the domain of U_0 is $\rho' \cdot \aleph_0$, and since U_0 and U_1 have the some domain, it follows that the dimension of the domain of W_1 is not more than $\rho' \cdot \aleph_0$. If $\rho' > \aleph_0$, then break up W_1 into ρ' direct summands, each on a space of dimension $\aleph_0$, and match these summands with the ρ' simple direct summands of U_0 and U_1. The result of this procedure is to reduce the problem to the problem of joining a simple shift U to the direct sum of a simple shift V and a unitary operator W on a space of dimension $\aleph_0$ or smaller.

If the dimension of the domain of W is n ($< \aleph_0$), the problem is easy to describe and to solve in terms of matrices. The shift U is unitarily equivalent to (and therefore it can be joined to) an operator with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & & \ddots \end{pmatrix},$$

and, similarly, the direct sum $V \oplus W$ can be joined to an operator with matrix

$$\begin{pmatrix} 1 & 0 & 0 & & 0 & 0 & & & & & \\ 0 & 1 & 0 & \cdots & 0 & 0 & & & & & \\ 0 & 0 & 1 & & 0 & 0 & & & 0 & & \\ & \vdots & & & \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & & & & & \\ 0 & 0 & 0 & & 0 & 1 & & & & & \\ & & & & & & 0 & 0 & 0 & 0 & \\ & & & & & & 1 & 0 & 0 & 0 & \\ & & & 0 & & & 0 & 1 & 0 & 0 & \\ & & & & & & 0 & 0 & 1 & 0 & \\ & & & & & & & & & & \ddots \end{pmatrix}.$$

It remains to prove that the first of these two matrices can be joined to the second. For this purpose, note that the (unitary) permutation matrix (with $n+1$ rows and columns)

$$\begin{pmatrix} 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

can be joined to the identity matrix (with $n + 1$ rows and columns). Let $t \to M_t$ be a continuous path of unitary matrices that joins them, and let P be the projection matrix (with $n + 1$ rows and columns)

$$\begin{pmatrix} 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The "product" path $t \to M_t P$ joins

$$\begin{pmatrix} 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

to P. Use this path in the northwest corner (of size $n + 1$) of the infinite matrices to obtain a path joining the matrix of U to the matrix of $V \oplus W$.

If the dimension of the domain of W is $\aleph_0$, the solution is easier. It is easy to verify that the operator matrix $\begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix}$ (considered as an operator on the direct sum of the underlying space with itself) is unitarily equivalent to U, and the operator matrix $\begin{pmatrix} W & 0 \\ 0 & U \end{pmatrix}$ is unitarily equivalent to $V \oplus W$. Since W can be joined to the identity by a continuous path, it remains to prove that $\begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix}$ can be joined to $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$. If $t \to \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix}$ is a continuous path of numerical unitary matrices that joins $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $t \to \begin{pmatrix} \alpha_t & \beta_t U \\ \gamma_t & \delta_t U \end{pmatrix}$ is a continuous path of partial isometries that joins $\begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$.

What we have proved so far (after successive reductions) implies that any two isometries can be joined by a continuous path, i.e., that the set $\boldsymbol{P}(\rho,0,\rho')$ is arcwise connected.

To prove that $\boldsymbol{P}(\rho, \nu, \rho')$ is always arcwise connected, it is sufficient to consider the case $\nu \leqq \rho'$. (Argue by adjoints.) If U_0 and U_1 are in $\boldsymbol{P}(\rho, \nu, \rho')$, then they can be enlarged to isometries V_0 and V_1. Such enlargements are far from unique; what is important for our purposes is that V_0 and V_1 can be found so that they have the same co-rank. If P_0 and P_1 are the projections onto the initial spaces of U_0 and U_1 (i.e., $P_0 = U_0^* U_0$ and $P_1 = U_1^* U_1$), then P_0 and P_1 have the same rank and co-rank. It follows that there exist paths $t \rightarrow V_t$ and $t \rightarrow P_t$ joining V_0 to V_1 and P_0 to P_1. Since $U_0 = V_0P_0$ and $U_1 = V_1P_1$, this implies that $t \rightarrow V_tP_t$ is a continuous path joining U_0 to U_1. The proof of Theorem 7 is complete.

The following consequence of Theorems 6 and 7 is trivial, but worth making explicit: the components of P are exactly the sets $\boldsymbol{P}(\rho, \nu, \rho')$.

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MAXIMUM MODULUS ALGEBRAS AND LOCAL APPROXIMATION IN C^n

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1. In [4] W. Rudin established an important result concerning maximum modulus algebras A of continuous complex-valued functions defined on the closure K of a Jordan domain in the complex plane (see also [5]). Rudin's result states, under the assumptions (a) A contains a function Ψ which is schlicht on K, and (b) A contains a non-constant function ϕ which is analytic in the interior, int K, of K, that every function in A is analytic in int K. In this note we will establish conditions under which assumption (b) alone yields the desired conclusion in a slightly more general setting. We assume that K is a compact set, with interior, of a Riemann surface, but also assume that int K is essentially open in the maximal ideal space Σ_A of A (A being regarded as a Banach algebra with the sup norm $\|f\| = \sup_{p\in K}|f(p)|$; see [2]). This means that each point of int K, excepting a set of points having no limit point in int K, has a neighborhood in int K which is open in Σ_A under the natural mapping of K into Σ_A. Under these assumptions it is easy to show, using the Local Maximum Modulus Principle of H. Rossi [3; Theorem 6.1] and Rudin's results, that (b) is sufficient to guarantee that A consists only of analytic functions. Our main purpose, however, is to establish the result by a geometric method, independent of Rudin's work, which is based on an appropriate local approximation in C^n. Unfortunately the geometric approach being used here only allows us to make the desired conclusion for twice continuously differentiable functions in A whereas the use of Rubin's results would give a proof valid for any function in A. However it is hoped that our method will be of some interest in itself.

The basic idea of the proof is as follows. For simplicity let K be the unit circle $\{z \in C: |z| \leqq 1\}$ in the complex plane, and let f and g be nonconstant functions in the maximum modulus algebra A. Suppose that $\Sigma_A = K$. Use f and g to map K into C^2 (the space of 2 complex variables) in the obvious way. If f and g are twice continuously differentiable in the neighborhood of a given point in int K then the image of this neighborhood in C^2 will be a two (real) dimensional surface possessing a tangent plane at the image p of the point. Let π be the two (real) dimensional tangent plane to this surface at p. If this plane is nonanalytic (Definition 1) then we can find a polynomial in the coordinates w_1 ane w_2 of C^2 which locally peaks [3] at p when

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restricted to the surface. The results of Rossi and the Arens-Calderon Theorem [1] then show that this will contradict the maximum modulus property. Thus π cannot be nonanalytic. This gives a relation between the complex derivatives of f and g which, in particular, implies that both functions are analytic at the pre-image of p if one of them is analytic there. In § 2 the essential geometric lemma is established and in § 3 it is used to prove the main result.

2. Let $F: M \to R^n$ be an immersion (a regular map in the sense of Whitney [6]) of a two (real) dimensional twice continuously differentiable manifold M into real Euclidean n-space R^n. Let $p \in M$ and let (U, h) give local coordinates about p, where U is an open set in M, and h is a homeomorphism from U onto $D = \{(u, v) \in R^2: u^2 + v^2 < 1\}$, with $h(p) = (0, 0)$. If x_j $(j = 1, \cdots, n)$ is a coordinate function in R^n then the functions $\phi_j(u, v) = x_j \circ F \circ h^{-1}(u, v)$ $(j = 1, \cdots, n)$ are differentiable and give a map $\Phi: D \to R^n$ defined by $\Phi(u, v) = (\phi_1(u, v), \cdots, \phi_n(u, v))$. Since F is an immersion, the 2 by n matrix

$$\left(\frac{\partial\phi_j}{\partial u}, \frac{\partial\phi_j}{\partial v}\right) = \begin{pmatrix} \dfrac{\partial\phi_1}{\partial u} & \dfrac{\partial\phi_1}{\partial v} \\ \vdots & \vdots \\ \dfrac{\partial\phi_n}{\partial u} & \dfrac{\partial\phi_n}{\partial v} \end{pmatrix}$$

has rank 2 and the mapping Φ is one-to-one in some disc $V = \{(u, v) \in R^2: u^2 + v^2 < r^2 < 1\}$. Further, the set $\Phi(V)$ is a surface element having a tangent plane at $\Phi(0, 0)$).

We can suppose for our purposes that $\Phi(0, 0)$ is the origin 0 in R^n. The tangent plane π to $\Phi(V)$ at 0 is then given parametrically by

$$x_j = \frac{\partial\phi_j}{\partial u} u + \frac{\partial\phi_j}{\partial v} v \qquad (j = 1, \cdots, n)\,, \tag{2.1}$$

where the derivatives are evaluated at $u = v = 0$. A change of local parameters from u and v to $u' = u'(u, v)$ and $v' = v'(u, v)$ with $\partial(u', v')/\partial(u, v) \neq 0$ (the inverse transformation being given by $u = u(u', v')$ and $v = v(u', v')$ in some neighborhood of $u = v = 0$) yield new functions $\phi_j'(u', v') = \phi_j(u(u', v'), v(u', v'))$ and a new parametrization of the tangent plane, namely,

$$\begin{aligned} x_j &= \frac{\partial\phi_j'}{\partial u'} u' + \frac{\partial\phi_j'}{\partial v'} v' \\ &= \left(\frac{\partial\phi_j}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial\phi_j}{\partial v}\frac{dv}{\partial u'}\right) u' + \left(\frac{\partial\phi_j}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial\phi_j}{\partial v}\frac{\partial v}{\partial v'}\right) v' \\ &\qquad\qquad (j = 1, \cdots, n)\,. \end{aligned}$$

Note that the rank of the matrix $(\partial\phi_j/\partial u, \partial\phi_j/\partial v)$ is the same as that of $(\partial\phi'_j/\partial u', \partial\phi'_j/\partial v')$ since $\partial(u', v')/\partial(n, v) \neq 0$.

Now u and v parametrize both the surface element $\Phi(V)$ and the tangent plane (given by (2.1)). Let η_j and η'_j $(j = 1, 2, \cdots n)$ denote the coordinates in R^n of the points B and B' on π and $\Phi(V)$, respectively, corresponding to the parameters u and v $(u^2 + v^2 < r^2)$. For sufficiently small u and v,

$$\eta'_j = \frac{\partial\phi_j}{\partial u}u + \frac{\partial\phi_j}{\partial v}v + \frac{1}{2}\left(\frac{\partial^2\phi_j}{\partial u^2}u^2 + 2\frac{\partial^2\phi_j}{\partial u\partial v}uv + \frac{\partial^2\phi_i}{\partial v^2}v^2\right)$$

where the first derivatives are evaluated at $u = v = 0$ and the second derivatives are evaluated at $u' = \theta u$, $v' = \theta v$ for some θ satisfying $0 < \theta < 1$. Since M is twice continuously differentiable, the second derivatives of ϕ_j are bounded in absolute value in some sufficiently small neighborhood of $(0, 0)$ and we obtain

$$\sum_1^m (\eta_j - \eta'_j)^2 \leqq K(|u| + |v|)^4$$

and so

$$|\eta_j - \eta'_j| \leqq L(|u| + |v|)^2 \tag{2.2}$$

where K and L are constants depending on these bounds and on n, and u and v are sufficiently small. These estimates will be used later.

Suppose now that $n = 2m$. One can define complex coordinates $w_j = x_{2j-1} + ix_{2j}$ making R^n into complex Euclidean space C^m. Also the (u, v)-plane can be formally complexified by writing $z = u + iv$, $\bar{z} = u - iv$. We then have a mapping Ψ: $V \to C^m$ defined by $\Psi(z, \bar{z}) = (w_1, \cdots, w_m)$ where

$$w_j = \Psi_j(z, \bar{z}) = \phi_{2j-1}\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + i\phi_{2j}\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) \qquad (j = 1, \cdots m)\,.$$

An elementary computation shows that in this formalism the tangent plane π to $\Psi(V)$ at the origin 0 is given parametrically by

$$w_j = \frac{\partial\Psi_j}{\partial z}z + \frac{\partial\Psi_j}{\partial\bar{z}}\bar{z} \qquad (j = 1, \cdots, m)$$

where the derivatives are evaluated at $z = \bar{z} = 0$. Furthermore, under a change of local coordinates in the parameter plane from z and $\bar{z}$ to $z' = u' + iv'$ and $\bar{z}' = u' - iv'$, the tangent plane is given parametrically by

$$\begin{aligned} w_j &= \frac{\partial\Psi'_j}{\partial z'}z' + \frac{\partial\Psi'_j}{\partial\bar{z}'}\bar{z}' \\ &= \left(\frac{\partial\Psi_j}{\partial z}\frac{\partial z}{\partial z'} + \frac{\partial\Psi_j}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial z'}\right)z' + \left(\frac{\partial\Psi_j}{\partial z}\frac{\partial z}{\partial\bar{z}'} + \frac{\partial\Psi_j}{\partial\bar{z}}\frac{\partial\bar{z}}{\partial\bar{z}'}\right)\bar{z}' \end{aligned}$$

where $\Psi'_j(z', \bar{z}') = \Psi_j(z(z', \bar{z}'), \bar{z}(z', \bar{z}))$. Since, as a short calculation shows,

$$\frac{\partial(z, \bar{z})}{\partial(z', \bar{z}')} = \frac{\partial(u, v)}{\partial(u', v')} \neq 0 ,$$

the complex rank of $(\partial\Psi_j/\partial z, \partial\Psi_j/\partial\bar{z})$ remains unaffected by a parameter change. We now make the following definition.

Definition 1. The two (real) dimensional plane in C^m defined parametrically by $w_j = \alpha_j z + \beta_j \bar{z}$ $(j = 1, 2, \cdots, m)$ is said to be *nonanalytic* if the rank of the 2 by m (complex) matrix (α_j, β_j) is 2.

The preceding remarks show that if π is nonanalytic in one coordinate parametrization then it remains so under any change of coordinates in the parameter plane. We want to establish the following.

LEMMA. *Suppose the tangent plane π to $\Psi(V)$ at the origin in C^m is nonanalytic. Then there is a polynomial in the coordinates w_j whose absolute value takes on a local maximum at the origin when restricted to $\Psi(V)$.*

Proof. Since π is nonanalytic there exist new coordinates $w'_i = \sum_{j=1}^{m} \gamma_{ij} w_j$ $(i = 1, \cdots, m)$, where the matrix (γ_{ij}) is nonsingular, such that in the w'_i-coordinates π is given parametrically by $w'_1 = z$, $w'_2 = \bar{z}$, and $w'_j = 0$ $(m \geqq j \geqq 3)$. Now let B and B' be points on π and $\Psi(V)$, respectively, corresponding to the parameters u and v. Let η_j and η'_j $(j = 1, \cdots, 2n)$ be the real coordinates of B and B' (with C^m regarded as R^{2n}) in the new coordinate system. Clearly $\eta_1 = u$, $\eta_2 = v$, $\eta_3 = u$, $\eta_4 = -v$, and $\eta_j = 0$ for $5 \leqq j \leqq 2n$. Let $\eta'_j - \eta_j = \varepsilon_j$ $(j = 1, \cdots, 4)$.

Now consider the function $P(w_i) = 1 - w'_1 w'_2$ (a polynomial in $w_1, \cdots, w_m$). When restricted to π, $P(w_i)$ is real-valued and has a maximum in absolute value at the origin. We would like to show that $|P(w_i)|$ also has a local maximum at the origin when restricted to a sufficiently small neighborhood of the origin on $\Psi(V)$. This will be true essentially because π has a contact of order at least 1 with $\Psi(V)$ at the origin (here we will use the estimates (2.2)).

We have, at the point B',

$$\begin{aligned} |P(B')|^2 &= |1 - (\eta'_1 + i\eta'_2)(\eta'_3 + i\eta'_4)|^2 \\ &= 1 - 2(u^2 + v^2) + (u^2 + v^2) + 2Q(u, v)[u^2 + v^2 - 1] \\ &\quad + [Q(u, v)]^2 + [u(\varepsilon_2 + \varepsilon_4) + v(\varepsilon_3 - \varepsilon_1)]^2 \end{aligned}$$

where

$$Q(u, v) = u(\varepsilon_1 + \varepsilon_3) + v(\varepsilon_2 - \varepsilon_4) + \varepsilon_1\varepsilon_3 - \varepsilon_2\varepsilon_4 .$$

Using inequalities (2.2) for $|\varepsilon_j|$ $(j = 1, \cdots, 4)$ we obtain

$$\begin{aligned}|P(B')|^2 &\leqq 1 - 2(u^2 + v^2) + M(|u| + |v|)^3 \\ &\leqq 1 - [1 - M(|u| + |v|)](|u| + |v|)^2\end{aligned}$$

for u and v sufficiently small and some constant M. If $|u| + |v| < 1/M$ we see that $|P(B')| < 1$ unless $B' = 0$.

3. Let K be a compact subset, with nonempty interior, of a Riemann surfce M.

DEFINITION 2. An algebra A of continuous functions on K is said to be a maximum modulus algebra on K if for every $f \in A$ there is a point p on the boundary ∂K of K such that $|f(q)| \leqq |f(p)|$ for all $q \in K$.

As remarked in [4], we can suppose without loss of generality that A is uniformly closed and contains the constants and so is a Banach algebra with identity and norm $||f|| = \sup_{p \in K} |f(p)|$. It is well known that there is a natural continuous mapping $i: K \to \Sigma_A$, where Σ_A is the maximal ideal space of A (with the usual Gelfand topology), defined by point evaluation (which is not $1:1$ unless A separates points in Σ_A).

THEOREM. *Let A be a uniformly closed algebra of continuous functions, containing the constants, on the compact subset K (with nonempty interior) of the Riemann surface M. Suppose that there is a set D of points in* int K *having no limit point in* int K, *such that each $p \in \operatorname{int} K - D$ has a neighborhood U for which $i(U)$ is open in Σ_A. Suppose further that A contains one nonconstant analytic function $g = g^1 + ig^2$. Then any function $f = f^1 + if^2$ in A such that f^1 and f^2 are twice continuously differentiable is analytic in* int K.

Proof. Let S be the discrete subset of int K on which the differential dg vanishes. For any point p in int $K - S$ there is a neighborhood containing p and contained in int $K - S$ and in which g is one-to-one. Thus for any point $p \in \operatorname{int} K - (D \cup S)$ there exists a neighborhood U containing p which is mapped homeomorphically by i onto an open set W in Σ_A and hence local coordinates in U may be transferred to W. Define the mapping $F: \Sigma_A \to C^2$ by $F(q') = (f(q'), g(q'))$, $q' \in \Sigma_A$ (where we have used the letters f and g to denote the extension, via the Gelfand representation, of f and g, defined on $i(K)$, to Σ_A). For any point q' in W we have $f(q') = f(i^{-1}(q'))$ and $g(q') = g(i^{-1}(q'))$ so that F can be regarded as a mapping defined on U by $F(q) = (f(q), g(q))$, $q \in U$. F defines an immersion of W since in the local coordinates $z = u + iv$ the matrix

$$\begin{pmatrix} f_u^1 & f_u^2 & g_u^1 & g_u^2 \\ f_v^1 & f_v^2 & g_v^1 & g_v^2 \end{pmatrix}$$

(here the subscripts u and v denote partial differentiation) is of rank 2 due to the nonvanishing of the differential $dg = \partial g/\partial z\, dz$—apply the Cauchy-Riemann equations to the matrix

$$\begin{pmatrix} g_u^1 & g_u^2 \\ g_v^1 & g_v^2 \end{pmatrix}.$$

Since A contains the constants we can suppose without loss of generality that $F(p)$ is the origin 0 in C^2. We have thus a mapping $F\colon \Sigma_A \to C^2$ which maps a neighborhood W of $i(p)$ onto a two-dimensional surface element $F(W)$ having a tangent plane π at 0.

We now note that π cannot be nonanalytic. For if this were the case then by the lemma of § 2 there would be a polynomial in the coordinates w_1 and w_2 of C^2 taking on a local maximum in absolute value at 0 when restricted to $F(W)$. By the Arens-Calderon theorem [1; Theorem 3.3] there would then be a function $k \in A$ taking on a local maximum at $i(p)$, and finally, by Rossi's Local Peak-Point Theorem [3, Theorem 4.1] there would be a function $\tilde{k} \in A$ taking on its maximum value exactly at $i(p)$, contradicting the fact that A is a maximum modulus algebra.

Thus the rank of

$$\begin{pmatrix} \dfrac{\partial g}{\partial z} & 0 \\ \dfrac{\partial f}{\partial z} & \dfrac{\partial f}{\partial \bar{z}} \end{pmatrix}$$

(the derivatives being evaluated in the local coordinates at p) must be 1 and this implies that $\partial f/\partial \bar{z} = 0$. The same conclusion could be drawn for any $p \in \inf K - (D \cup S)$ and so by the theorem of Riemann on removable singularities, f is analytic in $\operatorname{int} K$.

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MODULE CLASSES OF FINITE TYPE

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1. Finite type. In this paper we consider only rings with minimum condition on left and on right ideals. Also, we only consider finitely generated modules over these rings (such modules always possess a composition series of submodules).

There have been several papers [3, 4, 5, 10, 11, 12] on the problem of constructing indecomposable modules over such rings. Most of these papers are devoted to showing that certain rings have an infinite number of non-isomorphic modules of a given composition length for each of an infinite number of composition lengths. In this paper we shall consider a finiteness condition, not on the class of all finitely generated modules but on certain subclassses of that class.

DEFINITION. If C is a class of modules over the ring R we shall say that C is of finite type if for each integer n there are only a finite number of non-isomorphic modules in C of composition length less than n.

We shall study conditions under which the following classes of modules are of finite type:

1. LT the class of left modules which are submodules of projectives. From the results of [1], it is clear that these are the torsionless modules.
2. LW the class of left W-modules, these modules A for which $\mathrm{Ext}^1_R(A, R) = 0$
3. LN the non-torsionless left modules
4. LQ the torsionless left modules which are not duals of right modules.
5. LD the class of duals of right modules.
6. LR the class of reflexive left modules [1].
7. LTW the class of torsionless W-modules.

In the above definitions the dual of a module A is $\mathrm{Hom}_R(A, R)$ denoted by A^*. Also, A is reflexive if the natural homomorphism $A \to A^{**}$ is an isomorphism. See [7].

The corresponding classes of right modules (RT, RW, etc.) are defined analogously. All the theorems we prove go through with left and right interchanged.

A useful tool in our study is the following theorem proved by Morita and Tachikawa in [9] and also mentioned by Brauer in [2].

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THEOREM A. *If P is projective and if the diagram*

$$
\begin{array}{c}
0 \to X \to P \to A \to 0 \\
\downarrow \theta \\
0 \to Y \to P \to B \to 0
\end{array} \tag{1}
$$

has exact rows and if θ is an isomorphism, then the diagram can be embedded in the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\mu"'] & P \arrow[r] \arrow[d, "\rho"'] & A \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & Y \arrow[r] & P \arrow[r] & B \arrow[r] & 0
\end{tikzcd} \tag{1'}
$$

where ρ and μ are also isomorphisms.

It should be noted that the proof of Theorem A requires our standing hypothesis that every module under consideration has a composition series.

Before we deduce some corollaries from Theorem A, we need some additional information. Let l be the left composition length of the ring R and let r be the right composition length. Note that l and r need not be equal. Let $C(A)$ be the composition length of the module A.

LEMMA 1.1. *If the left module A has $C(A) = n$ then there exists a free module F_n, the direct sum of n copies of R considered as a left module, of composition length ln such that $F_n \to A \to 0$ is exact.*

The proof, an in induction on n, is essentially the same as the proof of Lemma 2.6 of [6]. By dualizing the above sequence we obtain

LEMMA 1.2. *If the left module A has $C(A) = n$ then A^* has composition length $\leqq nr$.*

Proof. The sequence of Lemma 1.1 induces $0 \to A^* \to F_n^*$ exact. The module F_n^* is a direct sum of copies of R considered as a right module [7] and hence $C(F_n^*) = nr$. Since A^* is a submodule of F_n^*, $C(A) \leqq nr$.

LEMMA 1.3. *If the left module A is torsionless with $C(A) = n$, then A can be embedded in a free module F such that $C(F) \leqq nrl$.*

Proof. By Lemma 1.2 $C(A^*) \leqq nr$ and by Lemma 1.1 there exists a free right module F_0 (the direct sum of nr copies of R) of composition length nr^2 such that $F_0 \to A^* \to 0$ is exact. This dualizes to

$$0 \to A^{**} \to F_0^* \qquad \text{exact}\ ,$$

whereby the proof of Lemma 1.2 $C(F_0^*) = nrl$. But since $A \to A^{**}$ is a monomorphism, this can be used to embed A in F_0^*. The idea of the above proof is due to Bass [1], although, being in a more general situation he was not concerned there with composition length.

It should be noted that the inequalities of Lemma 1.2 and 1.3 are, for most rings, quite crude. Using the above lemmas, and Theorem A have the following results.

THEOREM 1.4. *If LN is of finite type then so is LQ.*

Proof. Suppose that for some n there were an infinite number of non-isomorphic modules $\{T_\alpha\}$ in LQ all of composition length n. Then by Lemma 1.3 we can embed them *all* as submodules of a free module F of composition length nlr. Consider the infinite collection of factors $\{F/T_\alpha\}$. By [1, 7] these are modules in LN all having composition length $n(lr-1)$.

But the hypotheses of the theorem require that there are only a finite number of non-isomorphic modules in LN of each composition length. Thus for some $\alpha \neq \beta$ $F/T_\alpha \cong F/T_\beta$ and by Theorem A we have $T_\alpha \cong T_\beta$. This contradicts the assumption that the collection $\{T_\alpha\}$ consists of non-isomorphic modules.

The following theorem is modeled on the duality Theorem 1.1 of [7].

THEOREM 1.5. *LT is of finite type if and only if RT is of finite type.*

Proof. By right-left symmetry it is sufficient to prove the statement in one direction only.

Suppose that RT is of finite type and $\{T_\alpha\}$ is an infinite collection of non-isomorphic torsionless left modules all of composition length n. By Lemma 1.1 there is a free module F of composition length ln and an infinite collection of short exact sequences

$$F \xrightarrow{\mu_\alpha} T_\alpha \longrightarrow 0\ .$$

Now form the dual exact sequences.

$$0 \longrightarrow T_\alpha^* \xrightarrow{\mu_\alpha^*} F^* \longrightarrow F^*/T_\alpha^* \longrightarrow 0\ .$$

The right modules F/T_α^* are torsionless right modules [1; statements 4.2 and 4.4] and each of these modules has composition length less than rn. Since RT is of finite type there exist two indices α and β such that $F^*/T_\alpha^* \xrightarrow{\theta} F^*/T_\beta^*$ is an isomorphism. Using Theorem A we construct the exact commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_\alpha^* & \xrightarrow{\mu_\alpha} & F^* & \longrightarrow & F^*/T_\alpha^* & \longrightarrow & 0 \\ & & \mu\big\downarrow & & \big\downarrow\rho & & \big\downarrow\theta & & \\ 0 & \longrightarrow & T_\beta^* & \xrightarrow[\mu_\beta]{} & F^* & \longrightarrow & F^*/T_\beta^* & \longrightarrow & 0 \end{array}$$

with vertical isomorphisms. This gives the commutative diagram

$$\begin{array}{ccc} F^{**} & \xrightarrow{\mu_\alpha^{**}} & T_\beta^{**} \\ \big\downarrow\rho^* & & \big\downarrow\mu^* \\ F^{**} & \xrightarrow[\mu_\beta^{**}]{} & T_\alpha^{**} \end{array} .$$

In this situation $Im\mu_\beta{}^{**}$ coincides with the natural image of T_β in T_β^{**} and the similar situation holds for the subscript α. Then commutativity then implies that T_α is isomorphic with T_β via the isomorphism μ^*. This contradicts the assumption that the collection $\{T_\alpha\}$ consisted of non-isomorphic modules.

2. A dual to Theorem A. A dual to Theorem A would state that if two submodules of a free module F were isomorphic, then the isomorphism can be extended to an automorphism of F. This is not, in general, true as we shall show by an example. However, by assuming enough extra conditions we can obtain the desired conclusion. Recall that X is a W-module if $\mathrm{Ext}_R^1(X, R) = 0$; see [8].

THEOREM 2.1. *If in the diagram*

$$\begin{array}{ccccccccc} 0 & \to & A & \to & F & \to & F/A & \to & 0 \\ & & \big\downarrow\theta & & & & & & \\ 0 & \to & B & \to & F & \to & F/B & \to & 0 \end{array}$$

θ is an isomorphism, F is a free module and F/A and F/B are W modules, then the diagram can be embedded in a commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & A & \to & F & \to & F/A & \to & 0 \\ & & \big\downarrow\theta & & \big\downarrow\rho^* & & \big\downarrow\mu & & \\ 0 & \to & B & \to & F & \to & F/B & \to & 0 \end{array}$$

with all the vertical maps isomorphisms.

Proof. Consider the dual sequences

$$(*)\qquad \begin{array}{ccccccccc} 0 & \to & (F/A)^* & \to & F^* & \to & A^* & \to & 0 \\ & & & & & & \uparrow{\scriptstyle \theta^*} & & \\ 0 & \to & (F/B)^* & \to & F^* & \to & B^* & \to & 0 \end{array}$$

The exactness at A^* and B^* comes from the fact that F/A and F/B are W-modules. Also θ^* is an isomorphism because θ is one. By Theorem A there exists an automorphism ρ of F^* so that the diagram

$$\begin{array}{ccccc} F^* & \to & A^* & \to & 0 \\ \uparrow{\scriptstyle \rho} & & \uparrow{\scriptstyle \theta^*} & & \\ F^* & \to & B^* & \to & 0 \end{array}$$

is commutative.

Now dualize again to obtain the commutative diagram

$$\begin{array}{ccccc} 0 & \to & A^{**} & \to & F^{**} \\ & & \downarrow{\scriptstyle \theta^{**}} & & \downarrow{\scriptstyle \rho^*} \\ 0 & \to & B^{**} & \to & F^{**} \end{array}$$

Since both A, B are torsionless and F is reflexive [1, 7] we can identify A and B with their images in A^{**} and B^{**}. Also the mappings with two stars on them, when restricted to these images, coincide with the original maps. Thus, identifying F with F^{**}, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & F & \to & F/A & \to & 0 \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \rho^*} & & \downarrow{\scriptstyle \mu} & & \\ 0 & \to & B & \to & F & \to & F/B & \to & 0 \end{array}$$

where ρ^* induces μ on F/A to F/B. All the vertical maps are isomorphisms.

COROLLARY 2.2. *If LT is of finite type then so is LW.*

Proof. Suppose $\{W_\alpha\}$ is an infinite collection of nonisomorphic W-modules such that $C(W_\alpha) = n$. By Lemma 1.1 they are all epimorphic images of a free module F, $F \xrightarrow{\pi_\alpha} W_\alpha \to 0$ and $C(F) = ln$. The submodules Ker π_α of F all satisfy $C(\ker \pi_\alpha) = (l-1)n$ and by the assumption that LT is of finite type there exist two indices $\alpha \neq \beta$

such that $\operatorname{Ker}\pi_\alpha \cong \operatorname{Ker}\pi_\beta$. Now Theorem 2.1 implies that $W_\alpha \cong W_\beta$ contradicting the assumption that the elements in the collection $\{W_\alpha\}$ were non-isomorphic.

COROLLARY 2.3. *LTW is of finite type if and only if LR is of finite type.*

Proof. For the "if" part of the proof we proceed exactly as in the proof of Corollary 2.2. We use this fact, proved in [8], that if W is a torsionless W-module and

$$0 \to \operatorname{Ker}\pi \to F \to W \to 0$$

is exact with F free then $\operatorname{Ker}\pi$ is reflexive. Then the proof of 2.2 with the class LR replacing LT works here.

Conversely, if LTW is of finite type and if $\{Q_\alpha\}$ is an infinite collection of reflexives with $C(Q_\alpha) = n$, then by Lemma 1.3 they can all be embedded in a free module F with $C(F) \leqq lnr$,

$$0 \to Q_\alpha \to F\,.$$

But by [8] this embedding of the reflexive Q_α results in F/Q_α being a torsionless W-module. Hence by assumption there exists $\alpha \neq \beta$ such that $F/Q_\alpha \cong F/Q_\beta$. Then Theorem A implies $Q_\alpha \cong Q_\beta$ contradicting the assumption that the collection $\{Q_\alpha\}$ consists of non-isomorphic modules.

We conclude with an example which shows that Theorem 2.1. does not hold without the hypothesis that F/A and F/B are W-modules. Let R be the ring of matrices

$$\begin{pmatrix} x & 0 & 0 \\ y & x & 0 \\ z & 0 & x \end{pmatrix}$$

with x, y, z in a field K having more than 2 elements. R is commutative and is an indecomposable free module over itself. The radical N of R is the direct sum of two simple modules, $N = S_1 \oplus S_2$. If α, β are two distinct nonzero elements of K there is an automorphism θ of N which is "multiplication by α on S_1 and multiplication by β on S_2". Any extension of θ to a K-linear transformation on R will have two distinct eigenvalues. However, since R is indecomposable every module endomorphism (or automorphism) has only one eigenvalue, therefor θ cannot be extended to R.

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ON DENSITIES OF SETS OF LATTICE POINTS

BETTY KVARDA

1. Introduction. Let A be a set of positive integers, and for any positive integer x denote by $A(x)$ the number of integers of A which are not greater than x. Then the Schnirelmann density of A is defined [4] to be the quantity

$$\alpha = \operatorname*{glb}_{x} \frac{A(x)}{x} .$$

For any k sets $A_1, \cdots, A_k$ of positive integers, $k \geqq 2$, let the sum set $A_1 + \cdots + A_k$ be the set of all nonzero sums $a_1 + \cdots + a_k$ for which each a_i, $i = 1, \cdots, k$, is either contained in A_i or is 0. Let kA be the set $A + \cdots + A$ with k summands.

Schnirelmann [4] and Landau [2] have shown that if A and B are two sets of positive integers with $C = A + B$, and if α, β, γ are the Schnirelmann densities of A, B, C, respectively, then $\gamma \geqq \alpha + \beta - \alpha\beta$, and if $\alpha + \beta \geqq 1$ then $\gamma = 1$. They have also shown that if A is a set of positive integers whose Schnirelmann density is positive then A is a basic sequence for the set of positive integers, or, in other words, there exists a positive integer k such that every positive integer can be written as the sum of at most k elements of A.

We will show that by using extensions of the methods employed by Schnirelmann and Landau the above results can be generalized to certain sets of vectors in a discrete lattice (for definition and discussion see [3, pp. 28–31] or [5, pp. 141–145]). Without loss of generality it may be assumed that the components of the vectors in such a lattice are rational integers. The usual identification of algebraic integers with lattice points then gives an immediate extension of these results to algebraic integers.

2. Notation and definitions. Let Q_n be the set of all n-dimensional lattice points $(x_1, \cdots, x_n)$, $n \geqq 1$, for which each x_i, $i = 1, \cdots, n$, is a nonnegative integer and at least one x_i is positive. Define the sum of subsets of Q_n in the same manner as was done for sets of positive integers, and for any subsets A and B of Q_n let $A - B$ denote the set of all elements of A which are not in B. If A and S are subsets of Q_n and S is finite let $A(S)$ be the number of elements in $A \cap S$.

DEFINITION 1. A finite nonempty subset R of Q_n will be called a

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fundamental subset of Q_n or, briefly, a *fundamental set*, if whenever an element $(r_1, \cdots, r_n)$ is in R then all elements $(x_1, \cdots, x_n)$ of Q_n such that $x_i \leqq r_i$, $i = 1, \cdots, n$, are also in R.

DEFINITION 2. Let A be any subset of Q_n. The *density* of A is defined to be the quantity

$$\alpha = \operatorname{glb} \frac{A(R)}{Q_n(R)}$$

taken over all fundamental sets R.

3. Extension of the Landau-Schnirelmann results. Throughout this section we let A and B be subsets of Q_n with $C = A + B$, and let α, β, γ be the densities of A, B, C, respectively.

THEOREM 1. *If $\alpha + \beta \geqq 1$ then $\gamma = 1$.*

Proof. Assume $\gamma < 1$. Then there exists a fundamental set R for which $C(R) < Q_n(R)$, which in turn implies that there exists an element $(x_1^0, \cdots, x_n^0)$ in $Q_n - C$. Let R_0 be the set of all elements $(x_1, \cdots, x_n)$ in Q_n for which $x_i \leqq x_i^0$, $i=1, \cdots, n$. Then for any $(x_1, \cdots, x_n)$ in R_0 either $(x_1, \cdots, x_n)$ is in A, or $(x_1, \cdots, x_n) = (x_1^0, \cdots, x_n^0) - (b_1, \cdots, b_n)$ for some $(b_1, \cdots, b_n)$ in $B \cap R_0$, or neither, but not both. In particular, $(x_1^0, \cdots, x_n^0)$ is neither. Hence,

$$A(R_0) + B(R_0) \leqq Q_n(R_0) - 1 ,$$

and

$$\alpha + \beta \leqq \frac{A(R_0) + B(R_0)}{Q_n(R_0)} < 1$$

which is a contradiction. Therefore $\gamma = 1$.

THEOREM 2. $\gamma \geqq \alpha + \beta - \alpha\beta$.

Proof. Let ω_i, $1 \leqq i \leqq n$, be that vector in Q_n for which the ith component is 1 and the other components, if any, are 0. If any one of the vectors $\omega_1, \cdots, \omega_n$ is missing from A then $\alpha = 0$ and the theorem is trivial. Hence we assume all the vectors $\omega_1, \cdots, \omega_n$ are in A. We must show

$$\frac{C(R)}{Q_n(R)} \geqq \alpha + \beta - \alpha\beta \tag{1}$$

for all fundamental sets R. If $C(R) = Q_n(R)$ then (1) holds, since

$(1-\alpha)(1-\beta) \geqq 0$ implies $1 \geqq \alpha + \beta - \alpha\beta$. Therefore we assume $C(R) < Q_n(R)$ and, consequently, $A(R) < Q_n(R)$.

Let $H = R - A$. We will show that there exist vectors $a^{(1)}, \cdots, a^{(s)}$ in A and sets $L_1, \cdots, L_s$ with the following properties.

(i) $L_i \subseteqq H$ and L_i is not empty, $i = 1, \cdots, s$.

(ii) The sets $L'_i = \{x - a^{(i)} \mid x \in L_i\}$ are fundamental sets.

(iii) $L_i \cap L_j = \phi$ for $i \neq j$.

(iv) $H = L_1 \cup \cdots \cup L_s$.

Let the elements of R be ordered so that $(x_1, \cdots, x_n) > (x'_1, \cdots, x'_n)$ if $x_1 > x'_1$ or if $x_1 = x'_1, \cdots, x_p = x'_p, x_{p+1} > x'_{p+1}$. For every $h = (h_1, \cdots, h_n)$ in H, let A_h be the set of all $(a_1, \cdots, a_n)$ in A such that each $a_i \leqq h_i$. The sets A_h are not empty since $\omega_i \in A$ for $i = 1, \cdots, n$. The A_h are finite sets, hence they contain (in our ordering) a largest vector. Let $a^{(1)}, \cdots, a^{(s)}$ be all the distinct vectors that are largest vectors in any A_h. Let L_i be the set of all vectors x in H such that $a^{(i)}$ is the largest vector in A_x.

That (i), (iii), and (iv) are satisfied follows immediately from this definition of the L_i. To prove (ii) consider a vector $y = (y_1, \cdots, y_n)$ such that

$$x_j \geqq y_j \geqq a_j^{(i)} , \tag{2}$$

where $x = (x_1, \cdots, x_n)$ is in L_i and $y \neq a^{(i)}$. Suppose $y \in L_k$, $k \neq i$. Then

$$x_j \geqq y_j \geqq a_j^{(k)} \tag{3}$$

and $a^{(k)} \geqq a^{(i)}$. But (2) and (3) and $x \in L_i$ imply $a^{(k)} \leqq a^{(i)}$, hence $a^{(k)} = a^{(i)}$. Similarly, $y \in A$ implies $y = a^{(i)}$. This proves (ii).

If $b \in B \cap L'_i$ then $a^{(i)} + b$ is in $C \cap L_i$, hence in $C - A$. Therefore,

$$\begin{aligned} C(R) &\geqq A(R) + B(L'_1) + \cdots + B(L'_s) \\ &\geqq A(R) + \beta[Q_n(L'_1) + \cdots + Q_n(L'_s)] \\ &= A(R) + \beta[Q_n(L_1) + \cdots + Q_n(L_s)] \\ &= A(R) + \beta[Q_n(H)] \\ &= A(R) + \beta[Q_n(R) - A(R)] \\ &= (1-\beta)A(R) + \beta[Q_n(R)] \\ &\geqq (1-\beta)\alpha[Q_n(R)] + \beta[Q_n(R)] , \end{aligned}$$

and

$$\frac{C(R)}{Q_n(R)} \geqq \alpha + \beta - \alpha\beta ,$$

which completes the proof.

COROLLARY 1. *Let $A_1, \cdots, A_k$ be any k subsets of Q_n, $k \geqq 2$, let α_i be the density of A_i for $i = 1, \cdots, k$, and let $d(A_1 + \cdots + A_k)$ be the density of $A_1 + \cdots + A_k$. Then*

$$1 - d(A_1 + \cdots + A_k) \leqq (1 - \alpha_1) \cdots (1 - \alpha_k) .$$

Proof. If $k = 2$ then Theorem 2 implies that $1 - d(A_1 + A_2) \leqq 1-\alpha_1-\alpha_2+\alpha_1\alpha_2=(1-\alpha_1)(1-\alpha_2)$. Hence assume $1 - d(A_1 + \cdots + A_{k-1}) \leqq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})$. Then

$$\begin{aligned} 1 - d(A_1 + \cdots + A_{k-1} + A_k) &\leqq [1 - d(A_1 + \cdots + A_{k-1})](1 - \alpha_k) \\ &\leqq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})(1 - \alpha_k) . \end{aligned}$$

COROLLARY 2. *If A is any subset of Q_n with density $\alpha > 0$ then there exists an integer $k > 0$ such that $kA = Q_n$.*

Proof. There exists an integer $m > 0$ such that $(1 - \alpha)^m \leqq 1/2$. Let $d(mA)$ be the density of mA. Then Corollary 1 implies that $1 - d(mA) \leqq (1 - \alpha)^m \leqq 1/2$, or $d(mA) \geqq 1/2$. From Theorem 1, $d(mA) + d(mA) \geqq 1$ implies $d(2mA) = 1$, or $2mA = Q_n$.

4. Remark. We may identify Q_2 with the set of nonzero Gaussian integers $x + yi$ for which x and y are both nonnegative rational integers. Luther Cheo [1] defined density for subsets of this Q_2 as follows, using our notation.

DEFINITION 3. Let $x_0 + y_0 i$ be any element of Q_2 and S the set of all $x + yi$ in Q_2 such that $x \leqq x_0$ and $y \leqq y_0$. Then for any subset A of Q_2 the *density* of A is the quantity

$$\alpha_c = \operatorname*{glb}_S \frac{A(S)}{Q_2(S)} .$$

Cheo proved Theorem 1 for his density and also a theorem which implies that if ji is in A for all $j = 1, 2, \cdots$, and if $\alpha_c, \beta_c, \gamma_c$ are the Cheo densities of $A, B, C = A + B$, respectively, then

$$\gamma_c \geqq \alpha_c + \beta_c - \alpha_c\beta_c .$$

We cannot remove the requirement that all ji be in A by means of an argument like that used to establish Theorem 2 since it would be necessary to partition H in such a way that the sets L'_j are of the type S used in defining the Cheo density, and this is not always possible. Consider, for example, the set $R = \{x + yi: x + yi \text{ is in } Q_2, x \leqq 4, y \leqq 3\}$, and let $A \cap R = \{1, i, 3 + 3i\}$. Then $H = R - A$ cannot be so partitioned, as the reader can easily verify.

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A GEOMETRIC CHARACTERIZATION FOR A CLASS OF DISCONTINUOUS GROUPS OF LINEAR FRACTIONAL TRANSFORMATIONS

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Let $\mathfrak{B} = \{V_i \mid V_i z = (a_i z + b_i)/(c_i z + d_i);\ a_i d_i - b_i c_i = 1, i = 1, 2, \cdots\}$ be a group of linear fractional transformations, where a_i, b_i, c_i, d_i ($i = 1, 2, \cdots$) denote complex numbers. As indicated we use V_i and $V_i z$ to denote transformations and we use (linear) transformation in short for linear fractional transformation. A point z of the plane (by plane we mean, of course, plane of complex numbers) is called a limit point of $\mathfrak{B}$ if there exists a point z_0 and an infinite sequence of distinct transformations of $\mathfrak{B}$, say, $\{U_i\}$ such that $U_i z_0 \to z$ as $i \to \infty$. A point of the plane which is not a limit point is called an ordinary point of the group. A discontinuous group is one for which there exists an ordinary point. If $c_i \neq 0$, we define $I(V_i) = \{z \mid |c_i z + d_i| = 1\}$ and $K(V_i) = \{z \mid |c_i z + d_i| < 1\}$, called the isometric circle and isometric disk of V_i, respectively. The main result is contained in the following theorem which is proved in Part I of this paper.

THEOREM 1. *Let $\mathfrak{B}$ be a group of linear fractional transformations all of whose elements (except the identity) possess isometric circles whose radii are bounded. Then $\mathfrak{B}$ is discontinuous if and only if there exists an open set of points in the plane that is exterior to the union of all isometric circles.*

According to the theorem discontinuity for the class of groups in question could be defined in terms of the geometry of the isometric circles. In addition, it will be shown that the set of points exterior to the union of all isometric circles could be used to construct a fundamental region for these groups. This last result removes certain restrictions on a known result which is found in [1] (p. 39–49). There Ford shows that if a group is discontinuous and if infinity is an ordinary point, then the radii of all isometric circles are bounded and some neighborhood of infinity is exterior to the union of all isometric circles. The set of points exterior to the isometric circles he uses to construct a fundamental region for the group. In Ford's proof the fact that infinity is an ordinary point is crucial. For the

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class of discontinuous groups characterized by Theorem 1 we remove that distinguished role of infinity. We would like to mention that Ford uses the concept of 'proper discontinuity' rather than discontinuity as defined here. However the results carry over, since the two concepts are equivalent ([2]).

In Part II of this paper we give an example for a group $\mathfrak{V}$ of linear transformations for which the closed disks $\bar{K}(V_i)$ $(i = 1, 2, \cdots)$ cover the plane. By Theorem 1 it follows that $\mathfrak{V}$ is not discontinuous. This shows that the set of groups of linear transformations which satisfy the hypotheses of Theorem 1 and whose isometric circles cover the plane is not empty. This group is also discrete; it is therefore, like the Picard group, an example of a discrete group which is not discontinuous.

PART I

We say that the plane is almost covered by closed disks if the points that are exterior to the union of all the disks do not comprise an open set of the plane. In the following for cover or almost cover we write in short 'cover'. First we prove

THEOREM 2. *Let $\mathfrak{V}$ be a group of linear transformations all of whose elements save the identity possess isometric circles. If the isometric disks $\bar{K}$ 'cover' the plane and if their radii are bounded, then $\mathfrak{V}$ is not discontinuous.*

It is no restriction to assume that the number of transformations in $\mathfrak{V}$ is denumerable, since this is a necessary condition for discontinuity of $\mathfrak{V}$. First we prove six lemmas about the group $\mathfrak{V}$. We assume throughout that the hypotheses of Theorem 2 hold.

LEMMA 1. *If z_0 is an elliptic fixed point of order $n > 1$, then every point z equivalent to z_0 under $\mathfrak{V}$ is an elliptic fixed point of order n.*

The proof is easy and we omit it here.

LEMMA 2. *Let $\{I_n\}$ be a sequence of isometric circles of infinitely many distinct transformations of $\mathfrak{V}$ with radii r_n $(n = 1, 2, \cdots)$. If the centers of the I_n converge to the finite point δ, then the sequence $\{r_n\}$ is a nullsequence.*

Proof. Let $U_n z = (a_n z + b_n)/(c_n z + d_n)$ be the elements of $\mathfrak{V}$ for which $I(U_n) = I_n$ $(n = 1, 2, \cdots)$. Since the radii are bounded, the sequence of positive constants $\{r_n\}$ has at least one limit point. Let r be a limit point, and let us assume that $r > 0$. Then, on a subsequence, we have $\lim_{j\to\infty} r_j = \lim_{j\to\infty} (1/|c_j|) = r$. The sequence $\{c_j\}$, where $c_j = (1/r_j)e^{i\varphi_j}$, has a finite limit point $c \neq 0$; and, on a subsequence, $\lim_{k\to\infty} c_k = c$. To every c_k of the last subsequence corresponds a U_k whose isometric circle has center $-d_k/c_k$. By hypothesis we have $\lim_{k\to\infty}(-d_k/c_k) = \delta$. On noting that $U_k U_{k+1}^{-1} \neq I$ and that the matrix of the transformation $U_k U_{k+1}^{-1}$ is of the form $\begin{pmatrix} x & x \\ c_k d_{k+1} - c_{k+1}d_k & x \end{pmatrix}$, where the x stands for certain complex numbers, we deduce that for sufficiently large k $|c_k d_{k+1} - c_{k+1}d_k| = |(d_{k+1}/c_{k+1} - d_k/c_k)c_k c_{k+1}| < \varepsilon$, where ε is arbitrarily small and positive. But this is impossible, since all elements of $\mathfrak{B}$ (except I) possess isometric circles whose radii are bounded.

LEMMA 3. *If z_0 lies within infinitely many isometric circles, then it is a limit point of $\mathfrak{B}$.*

Proof. Since the radii of the isometric circles are bounded, every neighborhood of infinity contains centers of isometric circles. Thus infinity as an accumulation point of such centers is a limit point of $\mathfrak{B}$. This in turn implies that the centers of all isometric circles are limit points of the group.

Let g_n denote the center of I_n $(n = 1, 2, \cdots)$. Let K be a positive real number such that $r_n < K$ for $n = 1, 2, \cdots$, and let $\{g_j\}$ be a sequence of centers of those isometric circles that satisfy the hypotheses of the lemma. Since $|z_0 - g_j| < K$ or $|g_j| < K + |z_0|$, the sequence $\{g_j\}$ is bounded. We pick a limit point δ. Then, on a subsequence, we have $\lim_{k\to\infty} g_k = \delta$. Let the sequence $\{I_k\}$ correspond to the last subsequence. Since $|z_0 - g_k| < r_k$ and since by Lemma 2 $\lim_{k\to\infty} r_k = 0$, we deduce $\lim_{k\to\infty} g_k = z_0$. Thus z_0 as accumulation point of limit points of $\mathfrak{B}$ is itself a limit point.

LEMMA 4. *If every neighborhood of a point z_0 contains arcs of infinitely many isometric circles, then z_0 is a limit point of $\mathfrak{B}$.*

Proof. Since the lemma certainly holds when z_0 is an accumulation point of centers of isometric circles, we assume that these centers are bounded away from z_0. Let C be a circle with center z_0 and of radius ρ so that the centers of all isometric circles lie outside C; let C' be a circle with center z_0 and of radius $\rho/2$. We consider the infinite set of isometric circles $\varphi = \{I_n \mid I_n \cap C' \neq \Lambda; n = 1, 2, \cdots\}$, where Λ denotes the empty set. Their radii $r_n > \rho/2$ $(n = 1, 2, \cdots)$. The sequence $\{g_n\}$ consisting of the centers of the isometric circles in φ is bounded (see proof of Lemma 3). If δ denotes a limit point of $\{g_n\}$

then $|\delta - z_0| \geqq \rho$, and, on a subsequence, we have $\lim_{k\to\infty} g_k = \delta$. To this subsequence corresponds the sequence of isometric circles $\{I_k\}$ whose centers accumulate at δ only. By Lemma 2 the sequence $\{r_k\}$ is a null sequence, which contradicts $r_n > \rho/2$ for all n.

When every neighborhood of a point z_0 contains arcs of isometric circles we say that z_0 is an accumulation point of arcs of isometric circles. We observe that Lemma 4 includes the case where infinitely many of the circles pass through z_0. In view of the hypothesis that the isometric circles 'cover' the plane, a consequence of Lemma 4 is

LEMMA 5. *If z_0 is exterior to all isometric circles, then it is a limit point of* $\mathfrak{B}$.

LEMMA 6. *If z_0 is not an accumulation point of arcs of isometric circles and if it does not lie within an isometric circle, then z_0 lies on at least three isometric circles.*

Proof. Clearly, z_0 cannot be exterior to all isometric circles. Thus it lies on at least one isometric circle. If only one or two circles were to pass through z_0 we could construct a neighborhood of z_0 sufficiently small so that all other isometric circles lie outside this neighborhood. In either case the neighborhood contains an open set that is exterior to all isometric circles. Observing that three circles passing through z_0 can be arranged so that all points in a sufficiently small deleted neighborhood of z_0 lie within a circle the lemma follows.

These preliminary results we use now in the proof of Theorem 2. Let $\mathscr{L}$ be the set of limit points of $\mathfrak{B}$ on the Riemann sphere. Then $\mathscr{L}$, which is a closed set ([1], p. 43), contains

(i) the centers of all isometric circles (see proof of Lemma 3),

(ii) the nonelliptic fixed points of all transformations of $\mathfrak{B}$ (we assume that $\mathfrak{B}$ contains elliptic transformations of finite order only, since, if that were not the case, the theorem would be trivial),

(iii) the points that lie outside all isometric circles (see Lemma 5),

(iv) the points which are accumulation points of arcs of isometric circles (see Lemma 4).

Suppose that there is a point $z \notin \mathscr{L}$. Since then every point in a sufficiently small neighborhood of z is an ordinary point of $\mathfrak{B}$ (the set of ordinary points on the Riemann sphere is open, since $\mathscr{L}$ is closed) and since the number of elliptic fixed points is at most denumerable, it is no restriction to assume that z lies within an isometric circle and that z is not an elliptic fixed point. The isometric circle within which z lies we denote by $I(U_1)$. U_1 carries z into z_1, where $z_1 \neq z$ and where z_1 lies outside $I(U_1^{-1})$. Furthermore, $z_1 \notin \mathscr{L}$, since an ordinary point is not mapped on a limit point. Either z_1 lies within isometric circles, in which case we pick one of them and call it $I(U_2)$, or, according to Lemma 6, it lies on at least three isometric circles. In the latter case $I(U_2)$ denotes any one of them. Certainly $U_2 \neq U_1^{-1}$. In the same manner we proceed with $z_2 = U_2 z_1 = (U_2 U_1)z$. Again, because of Lemma 1 $z_2 \neq z_1$, and $z_2 \notin \mathscr{L}$. If z_2 does not lie within an isometric circle and if z_1 lies on $I(U_2)$, then z_2 lies on $I(U_2^{-1})$. By Lemma 6, it is then possible to pick $I(U_3) \neq I(U_2^{-1})$, and hence $U_3 \neq U_2^{-1}$. Continuing in this manner we obtain an infinite number of transformations $W_n = U_n U_{n-1} \cdots U_1$ $(n = 1, 2, \cdots)$, where $U_i U_{i-1} \neq I$ $(i = 2, 3, \cdots)$. Because of Lemma 3 the proof of Theorem 2 will be complete if we can show that z lies within $I(W_n)$ for $n = 1, 2, \cdots$.

Let $U_i z = (a_i z + b_i)/(c_i z + d_i)$, and let $\delta(U_i, z) = |c_i z + d_i|^{-2}$, called the deformation of U_i ([2]). Then $\delta(U_i, z)$ is greater than, equal to, or less than one according as z lies within, on, or outside $I(U_i)$. It is readily verified that

$$\delta(U_j U_i, z) = \delta(U_j, U_i z)\delta(U_i, z) , \tag{1}$$

and by an induction argument the formula can be extended to a product of any number of transformations. For $n \geqq 1$ we have $\delta(W_n, z) = \delta(U_n \cdots U_1, z) = \delta(U_1, z)\delta(U_2, U_1 z) \cdots \delta(U_n, U_{n-1} \cdots U_1 z) > 1$, since $\delta(U_1, z) > 1$ and every other factor $\delta(U_k, U_{k-1} \cdots U_1 z) \geqq 1 (k = 2, \cdots, n)$. This implies that z lies within $I(W_n)$ $(n = 1, 2, \cdots)$. Hence $z \in \mathscr{L}$; a contradiction.

If, however, the isometric disks $\bar{K}$ of $\mathfrak{B}$ do not 'cover' the plane we have the following theorem.

THEOREM 3. *Let $\mathfrak{B}$ be a group of linear fractional transformations all of whose elements (except the identity) possess isometric circles. If there exists an open set of points that is exterior to all isometric circles, then $\mathfrak{B}$ is discontinuous.*

Proof. Let $\mathscr{O}$ be the open set in the hypothesis. Pick z_0 in $\mathscr{O}$, where z_0 is finite. There exists a ε-neighborhood N_ε of z_0 such that $z \in N_\varepsilon$ implies $z \in \mathscr{O}$. For any transformation V in $\mathfrak{B}$ $(V \neq I)$ Vz_0 lies within $I(V^{-1})$, or $|Vz_0 - z_0| > \varepsilon$. Thus z_0 is a standard point of $\mathfrak{B}$ ([3], p. 38). Since every standard point is an ordinary point ([3], p. 47), $\mathfrak{B}$ is discontinuous.

This completes the proof of Theorem 1, since Theorems 2 and 3 imply the former.

Next, the remark about the fundamental region in the introduction calls for further elucidation. Let $\mathfrak{B}$ be a discontinuous group that satisfies the hypothesis of Theorem 1 and for which infinity is a limit point (for the case in which infinity is an ordinary point the following is well known). Let K_i ($i = 1, 2, \cdots$) be the isometric disks of $\mathfrak{B}$, let $\mathcal{O}$ be the set of points exterior to the set $\bigcup_i K_i$, and let $\mathcal{M}$ denote the set of ordinary points of $\mathfrak{B}$. Then $\mathcal{O}$ and $\mathcal{M}$ are open sets. Furthermore, $\mathcal{O} \subset \mathcal{M}$ is an immediate consequence of the proof of Theorem 3. Here, we do not intend to give a definition for a fundamental region. However, we want to show that $\mathcal{O}$ has the two properties that are customarily used in any definition; namely,

(i) no two points of $\mathcal{O}$ are equivalent under $\mathfrak{B}$ and

(ii) every point in $\mathcal{M}$ is equivalent to some point in $\bar{\mathcal{O}}$, the closure of the set $\mathcal{O}$.

As for (i) we note that if $z \in \mathcal{O}$ and $V \in \mathfrak{B}$ ($V \neq 1$), then Vz lies within $I(V^{-1})$ and hence is exterior to $\mathcal{O}$.

The gist of our proof of (ii) is the same as that of the corresponding proof in [2], where infinity is considered to be an ordinary point of the group. In our proof we make use of the following lemma, where we use primes to denote derivatives.

LEMMA 7. *Let $f_i(z)$ ($i = 1, \cdots, k$), where k is an integer greater than* 1, *be nonvanishing holomorphic functions in a domain $\mathcal{D}$, and let for $j \neq i$ $f_i'(z_0)f_j(z_0) - f_j'(z_0)\ f_i(z_0) \neq 0$ and $|f_i(z_0)| = |f_j(z_0)|$ $(i, j = 1, \cdots, k)$ for some point z_0 in $\mathcal{D}$. Then every neighborhood of z_0 in $\mathcal{D}$ contains a point z^* such that $|f_i(z^*)| \neq |f_j(z^*)|$ for $j \neq i$ $(i, j = 1, \cdots, k)$.*

Proof. For $i, j = 1, \cdots, k$ and $i < j$ we define the functions $f_{ij}(z) = f_i(z)/f_j(z)$. We observe that $|f_{ij}(z_0)| = 1$ and $f_{ij}'(z_0) \neq 0$. We choose the (circular) neighborhood $N(z_0)$ of z_0 so small that the mappings $f_{ij}(z)$ are one-to-one. For each function $f_{ij}(z)$ the level curve $|f_{ij}(z)| = 1$ consists of a finite number of disjoint analytic arcs in $N(z_0)$. If we pick a point z^* in $N(z_0)$ that does not lie on any level curve the conclusion of the lemma holds.

Let $z_0 \in \mathcal{M}$ and $z_0 \notin \bar{\mathcal{O}}$. In view of Lemma 3 z_0 lies within or on a finite number of isometric circles. Let U_i ($i = 1, \cdots, n$) denote the transformations in $\mathfrak{B}$ whose isometric circles $I(U_i)$ have this property. Since every element of $\mathfrak{B}$ (save the identity) possesses an isometric circle, no two of the $I(U_i)$'s coincide.

We divide the proof into two parts.

(i) We assume $\delta(U_1, z_0) > \delta(U_i, z_0)(i = 2, \cdots, n)$. Then U_1z_0 lies in $\mathcal{O}$. For suppose that U_1z_0 lies within or on some isometric circle $I(V)$. Using (1) for the deformation of the transformation VU_1 we deduce $\delta(VU_1, z_0) = \delta(V, U_1z_0)\ \delta(U_1, z_0) \geqq \delta(U_1, z_0)$. This would imply that z_0 lies within or on $I(VU_1)$. Hence $VU_1 = U_i$ for some i with $1 < i \leqq n$; which contradicts the maximum property of $\delta(U_1, z_0)$. We remark that the proof still holds for $n = 1$.

(ii) We assume $\delta(U_1, z_0) = \delta(U_2, z_0) = \cdots = \delta(U_k, z_0) > \delta(U_i, z_0)$ for $k < i \leqq n (1 < k \leqq n)$. Let $N(z_0)$ be a neighborhood of z_0 containing only ordinary points of $\mathfrak{V}$ and being so small that $\delta(U_j, z) > \delta(U_i, z)$ $(j = 1, \cdots, k; i = k + 1, \cdots, n)$ holds for all z in $N(z_0)$ and that $N(z_0)$ does not intersect any isometric disk other than the $K(U_i)$ $(i = 1, \cdots, n)$. As one readily verifies the functions $U_j'(z)$ $(j = 1, \cdots, k)$, where the prime denotes the derivative, satisfy the hypothesis of Lemma 7 with $N(z_0)$ in place of $\mathscr{D}$. Thus we conclude that every neighborhood of z_0 contains a point z^* having the property that, for some integer m with $1 \leqq m \leqq k$, $\delta(U_m, z^*) > \delta(U_j, z^*)$ $(j = 1, \cdots, k; j \neq m)$. By part (i) of this proof it follows that $U_m(z^*) \in \mathscr{O}$; and by continuity we have $U_{m^*}(z_0) \in \mathscr{O}$ for some suitable m^* with $1 \leqq m^* \leqq k$. This completes the proof about the two properties of $\mathscr{O}$.

We conclude this part with a remark. Let $\mathfrak{V}$ be a nondiscontinuous group of linear transformations satisfying the hypotheses of Theorem 2. For $V \in \mathfrak{V}$ we denote by V^* the 2×2 matrix that can be associated with Vz. Then $\mathfrak{V}^* = \{\pm V^* \mid V \in \mathfrak{V}\}$ is a group under matrix multiplication. Since every element of $\mathscr{V}$ (save the identity) possesses an isometric circle and since all the radii are bounded, in every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $\mathfrak{V}^*$ (save the two identity matrices) $c \neq 0$ and all the c's are bounded away from 0. This implies that $\mathfrak{V}^*$ is discrete; that is, $\mathfrak{V}^*$ does not contain a sequence of distinct matrices $\{V_n^*\}$ such that $V_n^* \to I^*$ as $n \to \infty$, where $I^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus all the nondiscontinuous groups characterized in Theorem 2 are such that the corresponding groups of matrices are discrete. That nondiscontinuous groups, as considered here, exist is not a trivial fact. We dovote the second part of this paper to the construction of a group of this type.

PART II

Here we give an example of a group of linear transformations that contains only elements with isometric circles which have the property that the closed isometric disks cover the plane. We divide the construction into three parts.

1. We construct a covering of the plane by closed disks $\bar{K}$ such that the open disks K are mutually disjoint. To begin with, we draw circles of radius unity and with centers at the points with coordinates $(2m + 1, 2n + 1)$ $(m, n = 0, \pm 1, \pm 2, \cdots)$. After drawing circles of radius $(\sqrt{2} - 1)$ units and with centers $(2m, 2n)$ $(m, n = 0, \pm 1, \pm 2, \cdots)$, there remain the interiors of congruent triangles whose sides are circular arcs uncovered. Within every triangle we construct a circle touching all three sides, and we continue in this manner. The follow-

ing proof[1] shows that by this construction every point of the plane lies within or on a circle.

Diagram 1 shows a triangle we encounter in our construction,

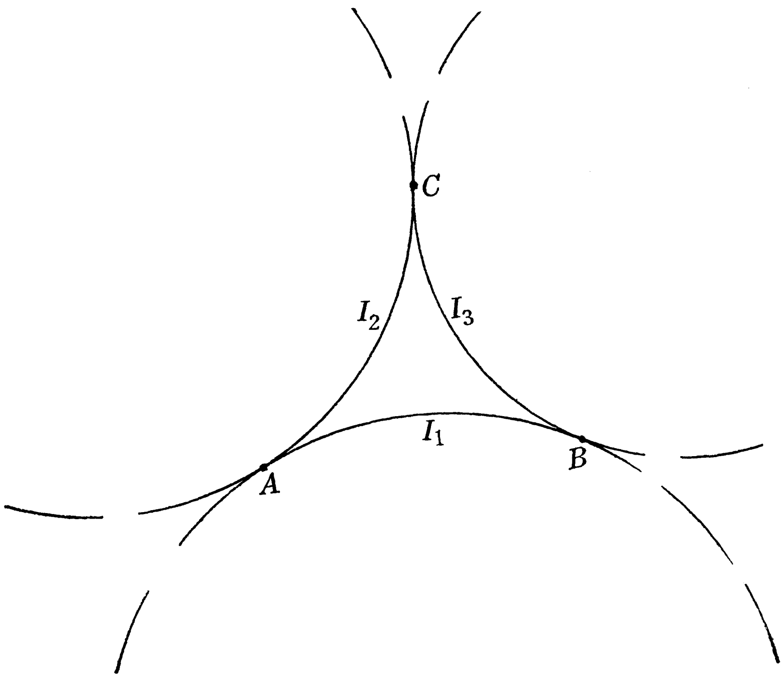

Diagram 1.

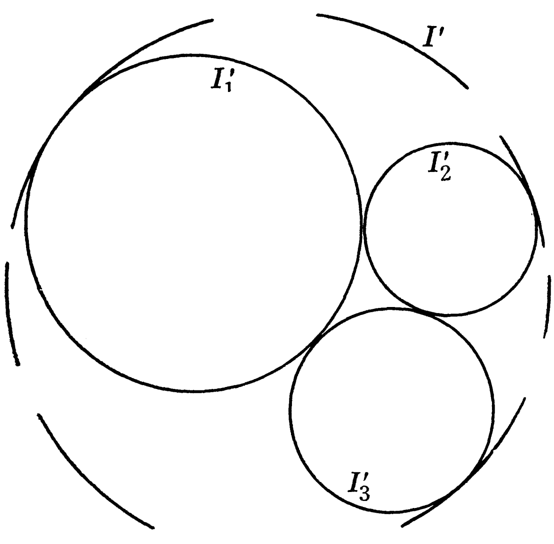

Diagram 2.

[1] This proof I owe to F. Herzog.

where I_1, I_2 and I_3 denote the circles whose arcs form the sides of triangle ABC. Let z_0 be any point in the interior of the triangle, and let S be a linear fractional transformation such that $Sz_0 = \infty$. Depending on the relative sizes of the image circles I'_k of I_k under $S(k = 1, 2, 3)$, we distinguish two cases. Either, (i) we can construct a circle I' such that the $I'_k (k = 1, 2, 3)$ are tangent internally to I' as indicated in Diagram 2. Then S^{-1} maps I' on a circle I that touches the three sides of triangle ABC and contains z_0 in its interior.

Or (ii) no circle I' as assumed in (i) exists. Then the configuration of the circles I'_k $(k = 1, 2, 3)$ resembles the one in Diagram 3, where triangle $A'B'C'$ is the image of triangle ABC under S.

Let t_1 be the common tangent to I'_1 and I'_3 through B', and let t_2 be the common tangent to I'_2 and I'_3 through C'. Clearly, the two tangents intersect in a point that lies within triangle $A'B'C'$. Let t be that common tangent to I'_1 and I'_2 which is shown in Diagram 3. We construct the circle K'_1 which is tangent externally to the I'_k

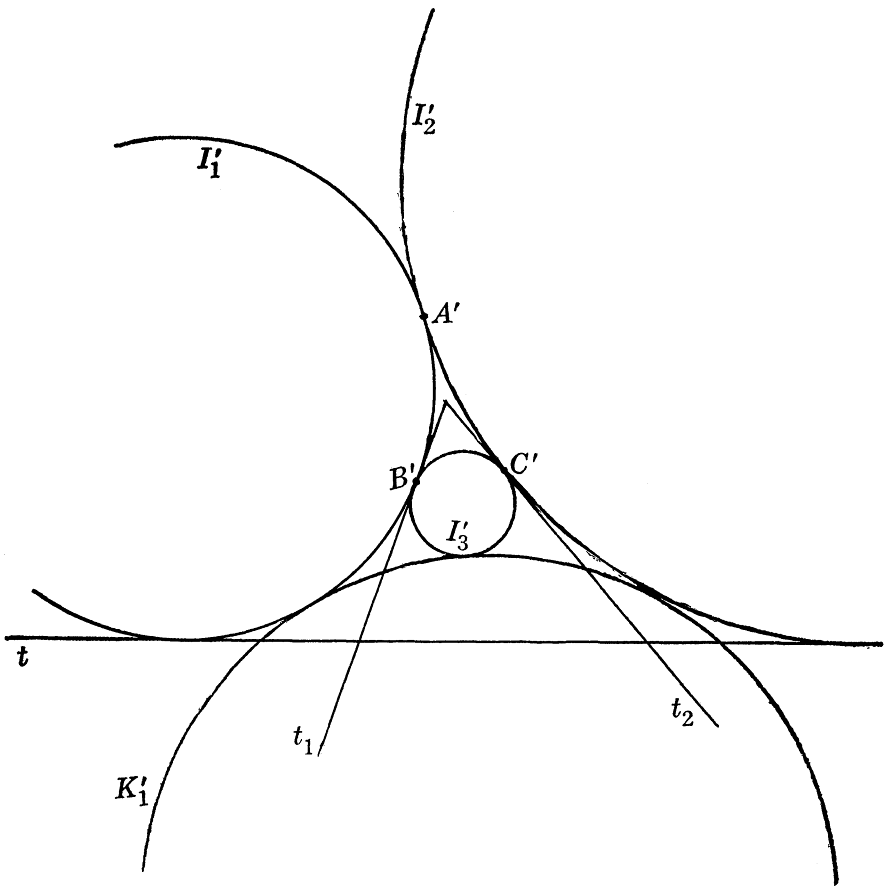

Diagram 3.

($k = 1, 2, 3$) and which lies outside triangle $A'B'C'$. Next we construct the circle K_2' which is tangent externally to I_1', I_2' and K_1'. We continue this process until we come across the first circle, say, K_s' that is tangent externally to I_1', I_2' and K_{s-1}' and that intersects or touches the line t. That the K_s' exists follows from the fact that the radii of successive circles K_i' ($i = 1, \cdots, s$) increase. Rigorous arguments for this we omit, in order not to lengthen the paper unduly. Finally we construct the circle I' which contains the circles I_1', I_2' and K_s' in its interior and which is tangent with each of them. In the case when K_s' touches the tangent line t, I' is the degenerate circle t. Under the mapping S^{-1} we obtain a chain of circles within triangle ABC, each K_i, where $K_i = S^{-1}K_i'$, being tangent externally to I_1, I_2 and K_{i-1} ($i = 2, \cdots, s$). $S^{-1}I' = I$ is the circle that is tangent externally to I_1, I_2 and K_s and that contains z_0 in its interior. In the degenerate case I will pass through z_0. This completes the proof.

2. In order to associate linear transformations with the covering circles we group them in pairs of circles of equal radii. The circles with centers $(2n, 0)$ ($n = 0, \pm 1, \pm 2, \cdots$) and with radii $(\sqrt{2} - 1)$ units we pair in some way so that all are used up. The remaining circles whose interiors intersect the y-axis are mapped under reflection in the x-axis on congruent circles. Each circle we pair with its image under this reflection. Under reflection in the y-axis all the remaining circles are mapped on congruent circles, and we pair them accordingly.

With every pair of circles I and I', with centers α and β, respectively, and with radii r, we associate a linear fractional transformation V such that I is the isometric circle of V and I' that of V^{-1}. It is readily verified that V is of the form

$$Vz = [(\beta/r)e^{i\varphi}z - ((\alpha\beta/r)e^{i\varphi} + re^{-i\varphi})]/[(1/r)e^{i\varphi}z - (\alpha/r)e^{i\varphi}] ,$$

where φ may be chosen arbitrarily and where we used the normalisation det. $V = 1$.

Since every circle contains a point with rational coordinates, the number of transformations is denumerable. We denote them by V_1, V_1^{-1}, V_2, V_2^{-1}, $\cdots$. Let $G_i = \{V_i^k \mid k = 0, \pm 1, \cdots\}$ ($V_i^0 = I$) and let $\mathfrak{V} = *\prod_i G_i$, the free product of the cyclic groups G_i. If $\bigcup_i G_i$ denotes the union of the G_i, then $\bigcup_i G_i = \mathfrak{V}$ ([2]).

3. There remains to be shown that each element ($\neq I$) of $\mathfrak{V}$ has an isometric circle and that the radii of all circles are bounded. These properties are consequences of the following lemma.

LEMMA 8. *Let $\{T_n\}$ be an infinite sequence of linear transformations which with every T_n contains T_n^{-1}, and all of whose elements* *possess isometric circles. Let* K_n *denote the isometric disk* $K(T_n)$ *and let* $\wedge$ *denote the empty set. If* $K_i \cap K_k = \wedge$ *for* $k \neq i$ $(i, k = 1, 2, \cdots)$, *then* $K_{n_1} \supset K$, *the isometric disk of* $W = T_{n_s} T_{n_{s-1}} \cdots T_{n_1}$, *where* $n_1, \cdots, n_s$ *are arbitrary positive integers, not necessarily distinct, except that* $T_{n_{i+1}} T_{n_i} \neq I$.

Proof. For $i = 1, \cdots, s$ we put $T_{n_i} = S_i$, $I_i = I(S_i)$, $I_i' = I(S_i^{-1})$, and $K_i' = K(S_i^{-1})$. Let z be any point outside I_1. Then $S_1 z$ lies in K_1', and hence outside I_2. $S_2(S_1 z)$ lies in K_2', and hence outside I_3. By an induction argument it follows that Wz lies in K_s'. Since at each step lengths in the neighborhood of z or its images are decreased the lemma follows.

Let $\{U_n\}$ be a sequence comprising the transformations V_1, V_1^{-1}, V_2, V_2^{-1}, $\cdots$. Then $\{U_n\}$ satisfies the hypotheses of Lemma 8 and, in addition, the radii of the isometric circles of the U_n do not exceed unity. Hence we conclude that the radii of all elements of the cyclic groups G_i $(i = 1, 2, \cdots)$ as well as those of $\mathfrak{B}$ do not exceed unity.

We close with a remark. That the free product $*\prod_i G_i$ is an isomorphic image of $\bigcup_i G_i$ is an easy consequence of Lemma 8.

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SIMPLE PATHS ON POLYHEDRA

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In Euclidean d-space ($d \geqq 3$) consider a convex polytope whose $n\,(n \geqq d+1)$ vertices do not lie in a $(d-1)$-space. By the "path length" of such a polytope is meant the maximum number of its vertices which can be included in any single simple path, i.e., a path along its edges which does not pass through any given vertex more than once. Let $p(n, d)$ denote the minimum path length of all such polytopes of n vertices in d-space. Brown [**1**] has shown that $p(n, 3) \leqq (2n+13)/3$ and Grünbaum and Motzkin [**3**] have shown that $p(n, d) < 2(d-2)n^{\alpha}$ for some $\alpha < 1$, e.g., $\alpha = 1 - 2^{-19}$ and they have indicated how this last value may be improved to $\alpha = 1 - 2^{-16}$. The main object of this note is to derive the following result which, for sufficiently large values of n, represents an improvement upon the previously published bounds.

THEOREM.
$$p(n, d) < (2d+3)\big((1 - 2/(d+1))n - (d-2)\big)^{\log 2/\log d} - 1 < 3d\, n^{\log 2/\log d}.$$

When $d = 3$ the example we construct to imply our bound is built upon a tetrahedron which we denote by G_0. Its 4 vertices, which will be called the 0th stage vertices, can all be included in a single simple path. Upon each of the 4 triangular faces of G_0 erect a pyramid in such a way that the resulting solid, G_1, is a convex polyhedron with 12 triangular faces. This introduces 4 more vertices, the 1st stage vertices, which can be included in a single simple path involving all 8 vertices of G_1. We may observe that it is impossible for a path to go from a 1st stage vertex to another 1st stage vertex without first passing through a 0th stage vertex.

The convex polyhedron G_2 is formed by erecting pyramids upon all the faces of G_1. Of the 12 2nd stage vertices thus introduced at most 9 can be included in any single simple path since, as before, no path can join two 2nd stage vertices without passing through an intermediate vertex of a lower stage and there are only 8 such vertices available.

The procedure continues as follows: the convex polyhedron G_k, $k \geqq 2$, is formed by erecting pyramids upon the 4.3^{k-1} triangular faces of G_{k-1}. Making repeated use of the fact that the method of construction makes it impossible for a path to join two vertices of the jth stage, $j \geqq 2$, without first passing through at least one vertex of a lower stage we find that at most 9.2^{j-2} of the 4.3^{j-1} vertices of the

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jth stage, $j = 2, 3, \cdots, k$, can be included in a single simple path along the edges of G_k. This and the earlier remarks imply that G_k, $k \geqq 1$, has $2 \cdot 3^k + 2$ vertices and at most $9 \cdot 2^{k-1} - 1$ of these can be included in a single simple path.

For any integer $n > 4$ let k be the unique integer such that

$$(1) \qquad 2 \cdot 3^k + 2 < n \leqq 2 \cdot 3^{k+1} + 2 .$$

Next consider the convex polyhedron with n vertices which can be obtained by erecting pyramids upon $n - (2 \cdot 3^k + 2)$ faces of G_k. Then, from considerations similar to those given before, it follows, using (1), that

$$(2) \qquad p(n, 3) \leqq 9 \cdot 2^k - 1 < 9((n-2)/2)^{\log 2/\log 3} - 1 .$$

This suffices to complete the proof of the theorem when $d = 3$ since the result is trivially true when $n = 4$.

In the general case the construction starts with a d-dimensional simplex. Upon each of its $(d-1)$-dimensional faces is formed another d-dimensional simplex by the introduction of a new vertex on the side of the face opposite to the rest of the original simplex in such a way that the resulting polytope is convex. This process is repeated and the rest of the argument is completely analogous to that given for the case $d = 3$. It should be pointed out that the result of Grünbaum and Motzkin holds even for graphs all of whose vertices, but for a bounded number are incident with 3 edges, while in the polytopes described above the distribution of valences is quite different.

In closing we remark that the path length of any 3-dimensional convex polyhedron with n vertices is certainly greater than

$$(\log_2 n/\log_2 \log_2 n) - 1 .$$

Suppose that there exists a vertex, q say, upon which at least $\log_2 n/\log_2 \log_2 n$ edges are incident. Let the vertices at the other ends of these edges be $p_1, p_2, \cdots, p_t$, arranged in counterclockwise order. Each pair, (p_i, p_{i+1}), $i = 1, \cdots, t-1$, of successive vertices in this sequence determines a unique polygonal face containing the edges $\overline{p_{i+1}q}$ and $\overline{qp_i}$. Traversing this face in a counterclockwise sense gives a path from p_i to p_{i+1} involving at least one edge. Since these faces all lie in different planes it is not difficult to see that these paths may be combined to give a simple path from q to p_1 to p_t whose length is at least $t \geqq \log_2 n/\log_2 \log_2 n$. If there is no vertex upon which this many edges are incident then the required result follows from the type of argument used by Dirac [**2**; Theorem 5] in showing that the path length is at least of the magnitude of $\log n$ if only a bounded number of edges are incident upon any vertex.

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REPRESENTATION OF A POINT OF A SET AS SUM OF TRANSFORMS OF BOUNDARY POINTS

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In a previous paper [1] we established a condition (Theorem I) for real numbers such that, in a linear space of dimension at least 2, every point of a 2-bounded set can always be represented as a sum of boundary points of the set, multiplied by these numbers. It is natural to ask for the corresponding condition in the case of complex numbers. Multiplication of a point by a real or complex number can be regarded as a special similarity. A more general theorem in which these similarities are replaced by linear transformations, or operators, will be proved in the present paper.

DEFINITION. Let B be a real Banach space with conjugate space B'. Let $S \subset B$ and $x' \in B'$, $\|x'\| = 1$. The *x'-width* of S is

$$w_{x'}(S) = \sup_{x,y \in S} (x - y)x' , \qquad w_{x'}(\phi) = -\infty .$$

The *width* of S is $w(S) = \inf w_{x'}(S)$.

Let $\mathfrak{A}$ be a linear transformation of B and $\mathfrak{A}^*$ the adjoint operation on B' defined by $x(x'\mathfrak{A}^*) = (x\mathfrak{A})x'$. Then $x'\mathfrak{A}^* = 0$ or we can define $x'_{\mathfrak{A}} = x'\mathfrak{A}^*/\|x'\mathfrak{A}^*\|$.

In the following all sets are assumed to be in a real Banach space.

LEMMA 1. (1) *If S is bounded then $w_{x'}(S)$ is a continuous function of x'.*

(2) $w_{x'}(S + T) = w_{x'}(S) + w_{x}(T)$ (*with the proviso that $-\infty$ added to anything—even $+\infty$—is $-\infty$*).

(3) *If S has interior points then $w(S) > 0$.*

(4) $$w_{x'}(S\mathfrak{A}) = \begin{cases} 0 & \text{if } x'\mathfrak{A}^* = 0 ; \\ w_{x'_{\mathfrak{A}}}(S) \cdot \|x'\mathfrak{A}^*\| & \text{if } x'\mathfrak{A}^* \neq 0 . \end{cases}$$

The proofs are all obvious.

LEMMA 2. *Let T be a connected set so that no translate of $-T$ is contained in the interior of S, then $S + T \subset T + \operatorname{bd} S$.*

Proof. Let $s \in S$, $t \in T$; then $s + t - T$ contains $s \in S$ but is not contained in the interior of S. Hence $(s + t - T) \cap \operatorname{bd} S$ is not empty and $s + T \subset T + \operatorname{bd} S$.

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LEMMA 3. *If S is bounded and $-\mathrm{cl} S \subset \mathrm{int}\, T$ then no translate of $-\mathrm{cl}\, T$ is contained in* $\mathrm{int}\, S$.

Proof. For one-dimensional spaces this is obvious since the hypothesis implies $\mathrm{diam}\, S < \mathrm{diam}\, T$. If the lemma were false then $a - \mathrm{cl}\, T \subset \mathrm{int}\, S$ for some point a. The mapping $x \to a - x$ leaves the lines through $a/2$ invariant and the contradiction follows from the fact that the inclusion is false for the intersection of the sets with such lines l for which $l \cap \mathrm{int}\, S \neq \phi$.

LEMMA 4. *Let $w_{x'}(S) < \infty$, let T be a connected set, and let $U = (S + T)\backslash(T + \mathrm{bd}\, S)$, then*

$$w_{x'}(U) \leqq w_{x'}(S) - w_{x'}(T) .$$

Proof. If $w_{x'}(T) = \infty$ then $S + T \subset T + \mathrm{bd}\, S$ by Lemma 2. If $w_{x'}(T) < \infty$ let $a = \inf_{s \in S} sx'$, $b = \sup_{s \in S} sx'$, $c = \inf_{t \in T} tx'$, $d = \sup_{t \in T} tx'$. If $s \in S$, $t \in T$ so that $(s + t)x' < a + d$ then $s + t - T$ contains s in S and $\inf_{t_1 \in T} (s + t - t_1)x' < a$ so that $s + t - T$ contains points in the complement of S. Since $s + t - T$ is connected it follows that $(s + t - T) \cap \mathrm{bd}\, S \neq \phi$ or $s + t \in T + \mathrm{bd}\, S$. Thus $\inf_{u \in U} ux' \geqq a + d$.

Similarly, if $s \in S$, $t \in T$ and $(s + t)x' > b + c$ then $s + t - T$ contains $s \in S$ while $\sup_{t_1 \in T} (s + t - t_1)x' > b$ so that $s + t - T$ contains points in the complement of S. Hence $(s + t - T) \cap \mathrm{bd}\, S \neq \phi$ and $s + t \in T + \mathrm{bd}\, S$. Thus $\sup_{u \in U} ux' \leqq b + c$, and hence

$$\begin{aligned} w_{x'}(U) &= \sup_{u \in U} ux' - \inf_{u \in U} ux' \leqq (b + c) - (a + d) = (b - a) - (d - c) \\ &= w_{x'}(S) - w_{x}(T) . \end{aligned}$$

DEFINITION. Let S be a bounded connected set in B. The *outer set, oS, of S* is the complement of the unbounded component of the complement of S and the *outer boundary*, $\mathrm{obd}\, S$, of S is the boundary of oS. Clearly $\mathrm{obd}\, S \subset \mathrm{bd}\, S$ and if $\dim B \geqq 2$ then $\mathrm{obd}\, S$ is connected.

THEOREM 1. *Let $S_1, S_2, \cdots, S_n$ be bounded connected sets in B with $\dim B \geqq 2$ so that no translate of $-\mathrm{cl}\ oS_1$ is contained in $\mathrm{int}\, oS_i$ $(i = 2, \cdots, n)$. Then*

$$\begin{aligned} &w_{x'}((S_1 + S_2 + \cdots + S_n)\backslash(\mathrm{obd}\, S_1 + \mathrm{obd}\, S_2 + \cdots + \mathrm{obd}\, S_n)) \\ &\qquad \leqq w_{x'}(S_1) - w_{x'}(S_2) - \cdots - w_{x'}(S_n) . \end{aligned}$$

Proof. By repeated application of Lemma 2 we have $S_1 + \cdots + S_n \subset oS_1 + \cdots + oS_n \subset oS_1 + \mathrm{obd}\, S_2 + \cdots + \mathrm{obd}\, S_n$ and the theorem follows from Lemma 4 where oS_1 plays the role of S and $\mathrm{obd}\, S_2 + \cdots + \mathrm{obd}\, S_n$ that of T.

COROLLARY. *If $S_1, \cdots, S_n$ satisfy the conditions of Theorem* 1 *and in addition for each i there is an x'_i so that $w_{x'_i}(S_i) < \sum_{j \neq i} w_{x'_i}(S_j)$ then $S_1 + \cdots + S_n \subset \operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n$.*

DEFINITION. Let B be a real Banach space with $\dim B \geqq 2$. A set of bounded linear operators $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is *admissible* if for every bounded set $S \subset B$ and every point $p \in S$ there exist outer boundary points $x_1, \cdots, x_n \in \operatorname{obd} S$ such that

$$p = x_1\mathfrak{A}_1 + \cdots + x_n\mathfrak{A}_n .$$

THEOREM 2. *If a set $\mathfrak{A}$ of operators $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ is admissible then*
(i) $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n = \mathscr{I}$, *the identity.*
(ii) *For each i there exists an $x' \in B'$, $x' \neq 0$ such that*

$$\| x'\mathfrak{A}_i^* \| \leqq \sum_{j \neq i} \| x'\mathfrak{A}_j^* \| .$$

If B is finite dimensional, $\dim B \geqq 2$, and $\mathfrak{A}$ satisfies (i) *and*

$$\text{(ii}'\text{)} \qquad \| x'\mathfrak{A}_i^* \| \leqq \sum \| x'\mathfrak{A}_j^* \| , \qquad i = 1, \cdots, n$$

for all $x' \in B'$ then $\mathfrak{A}$ is admissible.

Proof. The necessity of (i) and (ii) is nearly obvious. If $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n \neq \mathscr{I}$, let $p \in B$ be a point which is not invariant under $\mathfrak{A}_1 + \cdots + \mathfrak{A}_n$ and let $S = \{p\}$.

If S is the unit ball of B and

$$0 = x_1\mathfrak{A}_1 + \cdots + x_n\mathfrak{A}_n , \qquad \| x_1 \| = \cdots = \| x_n \| = 1$$

then

$$\| x_i\mathfrak{A}_i x' \| \leqq \sum_{j \neq i} \| x_j\mathfrak{A}_j x' \|$$

or

$$\| x_i x'\mathfrak{A}_i^* \| \leqq \sum_{j \neq i} \| x_j x'\mathfrak{A}_j^* \| .$$

Now if $\inf_{\|x\|=1} \| x\mathfrak{A}_i \| = 0$, then for every $\varepsilon > 0$ there exists an x' with $\| x' \| = 1$ and $\| x'\mathfrak{A}_i^* \| < \varepsilon$ and (ii) is trivial. If $\inf_{\|x\|=1} \| x\mathfrak{A}_i \| > 0$ then $\mathfrak{A}_i^*$ is onto and we can pick x' so that $\| x_i x'\mathfrak{A}_i^* \| = \| x'\mathfrak{A}_i^* \|$ and hence $\| x'\mathfrak{A}_i^* \| \leqq \sum_{j \neq i} \| x_j x'\mathfrak{A}_j^* \| \leqq \sum_{j \neq i} \| x'\mathfrak{A}_j^* \|$.

To prove the sufficiency of (i) and (ii′) we may restrict attention to connected sets since we may consider the component of p in S. Let $S_i = S\mathfrak{A}_i$. If for each S_i there is an S_j so that $j \neq i$ and no translate of $-\operatorname{cl} S_j$ is contained in $\operatorname{int} S_i$ then according to Lemma 2 we have

$$\begin{aligned} S \subset S_1 + \cdots + S_n &\subset oS_1 + \cdots + oS_n \\ &\subset \operatorname{obd} S_1 + (oS_2 + \cdots + oS_n) \\ &\subset \operatorname{obd} S_1 + \operatorname{obd} S_2 + (oS_3 + \cdots + oS_n) \subset \cdots \\ &\subset \operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n . \end{aligned}$$

Since B is finite dimensional we have $\operatorname{obd} S_i = (\operatorname{obd} S)\mathfrak{A}_i$ so that

$$S \subset (\operatorname{obd} S)\mathfrak{A}_1 + \cdots + (\operatorname{obd} S)\mathfrak{A}_n$$

which was to be proved. We may therefore assume that $-\operatorname{cl} S_j$ has a translate in $\operatorname{int} S_1$ for each $j = 2, \cdots, n$. Then according to Lemma 3 and Theorem 1

$$\begin{aligned} (1) \qquad & w_{x'}((S_1 + \cdots + S_n)\backslash(\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n)) \\ & \leqq w_{x'}(S_1) - w_{x'}(S_2) - \cdots - w_{x'}(S_n) . \end{aligned}$$

Since S_1 has an interior $\mathfrak{A}_1$, and hence $\mathfrak{A}_1^*$, are regular and we can choose x' so that $w_{x'_1}(S) = w(S)$ where $x'_1 = x'\mathfrak{A}_1^*/\| x'\mathfrak{A}_1^* \|$. By part (4) of Lemma 1 we have $w_{x'}(S_j) \geqq w(S) \cdot \| x'\mathfrak{A}_j \|$. Thus (1) becomes

$$\begin{aligned} w_{x'}((S_1 + \cdots S_n)\backslash(\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n)) &\leqq w(S)(\| x'\mathfrak{A}_1^* \| - \sum_{j \neq 1} \| x'\mathfrak{A}_j^* \|) \\ &\leqq 0 \end{aligned}$$

so that $(S_1 + \cdots + S_n)\backslash(\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n)$ has no interior points and is therefore empty since $\operatorname{obd} S_1 + \cdots + \operatorname{obd} S_n$ is closed. So we have again

$$\begin{aligned} S \subset S_1 + \cdots + S_n \subset \operatorname{obd} S_1 &+ \cdots + \operatorname{obd} S_n \\ &= (\operatorname{obd} S)\mathfrak{A}_1 + \cdots + (\operatorname{obd} S)\mathfrak{A}_n . \end{aligned}$$

REMARK. The hypothesis that B is finite dimensional can be dropped if we assume that the mappings $\mathfrak{A}_i$ are onto. If the $\mathfrak{A}_i$ are similarities of B onto itself then (ii) and (ii′) have the same simple form

$$(\text{ii}'') \qquad \| \mathfrak{A}_i \| \leqq \sum_{j \neq i} \| \mathfrak{A}_j \| \qquad i = 1, \cdots, n .$$

We thus have the following:

THEOREM 2′. *A set of similarities $\mathfrak{A}_1, \cdots, \mathfrak{A}_n$ of a Banach space B of dimension at least 2 onto itself is admissible if and only if it satisfies conditions* (i) *and* (ii″).

In the manner analogous to that used in [1] we can generalize the validity of Theorem 2 to a class of linear spaces which we define as follows.

DEFINITIONS. Let B be a linear space and let $\mathscr{F}$ be a family of linear transformations of B onto itself so that $\mathscr{F}$ is transitive on the nonzero elements of B. A *B-space* S is a linear subspace of a (finite or infinite) direct product of copies of B that is closed under simultaneous application of $\mathscr{F}$ to the components of a point. If x, $y \in S$ and $y \neq 0$ then $\{x + yF \mid F \in \mathscr{F}\}$ is a *B-subspace* of S. The B-subspaces can be given the topology of B by the association $x+yF \leftrightarrow zF$, $z \in B$, $z \neq 0$ where the choice of z is arbitrary due to the transitivity of $\mathscr{F}$. We can therefore define boundedness in B-subspaces (if boundedness is defined in B) and a set in S is *B-bounded* if through every point of the set there is a B-subspace whose intersection with the set is bounded.

THEOREM 3. *Theorem 2 remains valid for B-bounded sets in a B-space where B satisfies the conditions stated in Theorem 2. If B is one-dimensional then the same theorem holds for sets which are 2-bounded (in the sense of* [1]) *and satisfy the other conditions of Theorem 2.*

This is an immediate consequence of Theorem 2 if we consider the bounded intersection of S with a B-subspace through a point p of S.

Theorem 3 applied to the conditions of Theorem 2′ subsums the results of [1]. As one application we give the following:

THEOREM 4. *Let $f(z)$ be analytic in a proper subdomain D of the Riemann sphere and continuous in* $\operatorname{cl} D$. *Let $\alpha_1, \cdots, \alpha_n$ be complex numbers satisfying*

$$\alpha_1 + \cdots + \alpha_n = 1 \tag{i}$$

and

$$|\alpha_i| \leqq \sum_{i \neq j} |\alpha_j| . \tag{ii}$$

Then for every $z_0 \in D$ there exist $z_1, \cdots, z_n$ in $\operatorname{bd} D$ *such that*

$$f(z_0) = \alpha_1 f(z_1) + \cdots + \alpha_n f(z_n) .$$

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AN ANALOGUE OF KOLMOGOROV'S THREE-SERIES THEOREM FOR ABSTRACT RANDOM VARIABLES

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1. Introduction. $(\Omega, \mathscr{M}, P)$ is a probability space i.e. Ω is an abstract set of points w, $\mathscr{M}$ is a σ-field of subsets of Ω and P is a nonnegative countably additive set function defined on $\mathscr{M}$ such that $P(\Omega) = 1$. G is a locally compact Hausdorff abelian metric topological group. The group operation in G, as well as in the several other groups to be dealt with, will be denoted by $+$. Let e denote the identity element of G. By the Borel sets of G we mean the sets belonging to the σ-ring generated by the class $\mathscr{C}$ of compact subsets of G. Let $\mathscr{D}$ be the class of subsets of G whose intersection with every compact set is a Borel set. Notice that $\mathscr{D}$ is a σ-field containing the open subsets of G. The character group of G will be denoted by $\hat{G}$. A single valued mapping f of Ω into G will be called a generalised random variable (g.r.v.) if $f^{-1}(A) \in \mathscr{M}$ whenever $A \in \mathscr{D}$. An immediate consequence of this definition is that if f is a g.r.v. then $\eta(f)$ is an ordinary (complex valued) random variable for every $\eta \in \hat{G}$. A finite or a countably infinite collection of g.r.v.'s is said to be independent if and only if for every finite subset $\{X_i, i = 1, 2, \cdots, n\}$ of distinct members of the collection and for every choice of sets $A_j \in \mathscr{D}$, $j = 1, 2, \cdots, n$ it is true that $P\{w: X_i(w) \in A_i, i = 1, 2, \cdots, n\} = \prod_1^n P\{w: X_i(w) \in A_i\}$.

If G is the real line, $\hat{G}$ is the real line too. For $t \in \hat{G}$ and $x \in G$, $t(x) = \exp(itx)$. Given the random variable X and any real number $c > 0$ we define a new random variable $Y = t_0\alpha$ where $t_0 = c/\pi$ and α is the principal amplitude of $\exp(i\pi X/c)$. The two sets $\{w: -c < X(w) \leqq c\}$ and $\{w: X(w) \neq Y(w)\}$ are then seen to be equal. Denoting by N the interval $(-c, c]$, the classical three series theorem [2] may be stated thus: If $\{X_n, n = 1, 2, \cdots\}$ is a sequence of independent real valued random variables then $\sum_1^\infty X_n$ exists with probability 1 (a.e.) if and only if, for some $c > 0$, the following three series converge.

(i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$
(ii) $\sum_1^\infty EY_n$ and
(iii) $\sum_1^\infty \operatorname{var} Y_n$.

E and var denote respectively the mathematical expectation and

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variance. α_n is the principal amplitude of $\exp(i\pi X_n/c)$ and $Y_n = t_0\alpha_n$. The convergence of the above three series is easily seen to be equivalent to the convergence of

(i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$

(ii) $\sum_1^\infty E \log t(X_n)$ and

(iii) $\sum_1^\infty \operatorname{var} \log t(X_n)$ for every $t \in \hat{G}$, $\log t(X_n)$ being defined to be equal to $i\theta_n$ where θ_n is the principal amplitude of $\exp(itX_n)$. It is in this form the classical three series theorem lends itself for extension to the case of generalised random variables. In § 2 three lemmas are proved leading to the generalisation. In § 3 we give a neccessary and sufficient condition for the convergence almost everywhere of $\sum_1^\infty X_n$ in terms only of characters and not using characterstic functions.

The following two known results are quoted for the sake of completeness and ready reference.

THEOREM A. (Cor. (2.1) [4]).
If $\{h_n, n = 1, 2, \cdots\}$ is a sequence of continuous homomorphisms on a topological group G_1 to a toplogical group G_2 which converge pointwise to h throughout some Baire set of the second category then h is continuous.

THEOREM B. (§ 2.21 [3]).
Let G be a locally compact abelian group. Let N be a compact symmetric neighbourhood of e. Let G' be the subgroup of G generated by N. Then G' contains a discrete subgroup D with a finite number of generators such that the quotient group G'/D is compact and $D \cap (N + N + N) = \{e\}$.

2. For a sequence of real or complex numbers $g_n, n = 1, 2, \cdots$ we say that $\prod_1^\infty g_n$ exists if $\prod_n^\infty g_k$ is nonzero for sufficiently large n.

LEMMA 1. *For $\eta \in \hat{G}$, a necessary and sufficient condition that $\prod_1^\infty \eta(X_n)$ exists a.e. is that $\prod_1^\infty E\eta(X_n)$ exists.*

Proof. If $\prod_1^\infty \eta(X_n)$ exists a.e. then, by the bounded convergence theorem, $\prod_1^\infty E\eta(X_n)$ exists.

Conversely let $\prod_1^\infty E\eta(X_n)$ exist. Hence $\prod_1^\infty |E\eta(X_n)|$ exists. Let $\eta(X_n(w)) = \exp(i\theta_n(w))$ where $\theta_n(w)$ is the principal value of the amplitude. Hence $\theta_1, \theta_2, \cdots$ is a bounded, independent sequence of real valued random variables. Let θ'_n be the symmetrised version of θ_n and let θ'_n (1) be θ'_n truncated at 1. One has (p. 196, [2]) var θ'_n (1) $\leqq 3\{1 - |E\eta(X_n)|^2\}$. Hence $\sum_1^\infty$ var θ'_n (1) $< \infty$. By the classical three series theorem it follows that $\sum_1^\infty \theta'_n$ converges a.e. Consequently (p. 250, [2]) there exist constants α_n such that $\sum_1^\infty(\theta_n - \alpha_n)$ exists a.e. or equivalently $\prod_1^\infty \exp(-i\alpha_n)E\eta(X_n)$ exists. This implies the convergence of $\sum_1^\infty \alpha_n$ since $\prod_1^\infty E\eta(X_n)$ is assumed to converge. We now conclude $\sum_1^\infty \theta_n$ exists a.e. or, what is same, $\prod_1^\infty \eta(X_n)$ exists a.e.

LEMMA 2. *For a given $\eta \in \hat{G}$, the following two sets of conditions are equivalent.*

$$\prod_1^\infty E\eta(X_n) \textit{ exists}; \ \sum_1^\infty \operatorname{var} \eta(X_n) < \infty \tag{2.1}$$

$$\sum_1^\infty E\theta_n \textit{ converges}; \ \sum_1^\infty \operatorname{var} \theta_n < \infty \tag{2.2}$$

where $\eta(X_n) = \exp(i\theta_n)$, θ_n being the principal amplitude.

Proof. Suppose (2.2) holds. Therefore by the three series theorem on the line, $\sum_1^\infty \theta_n$ exists a.e. This implies that $\prod_1^\infty \eta(X_n)$ exists a.e. Hence $\prod_1^\infty E\eta(X_n)$ exists by the bounded convergence.

Let now $\alpha_n = E\theta_n$; $\beta_n = \operatorname{var} \theta_n$ and $\theta_n = \alpha_n + y_n$. As in the last lemma, $E\eta(X_n) = (1 + d_n\beta_n/2)\exp(i\alpha_n)$ where $|d_n| \leqq 1$.

$$\begin{aligned} E|\eta(X_n) - E\eta(X_n)|^2 &= E|\exp(iy_n) - (1 + d_n\beta_n/2)|^2 \\ &\leqq c\beta_n \text{ where } c \text{ is an absolute constant} \\ &= c \operatorname{var} \theta_n . \end{aligned}$$

Hence $\sum_1^\infty \operatorname{var} \eta(X_n) < \infty$.

Conversely, suppose (2.1) holds.

$$\begin{aligned} \operatorname{var} \eta(X_n) &= E|\exp(iy_n) - (1 + d_n\beta_n/2)|^2 \\ &= 1 + |1 + d_n\beta_n/2|^2 - 2 \text{ real part of } E\overline{(1 + d_n\beta_n/2)}\exp(iy_n) \\ &= 1 - |1 + d_n\beta_n/2|^2 . \end{aligned}$$

Hence $\sum_1^\infty \{1 - |1 + d_n\beta_n/2|^2\} < \infty$. Now, $|1 + d_n\beta_n/2|$ is the absolute value of the expectation $E\exp(iy_n)$ and hence is less than or equal to 1. It follows therefore that $\sum_1^\infty \{1 - |1 + d_n\beta_n/2|\} < \infty$ As $1 - |1 + d_n\beta_n/2| \geqq \beta_n/2$, this implies that

$$\sum_1^\infty \beta_n < \infty \quad \text{i.e.} \quad \sum_1^\infty \operatorname{var} \theta_n < \infty .$$

From the convergence of $\prod_1^\infty E\eta(X_n)$ and $\sum_1^\infty \beta_n$ and the relation $E\eta(X_n) = (1 + d_n\beta_n/2)\exp(i\alpha_n)$, we see that $\sum_1^\infty E\theta_n = \sum_1^\infty \alpha_n$ converges. Thus (2.1) implies (2.2).

LEMMA 3. *A necessary and sufficient condition that $\sum_1^\infty X_n$ exist a.e. is that $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$, and for some compact neighbourhood N of e*

(2.3) $$\sum_1^\infty P(w\colon X_n(w) \notin N) < \infty\ .$$

Proof. Suppose $\sum_1^\infty X_n$ exists a.e. Consequently, for every compact neighbourhood N of e, $P(w\colon X_n(w) \notin N$ i.o.[1]$) = 0$ or, equivalently, $\sum_1^\infty P\{w\colon X_n(w) \notin N\} < \infty$ by the Borel-Cantelli lemma. That $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$ follows from the continuity of the characters η.

Conversely, let N be any compact neighbourhood of e for which (2.3) is satisfied. Since $N - N \supseteqq N$, we have $P\{w\colon X_n(w) \notin N - N\} \leqq P\{w\colon X_n(w) \notin N\}$. Hence the symmetric neighbourhood $N - N$ of e also satisfies (2.3). Without loss of generality we may therefore assume that N in (2.3) is symmetric.

Denote by G^* the closed subgroup generated by N. Necessarily G^* is σ-compact. Further, by Theorem B, G^* contains a discrete subgroup D with a finite number of generators such that $G_1 = G^*/D$ is compact and $D \cap (N + N - N) = \{e\}$. Hence by the Borel-Cantelli lemma, (2.3) implies that $P\{w\colon X_n(w) \notin N \text{ i.o.}\} = 0$; that is, if $A_1 = \{w\colon X_n(w) \in N \text{ for all } n \geqq n_0(w)\}$ then $P(A_1) = 1$. Let σ be the natural mapping of G^* onto G_1 and write $Y_n(w) = \sigma X_n(w)$.

As G_1 is a compact, metric group, G_1 (and consequently $\hat{G}_1$) satisfies the second axiom of countablity. Also $\hat{G}_1$ is discrete, since G_1 is compact. Further $\hat{G}_1$ consists precisely of those elements of $\hat{G}$ which are identically one on D (cf: Theorem 34 [5]). In view of (2.3), we have $\prod_1^\infty \xi(Y_n)$ exists a.e. for each $\xi \in \hat{G}_1$. As $\hat{G}_1$ is countable we conclude that, with probability 1, $\prod_1^\infty \xi(Y_n)$ exists for *all* $\xi \in \hat{G}_1$. Observe that G_1, being a compact metric space, is a Baire set of the second category. It is now immediate from Theorem A that $\sum_1^\infty Y_n$ exists a.e.

Let A_2 be a set of probability 1 on which $\sum_1^\infty Y_n$ exists. If $A = A_1 \cap A_2$ then $P(A) = 1$. Let $w \in A$ and $n \geqq n_0(w)$. Hence

(2.4) $$X_n(w) + X_{n+1}(w) \in N + N\ .$$

As $\sigma(N)$ is a neighbourhood of the identity in G_1 and since $\sum_1^\infty Y_n(w)$ exists, it is clear that $Y_n(w) + Y_{n+1}(w) \in \sigma(N)$, if n is larger than a certain $n_1(w)$. That is

(2.5) $$X_n(w) + X_{n+1}(w) \in N + D \qquad \text{if } n \geqq n_1(w)\ .$$

From (2.4) and (2.5) and the property $D \cap (N + N - N) = \{e\}$, we conclude that $X_n(w) + X_{n+1}(w) \in N$ if $n \geqq \max(n_0, n_1)$. Repeating the argument a finite number of times it is seen that all finite tails of the series $\sum_1^\infty X_n(w)$ lie in N. By exactly similar reasoning, all finite tails lie in any preassigned neighbourhood M of e with $M \subseteqq N$. As N is compact, we can show (by arguments similar to the ones

[1] infinitely often

on p. 193 [1]) that $\sum_1^\infty X_n(w)$ exists. Thus on A, which is a set of probability 1, $\sum_1^\infty X_n$ exists. Combining these results, we have

THEOREM 1. *If* $\{X_n, n = 1, 2, \cdots\}$ *is an independent sequence of generalised random variables then* $\sum_1^\infty X_n$ *exists a.e. if and only if the series*

(i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$, *$N$ being any preassigned compact neighbourhood of e,*

(ii) $\sum_1^\infty E \log \eta(X_n)$ *and*

(iii) $\sum_1^\infty \operatorname{var} \log \eta(X_n)$ *converge for all* $\eta \in \hat{G}$. *Here* $\log(X_n)$ *is taken to be* $i\theta_n$ *where* θ_n *is the principal amplitude of* $\eta(X_n)$.

3. DEFINITION. The measure μ induced in $\mathscr{D}$ by a generalised random variable f will be called the distribution function of f. The distribution μ will be said to be symmetric if $\mu(A) = \mu(-A)$ for every $A \in \mathscr{D}$. It will be called regular if for every $A \in \mathscr{D}$, $\mu(A) = \sup\{\mu(C): C \subseteqq A, C \in \mathscr{C}\}$.

THEOREM 2. *If* $\{X_n, n = 1, 2, \cdots\}$ *is an independent sequence of generalised random variables with regular distributions, then* $\sum_1^\infty X_n$ *exists a.e. if and only if* $\prod_1^\infty \eta(X_n)$ *exists a.e. for every* $\eta \in \hat{G}$.

Proof. If $\sum_1^\infty X_n$ exists a.e. then $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$ by the continuity property of η.

Conversely, let $\prod_1^\infty \eta(X_n)$ exist a.e. for each $\eta \in \hat{G}$. The assertion is established through the following steps.

(i) Let G be compact. That the assertion is true in this case is seen by the same reasoning as for G_1 in Lemma 3.

(ii) Let G be discrete. The compact subsets of G are therefore only those subsets with a finite number of elements. As the distribution of each X_n is regular we can find a countable subgroup G_1 such that $P\{w: X_n(w) \in G_1, n = 1, 2, \cdots\} = 1$. Observe that $\hat{G}_1$ is the same as $\hat{G}$ restricted to G_1. Now let the X_n's have symmetric distributions. Hence, if $\varphi_n(\eta) = E\eta(X_n)$ then the φ_n's are real and $\varphi_n(-\eta) = \varphi_n(\eta)$. Now by Lemma 1, $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$, implies that $\prod_1^\infty \varphi_n(\eta)$ exists. Therefore $g(\eta) = \sum_1^\infty \{1 - \varphi_n(\eta)\}$ exists for every $\eta \in \hat{G}$. If $g_n(\eta) = \sum_1^n \{1 - \varphi_k(\eta)\}$ then the g_n's are continuous and $g_n(\eta)$ converges monotonically up to $g(\eta)$ as $n \to \infty$ for each η. Hence $\{\eta: g(\eta) \leqq a\} = \bigcap_1^\infty \{\eta: g_n(\eta) \leqq a\}$ is a closed set. $\hat{G}$ is a compact metric space and so is complete. Hence it is a set of the second category. Further, $\hat{G} = \bigcup_{n=1}^\infty \{\eta: g(\eta) \leqq n\}$ i.e. $\hat{G}$ is the union of a countable number of closed sets. Therefore by the Baire category theorem, one of these closed sets in the union, say the set $A = \{\eta: g(\eta) \leqq k\}$, has a nonnull interior V. Trivially g is bounded on V. By the positive definiteness and symmetry of ϕ_k,

$$1 - \phi_k^2(\xi) - \phi_k^2(\eta) + 2\phi_k(\xi)\phi_k(\eta)\phi_k(\xi + \eta) - \phi_k^2(\xi + \eta) \geqq 0 .$$

Let $a_k^2 = 1 - \phi_k(\xi)$, $b_k^2 = 1 - \phi_k(\eta)$ and $c_k^2 = 1 - \phi_k(\xi + \eta)$. Then the above inequality implies that

$$c_k^2 \leqq a_k^2 + b_k^2 - a_k^2 b_k^2 + a_k b_k \sqrt{(2 - a_k^2)(2 - b_k^2)} \leqq (a_k + b_k)^2 .$$

Consequently,

$$g(\xi + \eta) \leqq \{[g(\xi)]^{1/2} + [g(\eta)]^{1/2}\}^2 . \tag{3.1}$$

For any $\xi \in \hat{G}$ consider the open set $\xi - V$. From (3.1) it is immediate that g is bounded on $\xi - V$. The family $\xi - V$, $\xi \in \hat{G}$ is an open covering for the compact $\hat{G}$. Therefore there exists a finite subcover from this. As g is bounded on each member of this subcover it follows that g is bounded on $\hat{G}$.

Let m be the Haar measure of $\hat{G}$ with $m(\hat{G})=1$. As $P\{w: X_n(w) \neq e\} = \int_{\hat{G}} \{1 - \varphi_n(\eta)\} dm(\eta)$, we obtain $\sum_1^\infty P(w: X_n(w) \neq e\} = \int_{\hat{G}} g(\eta) dm(\eta) < \infty$. Since G is discrete this means that for the compact neighbourhood $N = \{e\}$ of e, $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$. That $\sum_1^\infty X_n$ exists a.e. follows from Lemma 3.

(iii) Let G be discrete but the distributions of the X_n's need not be symmetric.

Let Y_n, $n = 1, 2, \cdots$ be another independent sequence of g.r.v.'s and independent of the X_n's; let Y_n have the same distribution as X_n, $n = 1, 2, \cdots$.

Write $Z_n = X_n - Y_n$. The Z_n's therefore have symmetric distributions. Also the hypothesis yields that $\prod_1^\infty \eta(Z_n)$ exists a.e. for every $\eta \in \hat{G}$. Hence by (ii) above

$$\sum_1^\infty P\{w: Z_n(w) \neq e\} < \infty . \tag{3.2}$$

The distribution of each X_n is assumed to be regular. Hence there exists a countable set A such that $P\{w: Z_n(w) \in A \text{ for all } n\} = 1$. Now, if $p_n(a) = P\{w: X_n(w) = a\}$, we have

$$\begin{aligned} P\{w: Z_n(w) = e\} &= \sum_{a \in A} P\{w: X_n(w) = a\} P\{w: Y_n(w) = a\} . \\ &= \sum_{a \in A} p_n^2(a) \leqq \sup_{a \in A} p_n(a) \end{aligned}$$

Since there can only be a finite number of 'values' of X_n for which the associated probability is larger than any preassigned number, the supremum is attained. Let a_n be any one of the values taken by X_n with probability equal to this supremum. Therefore $P\{w: X_n(w) \neq a_n\} \leqq$

$P\{w: Z_n(w) \neq e\}$. Consequently, using (3.2), we obtain

$$\sum_1^\infty P\{w: X_n(w) \neq a_n\} < \infty \tag{3.3}$$

$$\text{or} \quad \sum_1^\infty P\{w: X_n(w) - a_n \notin N\} < \infty \ . \tag{3.4}$$

Where N is the compact neighbourhood of e consisting only of itself. From (3.3) we conclude that, with probability 1, $X_n = a_n$ except for a finite number of n's. This fact together with the hypothesis implies that $\prod_1^\infty \eta(a_n)$ exists for every $\eta \in \hat{G}$. That $\prod_1^\infty \eta(X_n - a_n)$ exists a.e for every $\eta \in \hat{G}$ is then immediate. Now using (3.4) we see by lemma 3 that $\sum_1^\infty (X_n - a_n)$ exists a.e. By Theorem A or by applying Lemma 3 to the random variables a_n we see however that $\sum_1^\infty a_n$ exists since $\prod_1^\infty \eta(a_n)$ exists, for every $\eta \in \hat{G}$. Hence $\sum_1^\infty X_n$ exists a.e., as was to be proved.

(iv) Let G be any metric abelian locally compact group. Let N be a compact symmetric neighbourhood of e and G^* the closed subgroup generated by N. Necessarily G^* is σ-compact and open. Let σ_1 be the natural mapping of G onto $G_1 = G/G^*$. As G^* is open, G_1 is discrete. Further $\hat{G}_1$ consists precisely of those elements of $\hat{G}$ which are identically one on G^*. Hence $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$ implies that $\prod_1^\infty \xi(Y_n)$ exists a.e. for each $\xi \in \hat{G}_1$, where $Y_n = \sigma_1 X_n$. By part (iii) above, $P\{w: Y_n(w) \neq e_1 \text{ i.o.}\} = 0$ where e_1 is the identity of G_1. That is

$$P\{w: X_n(w) \notin G^*\} = 0 \ . \tag{3.5}$$

In other words, there is probability 1 that all except a finite number of the X_n's lie in G^*.

As G^* is generated by a compact symmetric neighbourhood of e there exists, by Theorem B, a discrete group D with a finite number exists, by Theorem B, a discrete group D with a finite number of generators such that $G_2 = G^*/D$ is compact and $D \cap (N - N) = \{e\}$. Let e_2 be the identity element of G_2 and σ_2 the natural mapping of G^* onto G_2. Write $Z_n = \sigma_2 X_n$ if $X_n \in G^*$ and $= e_2$ if $X_n \notin G^*$. Hence Z_n, $n = 1, 2, \cdots$ is an independent sequence of g.r.v.'s in G_2. Recall that $\hat{G}^*$ consists of all the elements of $\hat{G}$ restricted to G^* and that $\hat{G}_2$ consists precisely of those elements of $\hat{G}^*$ which are identically 1 on D. Using the hypothesis and the equation (3.5) we get $\prod_1^\infty \xi(Z_n)$ exists a.e. for every $\xi \in \hat{G}_2$. Therefore we have

$$P\{w: Z_n(w) \notin \sigma_2(N) \text{ i.o.}\} = 0 \quad \text{i.e.} \quad P\{w: X_n(w) \notin N + D \text{ i.o.}\} = 0 \ .$$

Define $s_n = X_n$ if $X_n \in N + D$ and $s_n = e$ if $X_n \notin N + D$. Then for each s_n we have the unique decomposition $s_n = u_n \pm v_n$ where

$u_n \in N$ and $v_n \in D$. The u_n's form an independent sequence of g.r.v.'s and so do the v_n's. It is immediate from the hypothesis that $\prod_1^\infty \eta(s_n)$ exists a.e. for each $\eta \in \hat{G}$. Also, since $\prod_1^\infty \xi(Z_n)$ exists a.e. for each $\xi \in \hat{G}_2$, $\prod_1^\infty \eta(u_n)$ exists a.e. for each $\eta \in \hat{G}$. Hence $\prod_1^\infty \xi(v_n)$ exists a.e. for each $\xi \in \hat{D}$. As D is discrete we have, by part (iii), $P\{w: X_n(w) \neq e \text{ i.o.}\} = 0$. This is equivalent to saying $P\{w: s_n(w) \neq u_n(w) \text{ i.o.}\} = 0$. Or $P\{w: X_n(w) \notin N \text{ i.o.}\} = 0$ i.e. $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$. That $\sum_1^\infty X_n$ exists a.e. follows now by Lemma 3.

I thank the referee for his suggestions leading to a shorter proof of Lemma 1.

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ČEBYŠEV SUBSPACES OF FINITE CODIMENSION IN $C(X)$

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1. Introduction. Suppose that X is a compact Hausdorff space and that M is an n dimensional linear subspace of $C(X)$, the Banach space of all real-valued continuous functions f on X, with supremum norm. If f is in $C(X)$, the local compactness of M guarantees the existence of at least one function g in M such that $\|f - g\| = d(f, M) = \inf\{\|f - h\|: h \in M\}$, i.e., f has a *nearest point* g in M. A well-known extension of a classical theorem of Haar (see e.g. [2, Theorem 3.6]) states that M is a *Čebyšev subspace* of $C(X)$ (that is, there is a *unique* nearest point in M to f for every f in $C(X)$) if and only if each nontrivial function in M has most $n - 1$ zeros.[1] In the present note we intend to investigate infinite dimensional closed subspaces M of $C(X)$ in the hope of characterizing those having the Čebyšev property. Except for Proposition 3, our attention will be restricted to closed M of *finite codimension*, that is, those M for which the factor space $C(X)/M$ is finite dimensional. (The dimension of this factor space is the same as the dimension of the annihilator $M^{\perp}$ of M, the subspace of $C(X)^*$ consisting of all those continuous linear functionals on $C(X)$ which vanish on M.) There is, of course, an additional problem when dealing with infinite dimensional M. We have no assurance that a function f in $C(X)$ has even *one* nearest point in M. A subspace M with the property that each f in $C(X)$ contains at least one nearest point in M will be called a *Haar subspace* (or be said to have the *Haar property*). We know of no characterization of the Haar subspaces of $C(X)$. (A general necessary condition is given in Proposition 2, but we show by example that it is not sufficient.) Thus, most of our results are devoted to characterizing the Čebyšev subspaces from among the Haar subspaces of finite codimension.

Mairhuber was the first to show (see the discussion and references in [2, p. 253]) that if $C(X)$ contains a Čebyšev subspace of finite dimension $n, n > 1$, then X must be homeomorphic with a subset of the circle $|z| = 1$ in the complex z-plane. We show that if $C(X)$ contains a Čebyšev subspace of finite *codimension* $n, n > 1$, then X is totally disconnected; we also prove that X can contain at most countably many isolated points. The examples in § 4 show that, for certain X, $C(X)$ contains Čebyšev subspaces of codimension n, for $n = 1, 2, 3, \cdots$. In these examples, X is always *extremally* disconnected (that is, the

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[1] What we call "Čebyšev" subspaces were called "Haar" subspaces in [**2**], but we defer here to a more common usage. We now use the term "Haar subspace" (below) to replace the term "proximinal" used in [**2**].

closure of every open set in X is open), but we don't know whether this property is necessary for the existence of Čebyšev subspaces in $C(X)$. To obtain our examples we make use of the well known fact [1, p. 445] that the space L^∞ can be realized as $C(X)$ (for a certain extremally disconnected X).

As usual, we identify the space $C(X)^*$ with $rca(X)$, the space of regular countably additive real-valued finite measures μ on the Borel subsets of X [1, p. 265]. (Throughout the rest of this paper we will refer to an element of $rca(X)$ as simply "a measure on X.") If μ is a measure on X, then $\mu = \mu^+ - \mu^-$ (where μ^+ and μ^- are nonnegative measures on X), $|\mu| = \mu^+ + \mu^-$ and $||\mu|| = |\mu|(X)$. For f in $C(X)$, the value of μ at f is given by $(f, \mu) = \int_X f\,d\mu$. The *support* $S(\mu)$ of a measure μ is the closed set which equals the complement of the union of all open sets U for which $|\mu|(U) = 0$. Most of what we say about Čebyšev subspaces M will be in terms of $S(\mu)$ for μ in $M^\perp$; for instance, if M is a Haar subspace of finite codimension n in $C(X)$, then in order that M be a Čebyšev subspace it is sufficient that $S(\mu) = X$ for each μ in $M^\perp \sim \{0\}$, and it is necessary (for each μ) that $X \sim S(\mu)$ contain at most $n - 1$ points. (Since $X \sim S(\mu)$ is in some sense the "zero set" of μ, this latter property is dual to that discovered by Haar.) The above sufficient condition is necessary if X contains no isolated points, and the above necessary condition is sufficient if X contains n or more isolated points. Examples in § 4 show that for $n = 2$ and X having one isolated point, these converse statements are false.

2. General results. In Lemma 1 we give a well-known characterization of the Haar subspaces of codimension one in $C(X)$ (see, e.g. [3, p. 165] and references cited there; see [5] for the complex case) which is basic to much of what follows. In general, if E is a normed linear space and M is a subspace of codimension one (so that $M = L^{-1}(0)$ for some continuous linear functional L on E and $M^\perp = RL$, the one dimensional space of all real multiples of L), an application of the Hahn-Banach theorem shows that M is a Haar subspace if and only if there exists f in E such that $||f|| = 1$ and $L(f) = ||L||$; equivalently, L attains its supremum on the unit ball of E. The set of all linear functionals having this latter property is denoted by P. In the case $E = C(X)$, then, we are interested in measures which represent such functionals; the set of such measures is also denoted by P.

LEMMA 1. *A measure μ is in P if and only if the supports $S(\mu^+)$ and $S(\mu^-)$ are disjoint.*

Proof. If $S(\mu^+)$ and $S(\mu^-)$ are disjoint, then we choose f in $C(X)$,

$\|f\| = 1$, such that $f = 1$ on $S(\mu^+)$, $f = -1$ on $S(\mu^-)$, and hence $(f, \mu) = \|\mu\|$. Conversely, suppose $\|f\| = 1$ and $(f, \mu) = \|\mu\| = \mu^+(X) + \mu^-(X) = \|\mu^+\| + \|\mu^-\|$. If $f(x) < 1$ for some x in $S(\mu^+)$, then $f < 1$ in some open set U containing x; since $\mu^+(U) > 0$, it follows that $(f, \mu^+) < \|\mu\|$ (and $-(f, \mu^-) \leqq \|\mu^-\|$) so that $(f, \mu) < \|\mu\|$, a contradiction. Thus, $f = 1$ on $S(\mu^+)$ and (similarly) $f = -1$ on $S(\mu^-)$, so that these sets must be disjoint.

It is clear that a measure μ is in P if and only if every real multiple of μ is in P, so that a subspace M of codimension one in $C(X)$ has the Haar property if and only if $M^\perp \subset P$. The conjecture that this latter relation characterizes Haar subspaces of any finite codimension in $C(X)$ is disproved by Example 3 in § 4. One implication remains valid, however, as we now show.

PROPOSITION 2. *Suppose that M is a Haar subspace of finite codimension in the normed linear space E. Then $M^\perp \subset P$.*

Proof. If $L \in M^\perp$ we may represent L as a linear functional on E/M by $L(f + M) = L(f)$. Since E/M is finite dimensional there exists an element F of E/M such that $\|F\| = 1$ and $L(F) = \|L\|$. Since F is a translate of M and since M has the Haar property, there exists an element f in F of least norm, i.e., there exists f in E such that $F = f + M$ and $\|f\| = \|F\| = 1$. It follows that $L(f) = \|L\|$, which completes the proof.

(Note that the above proof is valid under the weaker assumption that E/M is a reflexive Banach space.)

In the next proposition the Čebyšev property of a subspace M of $C(X)$ is formulated in terms of functions in M and measures in $M^\perp$. The central idea is a slight extension of a construction due to V. Pták [4].

PROPOSITION 3. *Suppose that M is a Haar subspace of $C(X)$. Then M fails to have the Čebyšev property if and only if there exist f in $M \sim \{0\}$ and μ in $M^\perp \cap P \sim \{0\}$ such that $f(S(\mu)) = 0$.*

Proof. If the Haar subspace M does not have the Čebyšev property there exists h in $C(X)$ and f in $M \sim \{0\}$ with $d(h, M) = 1 = \|h\| = \|h - f\|$. By the Hahn-Banach theorem we can choose μ in $M^\perp$ such that $(h, \mu) = (h - f, \mu) = 1 = \|\mu\|$. As in the proof of Lemma 1 we see that $h = h - f = 1$ on $S(\mu^+)$ and $h = h - f = -1$ on $S(\mu^-)$. It follows that $f = 0$ on $S(\mu) = S(\mu^+) \cup S(\mu^-)$. To prove the converse, suppose there exist μ and f with the stated properties; we can assume $\|\mu\| = 1 = \|f\|$. Choose g in $C(X)$ such that $g = 1$ on $S(\mu^+)$, $g = -1$ on $S(\mu^-)$ and $\|g\| = 1$. Let $h = g(1 - |f|)$; since $h = g$ on $S(\mu)$, we have $(h, \mu) = 1$. Furthermore, $\|h\| = 1$ and $|h| + |f| = |g| + (1 - |g|)|f| \leqq 1$, so $||h - f|| = 1$. Finally, if $e \in M$, then $1 = (h, \mu) = (h - e, \mu) \leqq ||h - e||$, so $d(h, M) = 1$ and hence M does not have the Čebyšev property.

COROLLARY 4. *If M is a Haar subspace of $C(X)$ such that $S(\mu) = X$ for each μ in $M^\perp \cap P \sim \{0\}$, then M has the Čebyšev property.*

COROLLARY 5. *Suppose that M is a closed subspace of $C(X)$ of codimension one. Then M is a Čebyšev subspace if and only if $M^\perp \subset P$ and $S(\mu) = X$ for each μ in $M^\perp \sim \{0\}$.*

Proof. By Lemma 1, $M^\perp \subset P$ is equivalent to the fact that M is a Haar subspace. If $S(\mu) \neq X$ for some μ in $M^\perp \sim \{0\}$, then there exists f in $C(X) \sim \{0\}$ which vanishes on $S(\mu)$, and hence $f \in \{g: (g, \mu) = 0\} = M$, which shows (by Proposition 3) that M is not a Čebyšev subspace. The remainder of the proof is a consequence of Corollary 4.

(A result of the above nature was pointed out to us (without proof) by O. Hustad, who noted that it leads to an easy counterexample to the sufficiency portion of Theorem 3.4 of [2].)

PROPOSITION 6. *Suppose that M is a Čebyšev subspace of codimension $n > 0$ in $C(X)$. Then the set $X \sim S(\mu)$ contains at most $n - 1$ points for each μ in $M^\perp \sim \{0\}$*

Proof. Suppose that for some μ in $M^\perp \sim \{0\}$ the set $X \sim S(\mu)$ contains n or more points. Denote by N the subspace of $C(X)$ consisting of all functions which vanish on $S(\mu)$; N must have dimension n or greater. Choose a basis $\mu_1, \mu_2, \cdots, \mu_n$ for $M^\perp$, with $\mu_1 = \mu$. The subspace $M_1 = \{g: (g, \mu_i) = 0, i = 2, 3, \cdots, n\}$ has codimension $n - 1$, hence there exists f in $M_1 \cap N \sim \{0\}$. Since we also have $(f, \mu_1) = 0$, it follows that $f \in M \sim \{0\}$; by Proposition 3, M is not a Čebyšev subspace.

3. Main results.

THEOREM 7. *Suppose that $C(X)$ contains a Čebyšev subspace M of finite codimension n, $n \geqq 2$. Then X is totally disconnected.*

Proof. Suppose that X contains a connected subset K such that K contains more than one point (and hence contains infinitely many points). Note that we must have $K \subset S(\mu)$ for each μ in $M^\perp \sim \{0\}$. [Indeed, if $A = K \sim S(\mu)$ were nonempty, it would (by Proposition 6) be a finite set and therefore K (being infinite) would intersect both $S(\mu)$ and A. But A and $S(\mu)$ are disjoint closed sets whose union contains K, an impossibility.] Since, by Proposition 2, $S(\mu^+)$ and $S(\mu^-)$ are disjoint closed sets, we must have $K \subset S(\mu^+)$ or $K \subset S(\mu^-)$, so that

$\mu(K) \neq 0$ for each μ in $M^{\perp} \sim \{0\}$. But if μ_1 and μ_2 are linearly independent measures in $M^{\perp}$, we see that the nontrivial measure $\mu_1(K)\mu_2 - \mu_2(K)\mu_1$ vanishes on K, a contradiction which completes the proof.

THEOREM 8. *If $C(X)$ contains a Čebyšev subspace M of finite codimension $n\,(n \geqq 1)$, then X contains at most countably many isolated points.*

Proof. Choose μ in $M^{\perp} \sim \{0\}$; by Proposition 6, $X \sim S(\mu)$ contains only finitely many points. Now, $|\mu|(\{x\}) > 0$ for any isolated point x in X for which $x \in S(\mu)$. Since $|\mu|$ is a countably additive finite measure, its support can not contain uncountably many pairwise disjoint sets of positive measure. This shows that $S(\mu)$ (and hence X) contains at most countably many isolated points of X.

An example of a space $C(X)$ which contains no Čebyšev subspace of finite codimension may be obtained by letting X be a compactification of an uncountable discrete set.

THEOREM 9. *Suppose that X contains n or more isolated points. A Haar subspace M of codimension $n\,(n \geqq 1)$ in $C(X)$ is a Čebyšev subspace if and only if $X \sim S(\mu)$ contains at most $n-1$ points for each μ in $M^{\perp} \sim \{0\}$.*

Proof. The necessity portion follows from Proposition 6. To prove the sufficiency, suppose that M is not a Čebyšev subspace of $C(X)$; we will produce a measure ν in $M^{\perp} \sim \{0\}$ such that $X \sim S(\nu)$ contains n or more points. By Proposition 3 there exists f in $M \sim \{0\}$ and μ in $M^{\perp} \cap P \sim \{0\}$ such that $f = 0$ on $S(\mu)$. Thus, $X \sim S(\mu)$ is nonempty, and if it contained n or more points, our proof would be complete. Suppose that $X \sim S(\mu)$ contains fewer than n points. We will show that if $\nu_0 \in M^{\perp} \sim \{0\}$ is such that $S(\nu_0) \subset S(\mu)$ and $X \sim S(\nu_0)$ contains fewer than n points, then there exists ν_1 in $M^{\perp} \sim \{0\}$ such that $S(\nu_1)$ is a proper subset of $S(\nu_0)$. (Once we have shown this, an obvious induction will complete the proof.) Let X_0 denote the set $X \sim S(\nu_0)$; by assumption, X_0 contains exactly k points, $1 \leqq k \leqq n-1$. We first obtain an element ν_2 in $M^{\perp}$ for which $S(\nu_2) \subset S(\nu_0)$ and which is linearly independent of ν_0; this is done as follows: Choose a basis $\nu_0, \mu_1, \cdots, \mu_{n-1}$ for $M^{\perp}$ and let M_0 be the subspace of $C(X_0)^*$ spanned by the restrictions of the measures $\mu_1, \cdots, \mu_{n-1}$ to X_0. Since $S(\nu_0) \subset S(\mu)$, we have $f = 0$ on $S(\nu_0)$, while $(f, \mu_i) = 0$ for $i = 1, \cdots, n-1$; furthermore the restriction of f to X_0 is not identically zero. This shows that $M_0^{\perp}$ is a proper subspace of the k dimensional space $C(X_0)^*$, so that $M_0^{\perp}$ has dimension at most $k - 1 \leqq n - 2$. Hence there exists a nontrivial linear combination ν_2 of the measures μ_i which vanishes at each point of X_0, so that

$S(\nu_2) \subset S(\nu_0)$. By hypothesis, X contains at least n isolated points, so one of them, say x, is in $S(\nu_0)$. Since ν_0 and ν_2 are linearly independent, the measure $\nu_1 = \nu_2(\{x\})\nu_0 - \nu_0(\{x\})\nu_2$ is nontrivial, has support in $S(\nu_0)$, and vanishes at x. The latter property shows that $S(\nu_1) \neq S(\nu_0)$, which completes the proof.

Example 2 of § 4 shows that this theorem is invalid if the codimension of M is greater than the number of isolated points of X.

THEOREM 10. *Suppose that X contains no isolated points. A Haar subspace M of finite codimension in $C(X)$ is a Čebyšev subspace if and only if $S(\mu) = X$ for each μ in $M^{\perp} \sim \{0\}$.*

Proof. The sufficiency portion follows from Corollary 4. To complete the proof, suppose that M is a Čebyšev subspace and there exists μ in $M^{\perp} \sim \{0\}$ such that $X \sim S(\mu)$ is nonempty. By Proposition 6, this set contains only finitely many points; since their union is open, they are isolated points, a contradiction.

Example 1 of § 4 shows that this theorem fails to be true if X contains an isolated point. It is interesting to note that an argument similar to (but simpler than) the one in Theorem 7 shows that if $C(X)$ contains a Haar subspace M of finite codimension n ($n \geqq 2$) such that $S(\mu) = X$ for each μ in $M^{\perp} \sim \{0\}$, then X contains no isolated point.

4. Examples. As mentioned in the introduction, our examples are obtained by exploiting the fact that the space L^{∞} can be "realized" as $C(X)$. The connections between X and the measure space on which L^{∞} is defined are certainly well known, but we know of no explicit reference to them, so the next few paragraphs are devoted to a sketch of the material we need.

Suppose that (T, Σ, λ) is a σ-finite measure space; then there exists a compact Hausdoff space X_T and an isometry J from $L^{\infty}(T, \Sigma, \lambda)$ onto $C(X_T)$ which is linear, multiplicative (i.e., if $h = fg$ a.e. in L^{∞}, then $Jh = JfJg$ in $C(X_T)$) and carries a.e. nonnegative elements of L^{∞} into nonnegative functions in $C(X_T)$ (see, e.g. [1, p. 445[). Hence, if χ_E is the characteristic function of the measurable set E in T, then $J\chi_E$ is a characteristic function in $C(X_T)$ (since $J\chi_E = J(\chi_E^2) = (J\chi_E)^2$). Writing $J\chi_E = \chi_{\phi E}$, we have defined a map from the σ-algebra Σ (modulo sets of measure zero) onto the family of all open and closed subsets of X_T (the proof is the same as in [1, p. 312]). It is readily seen that ϕ maps the atoms of (T, Σ, λ) (i.e. those A in Σ of finite positive measure such that $B \subset A$ and B in Σ imply $\lambda(B) = 0$ or $\lambda(B) = \lambda(A)$) in a one-to-one fashion onto the isolated points of X_T; this is a consequence of the fact that the elements of $L^{\infty}(T, \Sigma, \lambda)$ are constant a.e. on each atom.

If $f \in L^1(T, \Sigma, \lambda)$, then f defines (in the natural way) a continuous linear functional on L^∞. Since $(L^\infty)^*$ and $C(X_T)^*$ are isometric there exists a unique measure $\nu(f)$ on X_T corresponding to the functional defined on L^∞ by f. (This correspondence may be described by the equation

$$\int_T gfd\lambda = \int_{X_T} Jgd\nu(f) , \quad g \in L^\infty .$$

Since J carries nonnegative elements of L^∞ onto nonnegative functions in $C(X_T)$, it follows that $\nu(|f|) = |\nu(f)|$. If we define the *support* $S(f)$ of f in L^1 to be the complement in T of the set on which $|f| = 0$ a.e., then $\phi(S(f)) = S(\nu(f))$.

We may now obtain Haar subspaces of $C(X_T)$ as follows: If N is a closed subspace of L^1, its annihilator M in L^∞ is weak*-closed and hence is a Haar subspace [2, p. 239]; it follows that $J(M)$ is a Haar subspace of $C(X_T)$. If N is finite dimensional, then we may identify N (by means of the natural embedding of L^1 into $(L^\infty)^*$) with $M^\perp = (N^\perp)^\perp$ in $(L^\infty)^*$. It follows that $(JM)^\perp$ in $C(X_T)^*$ consists of those measures of the form $\nu(f)$, for f in N. We now apply these remarks to the construction of two examples.

EXAMPLE 1. There exists a compact Hausdorff space X for which the following is true:

(i) X contains exactly one isolated point

(ii) $C(X)$ contains a Čebyšev subspace M of codimension 2, but $S(\mu) \neq X$ for some μ in $M^\perp \sim \{0\}$.

Proof. We will obtain X as the space X_T corresponding to (T, Σ, λ), where $T = [0, 1] \cup \{2\}$, Σ is the family of Borel subsets of T, and λ is Lebesque measure on the Borel subsets of $[0, 1]$, but $\lambda(\{2\}) = 1$. We define f_0 and f_1 in $L^1(T, \Sigma, \lambda)$ by $f_0 = 1$ on $T, f_1(x) = x$ if $0 \leqq x \leqq 1$, $f_1(2) = 0$. If N is the two dimensional space spanned by f_0 and f_1, then, as noted above, its annihilator M in L^∞ is a Haar subspace, and we may consider N and $M^\perp$ to be the same subspace of $C(X_T)^*$. The atom $\{2\}$ corresponds to an isolated point of X_T, and it is not in the support of the measure corresponding to f_1 (since it is not in $S(f_1)$. The subspace M is a Čebyšev subspace, however. If it were not, then by Proposition 3 there would exist g in $M \sim \{0\}$ such that $g = 0$ on the support of some measure in $M^\perp \sim \{0\}$. Equivalently, there would be g in $M \sim \{0\} \subset L^\infty$ and constants a_0 and a_1 such that $g = 0$ a.e. on $S(a_0 f_0 + a_1 f_1)$. If $a_0 = 0$, this implies $g = 0$ a.e. on $[0, 1]$; since $0 = (g, f_0) = \int_T g f_0 d\lambda = g(2)$, we see that $g = 0$ a.e. If $a_0 \neq 0$, then $S(a_0 f_0 + a_1 f_1) = T$ and therefore $g = 0$ a.e. Thus, no such g exists, which completes the proof.

EXAMPLE 2. There exists a compact Hausdorff space X for which the following is true:

(i) X contains exactly one isolated point.

(ii) For each $n \geqq 2$, $C(X)$ contains a Haar subspace M of codimension n which is not a Čebyšev subspace, although $X \sim S(\mu)$ is one point for each μ in $M^{\perp} \sim \{0\}$.

Proof. We let $X = X_T$, where (T, Σ, λ) is defined as in Example 1. Let N be the linear subspace of $L^1(T, \Sigma, \lambda)$ spanned by the functions $f_0, f_1, \cdots, f_{n-1}$, where $f_k(x) = x^k$ for $0 \leqq x \leqq 1$, and $f_k(2) = 0$, $k = 0, 1, \cdots, n-1$. As before, the annihilator M of N in L^∞ is a Haar subspace of codimension n; it is not a Čebyšev subspace, however, since it contains the function g which is zero on $[0, 1]$, while $g(2) = 1$. (This function is zero on $S(f_0)$, say.) Clearly, the isolated point of X_T corresponding to the atom $\{2\}$ is in $X_T \sim S(\mu)$ for each μ in $M^{\perp}$.

EXAMPLE 3. There exists a compact Hausdorff space X and a closed subspace M of codimension 2 in $C(X)$ such that $M^{\perp} \subset P$ but M is not a Haar subspace.

Proof. We take $C(X)$ to be the space c of all convergent sequences $f = \{f_n\}_{n=1}^{\infty}$ of real numbers (so that X is the one-point compactification of the integers). The space $C(X)^*$ is isometric with the space l of absolutely summable sequences, under the following correspondence: If $f \in c$ and if $\mu = \{\mu_n\}_{n=1}^{\infty} \in l$, then $(f, \mu) = \sum_{n=1}^{\infty} f_n \mu_n + \mu_0 \lim f_n$. Define measures μ^1 and μ^2 by $\mu_n^1 = 2^{-n}$, $n = 0, 1, \cdots$ and $\mu_n^2 = 4^{-n}$, $n = 1, 2, 3, \cdots$, $\mu_0^2 = 0$. For any real number a the sequence $(\mu^1 + a\mu^2)_n$ is eventually positive (and equals 1 at $n = 0$), so that the measure $\mu^1 + a\mu^2$ has disjoint positive and negative supports. It follows that the same is true for $b\mu^1 + a\mu^2$, a, b real. Let $M = \{f\colon (f, \mu^1) = 0 = (f, \mu^2)\}$; M is a closed subspace of c, and the previous remarks show that $M^{\perp} \subset P$. To see that M is not a Haar subspace it suffices to show that the translate M_1 of M defined by $M_1 = \{f\colon (f, \mu^1) = 1 = (f, \mu^2)\}$ does not contain a point of least norm. Let $m = \inf\{\|f\|\colon f \in M_1\}$, and suppose that there exists f in M_1 such that $\|f\| = m$. We can choose μ in $M^{\perp}$ such that $\|\mu\| = 1$ and $(f, \mu) = m$; letting $g = m^{-1}f$, we see that $\|g\| = 1 = (g, \mu)$. It follows that $g = 1$ on $S(\mu^+)$, $g = -1$ on $S(\mu^-)$ and $|g| \leqq 1$ elsewhere. Since $\mu \in M^{\perp}$, the sequence $\{\mu_n\}$ is eventually positive (and $\mu_0 > 0$) or it is eventually negative (and $\mu_0 < 0$). Letting $\varepsilon = \operatorname{sgn} \mu_0$, we see that there exists an integer $N > 0$ such that $f_n = \varepsilon m$ if $n \geqq N$, while $|f_n| \leqq m$ if $n < N$. Since $(\mu^1 - \mu^2)_n$ is eventually positive, we may assume that N is so large that $\mu_n^1 - \mu_n^2 > 0$ for $n \geqq N$. By assumption,

$$1 = (f, \mu^i) = \varepsilon m\Big(\mu_0^i + \sum_{n=N}^{\infty} \mu_n^i\Big) + \sum_{n=1}^{N-1} f_n \mu_n^i \qquad (i = 1, 2) .$$

Subtracting and dividing by ε, we obtain

$$m\Big[1 + \sum_{n=N}^{\infty} (\mu_n^1 - \mu_n^2)\Big] \leqq m \sum_{n=1}^{N-1} |\mu_n^1 - \mu_n^2|$$
$$\leqq m \sum_{n=1}^{\infty} (2^{-n} - 4^{-n}) < m ,$$

a contradiction which completes the proof.

The connection between $L^\infty(T, \Sigma, \lambda)$ and $C(X_T)$ described above may be used to obtain new proofs of Theorems 2.2 and 2.3 of [2]; they are immediate corollaries of Theorems 10 and 9 (respectively). For instance, Theorem 9 yields the following result, which is stronger than Theorem 2.3 of [2].

COROLLARY. *Suppose that (T, Σ, λ) is a σ-finite measure space containing at least n atoms, and that N is a subspace of dimension n in $L^1(T, \Sigma, \lambda)$. Then $N^\perp$ is a Čebyšev subspace of $L^\infty(T, \Sigma, \lambda)$ if and only if each f in $N \sim \{0\}$ vanishes on at most $n - 1$ atoms.*

Finally, the fact that for $n = 1, 2, 3, \cdots$ the space $L^1(T, \Sigma, \lambda)$ ($T = [0, 1]$, Σ = Borel sets, λ = Lesbegue measure) contains subspaces N of dimension n such that each f in $N \sim \{0\}$ is a.e. nonzero shows that $C(X_T)$ contains Čebyšev subspaces of each finite codimension.

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ON SUBGROUPS OF AN ABELIAN GROUP MAXIMAL DISJOINT FROM A GIVEN SUBGROUP

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1. Introduction. In [3], J. M. Irwin has introduced the concept of a high subgroup of an abelian group (A is high in G if A is maximal in G with respect to the property $A \cap (\bigcap_n nG) = 0$). Irwin and, subsequently, Irwin and Walker [4] have also considered N-high subgroups A of G (A is maximal in G with respect to $A \cap N = 0$). Among the properties of high subgroups is their purity in G ([1], [3] for p-groups, [3] for torsion groups and [4] for arbitrary abelian groups). In [5], S. Khabbaz has given a short proof of a theorem which implies the purity of high subgroups of a p-group. Irwin [3] raises the question of characterizing subgroups H of a group G for which every H-high subgroup is pure in G.

In this paper we consider pairs (H, M) of subgroups of an abelian group G with M maximal disjoint from H in G and ask what happens if M is not pure in G. The resulting information allows us to answer Irwin's question in various special cases. In particular we obtain the purity of high subgroups of arbitrary abelian groups and a generalization of the theorem of Khabbaz referred to above. We then consider various related questions and obtain a generalization of a theorem of Žuravskiĭ [7] on the splitting of mixed abelian groups.

Throughout the paper, G will denote an abelian group, H a subgroup of G and M a subgroup of G maximal with respect to $M \cap H = 0$. Following Irwin [3] we say that M is *H-high* in G. For any subgroup K of G and prime p, K_p denotes the set of all elements of K whose orders are a power of p, and $K[p]$ is the set of elements of K_p whose orders are $\leqq p$. The torsion subgroup of a group K will occasionally be denoted by K_t. For $x \in G$ we denote by $h_p(x) = \max\,[n \mid x \in p^n G]$ the height of the element x at p in G. Curly brackets denote the subgroup generated by the sets and elements inside. In particular, if M is a subgroup of G and $x \in G$ then $\{M, x\}$ is the subgroup of G generated by M and x. The set of rational integers will be denoted by Z, direct sums by $\oplus$ and not necessarily direct sums by $+$.

2. The main theorem. We remark first that if M is H-high in G then M is *neat* in G (cf. [2, pp. 91–92]); i.e, $M \cap pG = pM$ for each prime p. It is also easy to see that $G[p] = M[p] \oplus H[p]$ for any p.

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THEOREM 2.1. *Let M be an H-high subgroup of G. Then either M is pure in G or there exists a prime p and elements $\boldsymbol{m} \in M$, $h \in H[p]$ such that*

$$h_p(\mathbf{m}) = h_p(h) < h_p(h - \boldsymbol{m}) .$$

Proof. Suppose that M is not pure in G. Then there exist equations $nw = v$ with $n \in Z$, $w \in G$, $v \in M$ which have no solution $w \in M$. Among all such equations, let $nx = y (x \in G, y \in M)$ be one for which n is least positive. It is not hard to see that minimality of n implies that n is a power of some prime p, say $n = p^r$. By neatness of M, $r > 1$ and $p^r x = pm_1$, $m_1 \in M$ so that $p(p^{r-1}x - m_1) = 0$. Thus $p^{r-1}x - m_1 \in G[p]$ and since $G[p] = M[p] \oplus H[p]$, we have

$$p^{r-1}x - m_1 = m + h \qquad (m \in M[p], h \in H[p]) . \tag{1}$$

Suppose now that $h_p(h) \geqq r - 1$. Then $h = p^{r-1}z$ for some $z \in G$ which, from (1) and minimality of r yields $m_1 + m = p^{r-1}(x - z) = p^{r-1}m_2$ for some $m_2 \in M$. But this gives $p^{r-1}x = p^{r-1}m_2 + h$ or, $p^r x = p^r m_2 = y$ contrary to the choice of r. Thus $h_p(h) < r - 1$ and $p^{r-1}x - (m + m_1) = h$. With $\boldsymbol{m} = -(m + m_1)$ we now have

$$h_p(\mathbf{m}) = h_p(h) < h_p(h - \boldsymbol{m})$$

and the theorem follows.

COROLLARY 2.2. *If, for each p, either $M \subseteqq pG$ or H_p is divisible, then M is pure in G.*

Proof. Neatness of M and $M \subseteqq pG$ give $M = pM$. Thus, for each prime p, $m \in M$ and $h \in H_p$, either $h_p(m) = \infty$ or $h_p(h) = \infty$.

The author is indebted to the referee for the proof of Theorem 2.1 given above a proof which is shorter and less complicated than the author's original. The original proof, however, had a corollary which, at the suggestion of the referee, we include here. The proof requires that we outline the proof of Theorem 2.1 given originally. Therefore we state the result as

PROPOSITION 2.3. *Let H be a subgroup of G such that $H_t = G_t$ and let M be H-high in G. If M is not pure in G then there exists a prime p and elements $m \in M$, $h \in H[p]$ such that*

$$0 = h_p(m) = h_p(h) < h_p(h - m) .$$

Proof (in outline). Let p^r, x and y be as in the proof of Theorem

2.1. Then $\{M, x\} \cap H \neq 0$ so there exist nonzero elements of H of the form $u + nx$ with $u \in M$ and n a positive integer. Let c be the least positive integer such that $m + cx \in H, m + cx \neq 0$ for some $m \in M$. Then one can show that $c = p^k$ for some $k < r$, and with $h = m + p^k x$ we have $h \in H[p]$ and

$$(*) \qquad h_p(m) = h_p(h) < h_p(h - m) = k$$

At this point we have a proof of Theorem 2.1, since we have not yet used the hypothesis $H_t = G_t$. Now, $p^{r-1}(px) = y$ so that by choice of y there exists $v \in M$ such that $p^{r-1}v = y = p^r x$. Hence $p^{r-1}(px - v) = 0$ and $r - 1 > 0$. Using $G_t = H_t$ we conclude that $px - v \in H$. It is clear that $px - v \neq 0$ so that $k = 1$. Thus $(*)$ gives $h_p(m) = h_p(h) = 0$ as required.

3. Centers of purity.

DEFINITION 3.1. A subgroup H of an abelian group G will be called a *Center of Purity* in G if every H-high subgroup of G is pure in G.

Several classes of centers of purity can be obtained from the following proposition which is a corollary to Theorem 2.1.

PROPOSITION 3.2. If there exists a homomorphism f defined on G such that

(i) $H_t \subset \text{kernel } f$

(ii) $h_p(m) = h_p(f(m))$ for all $m \in M$ and primes p then M is pure in G.

Proof. For any prime p, $m \in M$ and $h \in H_p$ we have

$$h_p(m) = h_p(f(m)) = h_p(f(m - h)) \geqq h_p(m - h) = h_p(h - m)$$

so that the condition in Theorem 1 alternative to purity of M cannot hold. Hence M is pure.

The following corollary generalizes the theorem of Khabbaz [4] referred to in the introduction.

COROLLARY 3.3. *Let G be a p-group and put $p^\infty G = \bigcap_n p^n G$, $p^{\infty+1}G = 0$. Then any subgroup H of G such that $p^s G \supseteqq H \supseteqq p^{s+1}G$ for some s, $0 \leqq s \leqq \infty$, is a center of purity in G.*

Proof. Let f be the canonical homomorphism $f: G \to G/H$. Then $p^{s+1}(G/H) = 0$ (by definition if $s = \infty$) so $h_p(f(x)) \leqq s$ for all $x \in G$, $x \notin H$. Suppose $p^u f(y) = f(x)$ for some $u \in Z$ and $x \notin H$. Then $p^u y + h = x$ for some $h \in H$. Since $H \subseteqq p^s G$, $u \leqq s$ and $u < \infty$ there exists

$w \in G$ such that $p^u w = h$. Hence, $p^u(y + w) = x$. Thus $h_p(x) \geqq h_p(f(x))$. The other inequality being obvious we have $h_p(x) = h_p(f(x))$ for all $x \in G$, $x \notin H$ and the corollary follows.

COROLLARY 3.4. *For any abelian group G and subgroup H of G, if $H_t \subseteqq \bigcap_n nG$, then H is a center of purity in G. In particular, high subgroups are pure and torsion free subgroups are centers of purity. If the maximal torsion subgroup of G is divisible, every subgroup of G is a center of purity in G.*

Proof. As in Corollary 3.3 with f the canonical homomorphism $f: G \rightarrow G/H_t$.

One can ask with Irwin [3] for necessary and sufficient conditions on a subgroup H of a group G in order that H be a center of purity in G. We have not been able to find such conditions. In particular, we know of no centers of purity in a p-group other than those listed in Corollary 3.3 above but have not been able to show that there are no others. In one case, however, a decisive answer is readily obtained. We denote by T, in what follows, the maximal torsion subgroup of G.

LEMMA 3.5. *If $T \subseteqq H$ then H is a center of purity in G if and only if for all $g \in G$ and primes p, the conditions $\{g\} \cap H = 0$ and $h_p(g) = 0$ imply $h_p(g + t) = 0$ for all $t \in T$.*

Proof. If the condition is satisfied, then H is a center of purity by Proposition 2.3. Conversely, if H is a center of purity in G and $g \in G$ such that $\{g\} \cap H = 0$ and $h_p(g) = 0$ for some p then there exists a subgroup M of G maximal disjoint from H and containing g. For $t \in T$, if $h_p(t) > 0$ it is clear that $h_p(g + t) = h_p(g) = 0$. Suppose then that $t \in T$ and $h_p(t) = 0$. We can write $t = t_p + t'$ where t_p has order p^l for some $l \geqq 0$ and the order of t' is prime to p. Then $h_p(t') = \infty$ so that $h_p(t) = h_p(t_p) = 0$. Clearly also $h_p(g + t) = h_p(g + t_p) \geqq k$ say. Let $x \in G$ such that $p^k x = g + t_p$. Then, with $e = k + l$, we have $p^e x = p^l g$. By purity of M there exists $m \in M$ such that $p^e m = p^l g$. Hence $p^l(p^k m - g) = 0$. Now since $T \subseteqq H$ and $M \cap H = 0$ we have $p^k m = g$ so that $k = 0$ by hypothesis on g. Thus, $h_p(g + t_p) = h_p(g + t) = 0$ as required.

DEFINITION 3.6. A subgroup H of G containing T will be called a *special center of purity* if H is a center of purity and there exists $x \in G$ such that $x \neq 0$, $\{x\} \cap H = 0$. A mixed group G is said to be properly mixed if $0 \neq T \neq G$.

THEOREM 3.7. *For a properly mixed group G the following are* *equivalent*:

(i) *T is divisible.*

(ii) *Every subgroup of G is a center of purity in G.*

(iii) *G contains a special center of purity.*

Proof. (i) implies (ii) by Corollary 3.4. (ii) implies (iii) since if (ii) holds and G is properly mixed, then T is a special center of purity. To show that (iii) implies (i), let H be a special center of purity in G. If T is not divisible then T_p is not divisible for some p so that there exists $t \in T_p$ such that $h_p(t) = 0$. Let $x \in G$, $\{x\} \cap H = 0$ and $x \neq 0$ and put $g = px + t$. Clearly $\{g\} \cap H = 0$ and $h_p(g) = 0$. However. $h_p(g - t) \geqq 1$. This contradicts Lemma 3.5 and completes the proof.

4. Reduction theorems. If M is maximal disjoint from H in G, we consider here circumstances under which we can reduce the problem of the purity of M in G to an analogous problem in a subgroup of G or in a factor group of G. Again, the location of T with respect to H plays a role.

THEOREM 4.1. *Let M and R be 'subgroups of G such that $G = M + R$. Then*

(i) *M is maximal disjoint in G from a subgroup $H \subseteqq R$ if and only if $M \cap R$ is maximal disjoint from H in R.*

(ii) *If $M \cap R$ is pure in R then M is pure in G. Conversely if $T \subseteqq R$ and M is pure in G then $M \cap R$ is pure in R.*

Proof. (i) If M is maximal disjoint in G from $H \subseteqq R$ and $r \in R$, $r \notin M \cap R$ then $\{M, r\} \cap H \neq 0$. Hence there exist $m \in M$, $a \in Z$ such that $m + ar \in H$, $m + ar \neq 0$. Since $H \subseteqq R$, we have $m + ar \in R$ so that $m \in M \cap R$. Thus $\{M \cap R, r\} \cap H \neq 0$.

Conversely, if $M \cap R$ is maximal disjoint from H in R and $g \in G$, $g \notin M$ we have $g = m + r$ for some $m \in M$, $r \in R$ by hypothesis. Now, $g \notin M$ implies that $r \notin M \cap R$ so that $\{M \cap R, r\} \cap H \neq 0$. Let $m_1 \in M$, $b \in Z$ such that $h = m_1 + br \in H$, $h \neq 0$. Then $m_1 + bg = h + bm$ so $m_1 - bm + bg = h \in H$, $h \neq 0$, $m_1 - bm \in M$; i.e. $\{M, g\} \cap H \neq 0$. Now since $0 = M \cap R \cap H = M \cap H$, M is maximal disjoint from H as required.

(ii) If $M \cap R$ is pure in R and $ng = m \in M$, let $g = m_1 + r$, $m_1 \in M$, $r \in R$. Then $m = ng = nm_1 + nr$ so $m - nm_1 = nr \in M \cap R$. By purity of $M \cap R$ in R there exists $m_2 \in M \cap R$ such that $nm_2 = m - nm_1$. Hence $n(m_1 + m_2) = m$ with $m_1 + m_2 \in M$ as required.

Conversely, if $T \subseteqq R$ and M is pure in G, suppose $nr = m \in M \cap R$ for some $n \in Z$, $r \in R$. By purity of M in G there exists $m_1 \in M$ such that $nm_1 = m$. Then $n(m_1 - r) = 0$ so that $m_1 - r \in T \subseteqq R$; i.e. $m_1 \in M \cap R$. Thus $M \cap R$ is pure in R.

THEOREM 4.2. *Suppose $T \subseteqq H$ and let M be maximal disjoint from H in G. Then the following are equivalent*:

(i) *M is pure in G*

(ii) *$(M+T)/T$ is pure in G/T*

(iii) *$(M+T)/T$ is maximal disjoint from H/T in G/T.*

Proof. It is well known (cf. [2, p. 94]) that if M is pure in G then $\{M, T\}$ is pure in G and in our case the converse is true since $\{M, T\} = M \oplus T$. Now, since T is pure in G, $M + T$ is pure in G if and only if $M + T/T$ is pure in G/T so that (i) and (ii) are equivalent. Also, since G/T is torsion free, if $M + T/T$ is maximal disjoint from H/T then $M + T/T$ is pure in G/T so that (iii) implies (ii). Finally assume that M is pure in G and let $g + T \in G/T$, $g + T \notin M + T/T$. Then $g \notin M$ so there exist $m \in M$, $a \in Z$ such that $m + ag \in H$, $m + ag \neq 0$. If $m + ag - t \in T$ say $bt = 0$. Then $bm = -bag$ and by purity of M in G there exists $m_1 \in M$ such that $bam_1 = bm$. Then $b(am_1 - m) = 0$ so $am_1 = m$ since $M \cap T = 0$. Now we have $t = m + ag = a(m_1 + g) \in T$ whence $m_1 + g \in T$. But this contradicts $g + T \notin M + T/T$. We conclude that $(m + ag) + T \in H/T$, $m + ag + T \neq T$ so that, disjointness of $M + T/T$ from H/T being clear, $M + T/T$ is maximal disjoint from H/T in G/T.

5. On the splitting of mixed groups. As an immediate consequence of Theorem 4.2 and Proposition 3.2 we have

PROPOSITION 5.1. Let T be the maximal torsion subgroup of the mixed group G. Then the following are equivalent:

(i) $G = M \oplus T$.

(ii) M is maximal disjoint from T in G and pure in G.

(iii) M is maximal disjoint from T in G and the natural mapping $\nu: G \to G/T$ is height preserving on M; i.e. $h_p(m) = h_p(\nu(m))$ for all $m \in M$, and all primes p.

As a result, there exist groups at the opposite end of the spectrum from centers of purity; i.e. since there exist nonsplitting mixed groups, we have

COROLLARY 5.2. *There exist groups G containing subgroups H such that, if M is maximal disjoint from H in G then M is not pure.*

One is tempted to try to use Proposition 5.1 to obtain splitting criteria for mixed groups in terms of the structure of the groups. If, for example, G contains a subgroup M maximal disjoint from T and p-divisible for all p for which $T_p \neq 0$ then M is pure by Theorem 2.1 and hence G splits. A necessary condition for such a situation is of course that G/T be p-divisible for all p for which $T_p \neq 0$. Just this condition has recently been considered by V. S. Žuravskiĭ ([6], [7]).

Although we fail to apply Proposition 5.1 we can point out a generalization of the result of Žuravskiĭ and, since the proof is quite short, it may be worthwhile to include this here. First, we observe

LEMMA 5.3. *Let G be a mixed group and R the subgroup of G generated by a complete system of representatives of $G \bmod T$. If R splits, $R = S \oplus R \cap T$, then G splits, $G = S \oplus T$.*

The proof is immediate. Now, for a mixed group G, we say that G satisfies the *maximal element condition* if each coset of T in G contains an element x such that $h_p(x) = h_p(x + T)$ for all p. Evidently (either directly or by Proposition 5.1) this is a necessary condition for the splitting of G. Let π be the set of primes p for which $T_p \neq 0$.

THEOREM 5.4. *Let G be a mixed group and T its maximal torsion subgroup. Suppose that*

(i) *G satisfies the maximal element condition.*

(ii) *G/T is p-divisible for all $p \in \pi$.*

(iii) *$\bigcap_{p\in\pi} \bigcap_n p^n T$ is bounded.*

Then G splits.

Proof. In each coset of T in G select an element x such that $h_p(x) = h_p(x + T)$ for all p. Then, by (ii) $h_p(x) = \infty$ for all $p \in \pi$. Let R be the subgroup of G generated by the elements so selected. Then it is clear that $R \cap T \subseteqq \bigcap_{p\in\pi} \bigcap_n p^n T$ and hence, by (iii) $R \cap T$ is bounded. Thus R splits, so G splits also.

The case treated by Žuravskiĭ is that in which T is p-primary and G/T is rank one. He also constructs an example [7, p. 380, Theorem 3.4] of a nonsplitting mixed group G satisfying (i) and (ii) (with T a primary group and G/T of rank one) but not (iii) so in this sense, condition (iii) is necessary. We remark that, as stated, [7, p, 380, Theorem 3.4] seems to say that given conditions (i) and (ii), the condition (iii) is necessary and sufficient for the splitting of G, but this is obviously false and, equally obviously (from the proof of the theorem) not what the author intended.

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RICCATI MATRIX DIFFERENTIAL EQUATIONS AND NON-OSCILLATION CRITERIA FOR ASSOCIATED LINEAR DIFFERENTIAL SYSTEMS

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1. Introduction. For real scalar linear homogeneous differential equations of the second order which are non-oscillatory on some interval (a, ∞) the concept of a "principal solution at ∞" was introduced by Leighton and Morse [4; 5]. Several years later Hartman and Wintner [2] studied the same concept, and subsequently Hartman [3] extended the notion of a principal solution to a self-adjoint matrix differential equation of the second order, characterizing such solutions by a distinguishing property in the class of solutions non-singular on some neighborhood of ∞ and which are "prepared" in his terminology. For a self-adjoint matrix differential system of more general type than considered by Hartman, Reid [9] presented a generalized definition of principal solution that distinguishes such solutions in the class of all solutions that are non-singular on some neighborhood of ∞; the determination of principal solutions in [9] is based on variational methods which are applicable directly to differential systems with complex coefficients that are of the form of the accessory differential equations for a calculus of variations problem of Bolza type, (see, for example, Bliss [1, § 81]).

Recently S. Sandor [11] has considered properties of solutions of Riccati matrix differential equations, including a generalization of the classical anharmonic ratio property that in character is quite different from the generalization studied by Whyburn [12] and Reid [7]. Moreover, for a real self-adjoint matrix system equivalent to the equation considered by Hartman [3], Sandor has shown the equivalence of the existence of a principal solution at ∞ in the sense of Hartman and the existence of a "right-hand frontier solution" of the associated Riccati matrix differential equation. Evidently Sandor was unaware of the paper [9] of Reid, for there are many intimate relationships between the results of the two papers, although the method of attack is quite different.

The purpose of the present paper is to study in more detail the concept of a principal solution of a non-oscillatory linear matrix differential system, together with related problems for the associated Riccati matrix equation. In particular, certain aspects considered

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previously in variational context only are here divorced from such limitations. § 2 is devoted to basic relationships between linear matrix systems and Riccati equations, together with a representation theorem, (Lemma 2.1), which is derived under more general conditions than those employed by Sandor [11] in corresponding results, and which permits simplification in the ensuing proof of the anharmonic ratio property, of Sandor [11] and Levin [6]. The results in § 3 on the variation of solutions of a Riccati equation are prefatory to § 4 on the concepts of a "principal solution" for a non-oscillatory linear system, and the corresponding "distinguished solution" of the associated Riccati equation; in this discussion these concepts are not limited to the instance of self-adjoint linear systems, as has been the case in the above cited papers. § 5 is devoted to the case in which the involved linear system is self-adjoint, but of a more general character than those treated by Hartman [3], Reid [9], and Sandor [11]. Systems that are non-oscillatory on intervals of the form $(-\infty, a)$ or $(-\infty, \infty)$ are treated briefly in § 6, and § 7 is devoted to certain specific results for systems with constant coefficients.

For simplicity of treatment, throughout the discussion of non-oscillatory systems in § 4–7 it is assumed that the involved linear system is identically normal. For systems that are not identically normal, however, certain modifications of the basic theorems of § 4, 5 hold, and the author plans to further this study in a subsequent paper.

Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector $y = (y_\alpha)$, $(\alpha = 1, \cdots, n)$, the norm $|y|$ is given by $(|y_1|^2 + \cdots + |y_n|^2)^{1/2}$. The symbol E is used for the $n \times n$ identity matrix, while 0 is used indiscriminately for the zero matrix of any dimensions; the conjugate transpose of a matrix M is denoted by M^*. If M is an $n \times n$ matrix the symbol $|M|$ is used for the supremum of $|My|$ on the unit sphere $|y| = 1$. The notation $M \geqq N$, $\{M > N\}$, is used to signify that M and N are hermitian matrices of the same dimensions and $M - N$ is a nonnegative, {positive}, definite hermitian matrix. If $M \geqq 0$ then $M^{1/2}$ signifies the unique nonnegative definite square root of M; if $M > 0$ then $M^{-1/2}$ denotes the reciprocal of $M^{1/2}$. For an arbitrary square matrix M we set $M_{\mathfrak{R}} = \frac{1}{2}(M + M^*)$ and $M_{\mathfrak{I}} = \frac{1}{2}i(M^* - M)$, so that $M_{\mathfrak{R}}$ and $M_{\mathfrak{I}}$ are the hermitian matrices with the definitive property $M = M_{\mathfrak{R}} + iM_{\mathfrak{I}}$. If the elements of a matrix $M(x)$ are a.c. (absolutely continuous) on an interval $[c, d]$, then $M'(x)$ signifies the matrix of derivatives at values for which these derivatives exist and the zero matrix elsewhere; correspondingly, if the elements of $M(x)$ are (Lebesgue) integrable on $[c, d]$ then $\int_c^d M(x)dx$ denotes the matrix of integrals of respective elements of $M(x)$. If matrices $M(x)$ and $N(x)$ are equal a.e. (almost everywhere) on their domain of definition we write simply $M(x) = N(x)$. Finally, for brevity a matrix $M(x)$ is termed continuous, etc., when each element of the matrix possesses the specified property.

2. Related linear systems and Riccati equations. The linear vector differential systems to be considered are of the form

$$u' = A(x)u + B(x)v\,, \quad v' = C(x)u - D(x)v\,, \tag{2.1}$$

where $u(x)$ and $v(x)$ are n-dimensional vector functions, and $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are $n \times n$ matrices with complex elements which are (Lebesgue) integrable on arbitrary compact subintervals of a given interval X on the real line. A major portion of our discussion involves the corresponding matrix differential equations

$$U' = A(x)U + B(x)V\,, \quad V' = C(x)U - D(x)V\,, \tag{2.2}$$

where in general $U(x)$ and $V(x)$ are matrices of n rows and r, ($r \geqq 1$), columns. By a solution $(u;\, v)$ of (2.1), or a solution $(U;\, V)$ of (2.2), will be meant vector or matrix functions which are a.c. on arbitrary compact subintervals of X, and such that (2.1) or (2.2) hold a.e. on X. For brevity, we introduce the notations

$$\begin{aligned} L_1[U, V] &= U' - A(x)U - B(x)V\,, \\ L_2[U, V] &= V' - C(x)U + D(x)V\,, \end{aligned} \tag{2.3}$$

for general n-rowed matrices U, V so that (2.2) becomes $L_\alpha[U, V] = 0$, $\alpha = 1, 2$.

If $U(x)$, $V(x)$ are $n \times n$ matrix functions a.c. on compact subintervals of X, and $U(x)$ is non-singular on X, then the corresponding Riccati matrix differential operator

$$K[W] \equiv W' + WA(x) + D(x)W + WB(x)W - C(x) \tag{2.4}$$

satisfies the identity

$$U^*(x)K[VU^{-1}]U(x) \equiv U^*(x)\big(L_2[U, V] - V(x)U^{-1}(x)L_1[U, V]\big)\,. \tag{2.5}$$

Consequently, if $(U(x);\, V(x))$ is a solution of (2.2) on X with $U(x)$ nonsingular on this interval, then $W(x) = V(x)U^{-1}(x)$ is a solution of the Riccati matrix differential equation

$$K[W] = 0 \tag{2.6}$$

on this interval; that is, $W(x)$ is an $n \times n$ matrix which is a.c. on compact subintervals of X and (2.6) holds a.e. on X. Conversely, if $W(x)$ is a solution of (2.6) on X, and for $s \in X$ the matrix $U(x)$ is determined as the solution of

$$U' = [A(x) + B(x)W(x)]U\,, \quad U(s) = M\,, \quad M \textit{ nonsingular},$$

then $(U;\ V) = (U(x);\ W(x)U(x))$ is the solution of (2.2) satisfying $U(s) = M$, $V(s) = W(s)M$, and $W(x) = V(x)U^{-1}(x)$ on X.

If $W(x)$ and $W_0(x)$ are $n \times n$ matrices a.c. on compact subintervals of X, then $\Psi(x) = W(x) - W_0(x)$ satisfies the identity

$$K[W] - K[W_0] = \Psi' + \Psi(A + BW_0) + (D + W_0B)\Psi + \Psi B\Psi\,. \tag{2.7}$$

LEMMA 2.1. *If $W_0(x)$ is a solution of* (2.6) *on X, and for $s \in X$ the matrices $G(x) = G(x, s \mid W_0)$, $H(x) = H(x, s \mid W_0)$ are solutions of the linear differential systems*

$$G' + (D + W_0B)G = 0\,, \quad G(s) = E\,, \tag{2.8}$$

$$H' + H(A + BW_0) = 0\,, \quad H(s) = E\,, \tag{2.9}$$

and

$$\Theta(x, s \mid W_0) = \int_s^x H(t)B(t)G(t)dt\,, \tag{2.10}$$

then $W(x)$ is a solution of (2.6) *on X if and only if the constant matrix $\Gamma = W(s) - W_0(s)$ is such that $E + \Theta(x, s \mid W_0)\Gamma$ is nonsingular on X, and*

$$W(x) = W_0(x) + G(x, s \mid W_0)\Gamma[E + \Theta(x, s \mid W_0)\Gamma]^{-1}H(x, s \mid W_0)\,. \tag{2.11}$$

If $K[W_0] = 0$ on X, and for an arbitrary $W(x)$ we set $\Psi(x) = W(x) - W_0(x)$, in view of (2.7), (2.8), (2.9) it follows that W satisfies (2.6) on X if and only if the matrix $F(x)$ defined by $\Psi(x) = G(x)F(x)H(x)$ is a solution on X of the special Riccati matrix differential equation

$$F' + F[H(x)B(x)G(x)]F = 0, \quad F(s) = \Gamma \equiv W(s) - W_0(s)\,. \tag{2.12}$$

If $F(x)$ is a solution of (2.12) on X, and $\Theta(x, s \mid W_0)$ is defined by (2.10), then $F_1(x) = F(x)[E + \Theta(x, s \mid W_0)\Gamma] - \Gamma$ satisfies the linear homogeneous system

$$F_1' = -F(x)H(x)B(x)G(x)F_1\,, \quad F_1(s) = 0\,, \tag{2.13}$$

and consequently $F_1(x) \equiv 0$ on X. Moreover, if $r \in X$ and η is a vector such that $[E + \Theta(r, s \mid W_0)\Gamma]\eta = 0$, then $0 = F_1(r)\eta = -\Gamma\eta$, and hence $\eta = 0$. Consequently $E + \Theta(x, s \mid W_0)\Gamma$ is nonsingular throughout X, and

$$F(x) = \Gamma[E + \Theta(x, s \mid W_0)\Gamma]^{-1} \tag{2.14}$$

on this interval. Conversely, if Γ is a constant matrix such that

$E + \Theta(x, s \mid W_0)\Gamma$ is nonsingular throughout X, then $F(x)$ defined by (2.14) is the solution of (2.12) on X, and $W(x)$ given by (2.11) satisfies (2.6).

Since for arbitrary $n \times n$ matrices Θ, Γ the identity $(E + \Gamma\Theta)\Gamma = \Gamma(E + \Theta\Gamma)$ implies that $E + \Gamma\Theta$ is non-singular if and only if $E + \Theta\Gamma$ is non-singular, and $\Gamma(E + \Theta\Gamma)^{-1} = (E + \Gamma\Theta)^{-1}\Gamma$, the non-singularity of $E + \Theta(x, s \mid W_0)\Gamma$ on X is equivalent to the non-singularity of $E + \Gamma\Theta(x, s \mid W_0)$ on this interval, and an alternate form of (2.11) is

$$(2.11') \quad W(x) = W_0(x) + G(x, s \mid W_0)[E + \Gamma\Theta(x, s \mid W_0)]^{-1}\Gamma H(x, s \mid W_0) .$$

In particular, if $W_0(x)$ and $W(x)$ are solutions of (2.6) on X, and $\Gamma = W(s) - W_0(s)$ is non-singular, then (2.11) and (2.11′) each reduces to

$$(2.11'') \quad W(x) = W_0(x) + G(x, s \mid W_0)[\Gamma^{-1} + \Theta(x, s \mid W_0)]^{-1}H(x, s \mid W_0) .$$

For the special case of $\Gamma = W(s) - W_0(s)$ non-singular Sandor [11] obtained this latter formula, and in this instance he termed $W(x)$ *representable with the aid of* $W_0(x)$ by (2.11′). The above results presenting (2.11) and (2.11′) show that this concept of representability may be given a form independent of the non-singularity of $W(s) - W_0(s)$. Moreover, it is to be noted that (2.11) implies that throughout X the rank of $W(x) - W_0(x)$ is equal to that of Γ, thus presenting a new proof of the known result that the difference of two solutions of (2.6) is of constant rank throughout a common interval of definition, (see Reid [7; Theorem 2.1]). Finally, it is to be remarked that if $W_0(x)$ is a solution of (2.6) on an interval X, and $(U_0(x); V_0(x))$ is a solution of (2.2) such that $U_0(x)$ is nonsingular and $W_0(x) = V_0(x)U_0^{-1}(x)$ on this interval, then the solution $H(x, s \mid W_0)$ of (2.9) is given by

$$(2.15) \qquad H(x, s \mid W_0) = U_0(s)U_0^{-1}(x) .$$

LEMMA 2.2. *If* $W_0(x)$, $W_\alpha(x)$, $(\alpha = 1, \cdots, k)$, *are solutions of* (2.6) *on* X, $s \in X$, *and* $\Gamma_\alpha = W_\alpha(s) - W_0(s)$, *then*

$$(2.16) \quad \begin{aligned} W_\alpha(x) - W_\beta(x) = {} & G(x, s \mid W_0)[E + \Gamma_\beta\Theta(x, s \mid W_0)]^{-1} \\ & \cdot(\Gamma_\alpha - \Gamma_\beta)[E + \Theta(x, s \mid W_0)\Gamma_\alpha]^{-1}H(x, s \mid W_0) . \end{aligned}$$

In view of Lemma 2.1, $G = G(x, s \mid W_0)$, $H = H(x, s \mid W_0)$ and $\Theta = \Theta(x, s \mid W_0)$ are such that

$$W_\alpha - W_0 = G\Gamma_\alpha[E + \Theta\Gamma_\alpha]^{-1}H = G[E + \Gamma_\alpha\Theta]^{-1}\Gamma_\alpha H ,$$

and (2.16) is an immediate consequence of the relation

$$\begin{aligned} \Gamma_\alpha - \Gamma_\beta &= [E + \Gamma_\beta\Theta]\Gamma_\alpha - \Gamma_\beta[E + \Theta\Gamma_\alpha] , \\ &= [E + \Gamma_\beta\Theta](\Gamma_\alpha[E + \Theta\Gamma_\alpha]^{-1} - [E + \Gamma_\beta\Theta]^{-1}\Gamma_\beta)[E + \Theta\Gamma_\alpha] . \end{aligned}$$

If M_1, M_2, M_3, M_4 are $n \times n$ matrices with $M_3 - M_2$ and $M_4 - M_1$ non-singular, we introduce the notations

$$
\begin{aligned}
&\{M_1, M_2, M_3\} = (M_3 - M_1)(M_3 - M_2)^{-1}\,, \\
(2.17)\quad &\{M_1, M_2, M_3, M_4\} = \{M_1, M_2, M_3\}\{M_2, M_1, M_4\}\,, \\
&\qquad = (M_3 - M_1)(M_3 - M_2)^{-1}(M_4 - M_2)(M_4 - M_1)^{-1}\,.
\end{aligned}
$$

Clearly, $\{M_1, M_2, M_3, M_4\}$ is a direct matrix generalization of the scalar anharmonic ratio.

THEOREM 2.1. *If* $W_0(x)$, $W_\alpha(x)$, $(\alpha = 1, 2, 3, 4)$, *are solutions of* (2.6) *on* X *with* $W_3(x) - W_2(x)$ *and* $W_4(x) - W_1(x)$ *non-singular, and* $s \in X$, *then*

$$
\begin{aligned}
(2.18)\quad &\{W_1(x), W_2(x), W_3(x), W_4(x)\} \\
&= \Phi(x, s \mid W_0, W_1)\{W_1(s), W_2(s), W_3(s), W_4(s)\}\Phi^{-1}(x, s \mid W_0, W_1)\,,
\end{aligned}
$$

where $\Phi(x, s \mid W_0, W_1) = G(x, s \mid W_0)[E + \Gamma_1\Theta(x, s \mid W_0)]^{-1}$, *and* $\Gamma_1 = W_1(s) - W_0(s)$.

If $\Gamma_\alpha = W_\alpha(s) - W_0(s)$, $(\alpha = 1, 2, 3, 4)$, then from (2.16) it follows directly that $G = G(x, s \mid W_0)$ satisfies

$$
\begin{aligned}
\{W_1(x), W_2(x), W_3(x)\} &= G[E + \Gamma_1\Theta]^{-1}\{\Gamma_1, \Gamma_2, \Gamma_3\}[E + \Gamma_2\Theta]G^{-1}\,, \\
\{W_2(x), W_1(x), W_4(x)\} &= G[E + \Gamma_2\Theta]^{-1}\{\Gamma_2, \Gamma_1, \Gamma_4\}[E + \Gamma_1\Theta]G^{-1}\,,
\end{aligned}
$$

and (2.18) is an immediate consequence of these relations and

$$\{\Gamma_1, \Gamma_2, \Gamma_3\,\Gamma_4\} = \{W_1(s), W_2(s), W_3(s), W_4(s)\}\,.$$

The fact that the "anharmonic ratio" of four solutions of (2.6) is similar to a constant matrix has been established by Sander [11] and Levin [6]; it is to be noted that Levin's hypotheses are needlessly strong as he supposes that $W_\alpha(x) - W_\beta(x)$, $(\alpha, \beta = 1, 2, 3, 4;\ \alpha \neq \beta)$, is non-singular. In view of the generality of the result of our Lemma 2.2, however, the proof of the above Theorem 2.1 is more direct than that given by Sandor for his Theorem 1, which involved the determination of a particular solution $W_0(x)$ such that each of the constant matrices Γ_α, $(\alpha = 1, 2, 3, 4)$, is non-singular. Indeed, in the proof of Theorem 2.1 one might choose $W_0(x) = W_1(x)$, in which case $\Gamma_1 = 0$ and (2.18) reduces to

$$
\begin{aligned}
(2.19)\quad &\{W_1(x), W_2(x), W_3(x), W_4(x)\} \\
&= G(x,s \mid W_1)\{W_1(s), W_2(s), W_3(s), W_4(s)\}G^{-1}(x, s \mid W_1)\,.
\end{aligned}
$$

It is to be remarked that the above type of anharmonic ratio property of four solutions of (2.6) is quite different from the generalization of the anharmonic ratio considered by Whyburn [12] and Reid [7].

With the aid of (2.18) and (2.19) one may deduce that if $W_0(x)$ and $W_1(x)$ are solutions of (2.6) on an interval X then

$$G(x, s \mid W_1) = G(x, s \mid W_0)[E + \Gamma_1 \Theta(x, s \mid W_0)]^{-1}, \tag{2.20}$$
where
$$\Gamma_1 = W_1(s) - W_0(s).$$

Relation (2.20) is but one of the variational relations for solutions of (2.6) which will be established in the next section, however, so it will not be considered further here.

Of special significance is the class of systems for which the coefficient matrices satisfy the conditions

$$B(x) \equiv B^*(x), \quad C(x) \equiv C^*(x), \quad D(x) \equiv A^*(x), \tag{2.21}$$

since particular systems of this type occur as accessory systems for simple integral variational problems, (see, for example, Bliss [1, § 81], Reid [7]). In this instance, if $(U(x);\ V(x))$ is a solution of (2.2) on X then there exists a constant matrix K such that $U^*(x)V(x) - V^*(x)U(x) \equiv K$; in particular, if $U(x)$ is non-singular on X then $W(x) = V(x)U^{-1}(x)$ is a solution of (2.6) on X such that

$$W(x) - W^*(x) = U^{*-1}(x)KU^{-1}(x), \tag{2.22}$$

and $W(x)$ is hermitian if and only if $K = 0$. Moreover, if $s \in X$ then the solution $H = H(x,s \mid W)$ of the corresponding equation (2.9) satisfies

$$H^{*\prime} + (D + [W - U^{*-1}KU^{-1}]B)H^* = 0, \quad H^* = E \textit{ for } x = s,$$

and for the solution $G = G(x, s \mid W)$ of the corresponding equation (2.8) the relation (2.15) and the method of variation of parameters yields

$$G(x, s \mid W) = H^*(x, s \mid W)U^{*-1}(s)T^{*-1}(x, s;\ U)U^*(s), \tag{2.23}$$

where $T = T(x, s \mid U)$ is the solution of the differential system

$$T' = -U^{-1}(x)B(x)U^{*-1}(x)KT, \quad T(s) = E. \tag{2.24}$$

Consequently the function $\Theta(x, s \mid W)$ given by (2.10) has the form

$$\Theta(x, s \mid W) = U(s)S^*(x, s;\ U)U^*(s), \tag{2.25}$$

where

$$S(x, s;\ U) = \int_s^x T^{-1}(t, s;\ U)U^{-1}(t)B(t)U^{*-1}(t)dt \tag{2.26}$$

is the function introduced by Reid [9, equation (3.6)] for the general characterization of principal solutions of non-oscillatory self-adjoint differential systems.

Following the terminology used by Reid [8; 9], if the coefficient matrices satisfy (2.21) then two solutions $(u_1(x); v_1(x))$ and $(u_2(x); v_2(x))$ of (2.1) for which the constant value of $u_1^*(x)v_2(x) - v_1^*(x)u_2(x)$ is zero are said to be (mutually) conjoined solutions. As in Lemma 2.3 of Reid [8], one may prove for such systems (2.1) that the maximum dimension of a conjoined family of solutions is n, and that a given conjoined family of solutions of dimension less than n is contained in a conjoined family of dimension n.

3. Variation of solutions. Let $\boldsymbol{A}(x)$ denote the $2n \times 2n$ direct sum matrix

$$\boldsymbol{A}(x) = \begin{Vmatrix} A(x) & 0 \\ 0 & 0 \end{Vmatrix},$$

where 0 is the $n \times n$ zero matrix, with similar definitions for $\boldsymbol{B}(x)$, $\boldsymbol{C}(x)$, $\boldsymbol{D}(x)$ in terms of the corresponding $B(x)$, $C(x)$, $D(x)$. It may be verified directly that a $2n \times 2n$ matrix $\boldsymbol{W}(x)$ is a solution of the Riccati matrix differential equation

$$\boldsymbol{W}' + \boldsymbol{W}\boldsymbol{A}(x) + \boldsymbol{D}(x)\boldsymbol{W} + \boldsymbol{W}\boldsymbol{B}(x)\boldsymbol{W} - \boldsymbol{C}(x) = 0 \tag{3.1}$$

on an interval X if and only if

$$\boldsymbol{W}(x) = \begin{Vmatrix} W(x) & G(x) \\ H(x) & -\Theta(x) \end{Vmatrix}, \tag{3.2}$$

where $W(x)$, $G(x)$, $H(x)$ and $\Theta(x)$ are $n \times n$ matrices which satisfy on this interval the Riccati system

$$\begin{aligned} &W' + WA(x) + D(x)W + WB(x)W - C(x) = 0\,, \\ &G' + [D(x) + WB(x)]G = 0\,, \\ &H' + H[A(x) + D(x)W] = 0\,, \\ &\Theta' - HB(x)G = 0\,. \end{aligned} \tag{3.3}$$

This relation between a Riccati system (3.3) and the associated single Riccati equation (3.1) has been exploited previously by the author in the study of a different type of problem, (see Reid [10, § 4]).

In particular, if $W_0(x)$ is a solution of (2.6) on X, and $G(x, s \mid W_0)$, $H(x, s \mid W_0)$ and $\Theta(x, s \mid W_0)$ are defined by (2.8), (2.9) and (2. 10), then the solution $\boldsymbol{W} = \boldsymbol{W}_0(x)$ of (3.1) satisfying the initial condition

$$\boldsymbol{W}_0(s) = \begin{Vmatrix} W_0(s) & E \\ E & 0 \end{Vmatrix} \tag{3.4}$$

is given by

$$\boldsymbol{W}_0(x) = \begin{Vmatrix} W_0(x) & G(x, s \mid W_0) \\ H(x, s \mid W_0) & -\Theta(x, s \mid W_0) \end{Vmatrix}. \tag{3.5}$$

Moreover, for this solution $\mathbf{W}_0(x)$ of (3.1) the matrix functions $\mathbf{G}(x, s \mid \mathbf{W}_0)$, $\mathbf{H}(x, s \mid \mathbf{W}_0)$, $\boldsymbol{\Theta}(x, s \mid \mathbf{W}_0)$ determined by the corresponding equations (2.8), (2.9), (2.10) are computed readily to be

$$
\begin{aligned}
\mathbf{G}(x, s \mid \mathbf{W}_0) &= \begin{Vmatrix} G(x, s \mid W_0) & 0 \\ -\Theta(x, s \mid W_0) & E \end{Vmatrix}, \\
\mathbf{H}(x, s \mid \mathbf{W}_0) &= \begin{Vmatrix} H(x, s \mid W_0) & -\Theta(x, s \mid W_0) \\ 0 & E \end{Vmatrix}, \qquad (3.6) \\
\boldsymbol{\Theta}(x, s \mid \mathbf{W}_0) &= \begin{Vmatrix} \Theta(x, s \mid W_0) & 0 \\ 0 & 0 \end{Vmatrix}.
\end{aligned}
$$

If $\mathbf{W}_0(x)$ is the solution (3.5) of (3.1) on X, and $\mathbf{W}(x)$ is a second solution of this equation on X satisfying the initial condition

$$
\mathbf{W}(s) = \begin{Vmatrix} W(s) & E \\ E & 0 \end{Vmatrix}, \tag{3.7}
$$

then the associated equation (2.11) in $\mathbf{W}(x)$ and $\mathbf{W}_0(x)$, with

$$
\boldsymbol{\Gamma} = \begin{Vmatrix} \Gamma & 0 \\ 0 & 0 \end{Vmatrix}, \quad \Gamma = W(s) - W_0(s), \tag{3.8}
$$

yields (2.11) in $W(x)$, $W_0(x)$ and also the following additional equations of variation:

$$
\begin{aligned}
G(x, s \mid W) &= G(x, s \mid W_0)[E + \Gamma\Theta(x, s \mid W_0)]^{-1}, \\
H(x, s \mid W) &= [E + \Theta(x, s \mid W_0)\Gamma]^{-1}H(x, s \mid W_0), \qquad (3.9) \\
\Theta(x, s \mid W) &= [E + \Theta(x, s \mid W_0)\Gamma]^{-1}\Theta(x, s \mid W_0), \\
&= \Theta(x, s \mid W_0)[E + \Gamma\Theta(x, s \mid W_0)]^{-1}.
\end{aligned}
$$

In particular, if $\Theta(x, s \mid W_0)$ is non-singular on a subinterval X_0 of X then $\Theta(x, s \mid W)$ is non-singular on this subinterval also, and

$$
\Theta^{-1}(x, s \mid W) = \Theta^{-1}(x, s \mid W_0) + \Gamma. \tag{3.10}
$$

4. Principal solutions for non-oscillatory systems (2.1.) Two distinct points s and t on X are said to be *(mutually) conjugate*, (with respect to (2.1)) if there exists a solution $(u(x); v(x))$ of this system with $u(x) \not\equiv 0$ on the subinterval with endpoints s and t, while $u(s) = 0 = u(t)$. The system (2.1) is termed *non-oscillatory* on a given subinterval X_0 provided no two distinct points of this subinterval are conjugate; moreover, (2.1) will be called *non-oscillatory for large* {small} x if there exists a subinterval $[a, \infty)\{(-\infty, a_1]\}$ of X on which this system is non-oscillatory.

A system (2.1) is termed *identically normal on X*, or *normal on every subinterval* of X, if whenever $(u; v) = (0; v(x))$ is a solution of this system on a non-degenerate subinterval of X then also $v(x) \equiv 0$ on this subinterval. If (2.1) is identically normal on X, $s \in X$, and $(U(x);\ V(x))$ is a solution of (2.2) with $U(s) = 0$ and $V(s)$ non-singular, then the points t conjugate to s are those values for which $U(t)$ is singular; in particular, if such a system is non-oscillatory on an interval X, and $s \in X$, then $U(x)$ is non-singular on each of the subintervals $X_s^+ = \{x \mid x \in X,\ x > s\}$ and $X_s^- = \{x \mid x \in X,\ x < s\}$.

A basic result for non-oscillatory systems is the following theorem. It is to be emphasized that in contrast to the special case considered previously by the author in [9], the result of this theorem is not limited to self-adjoint systems of the form of accessory equations for problems of the calculus of variations, and the proof is independent of variational principles.

THEOREM 4.1. *If* (2.1) *is identically normal and non-oscillatory on an interval* X, *and* $W_0(x)$ *is a solution of* (2.6) *on this interval, then for* $s \in X$ *the matrix* $\Theta(x, s \mid W_0)$ *is non-singular on each of the subintervals* X_s^+ *and* X_s^-; *moreover,*

$$\Theta^{-1}(t, s \mid W_0) = W_0(s) - W_t(s), \quad t \in X_s^+ \ \text{or} \ t \in X_s^-, \tag{4.1}$$

where $W_t(x) = V_t(x) U_t^{-1}(x)$ *and* $(U_t(x);\ V_t(x))$ *is the solution of* (2.2) *determined by the initial conditions*

$$U_t(t) = 0, \quad V_t(t) = E. \tag{4.2}$$

Suppose that $t \in X_s^+$, and $(U_t(x);\ V_t(x))$ is the solution of (2.2) satisfying (4.2). In view of the condition that (2.1) is identically normal and non-oscillatory on X, the matrix $U_t(x)$ is non-singular on X_t^- and X_t^+. In particular, on X_t^- each of the matrices $W_0(x)$ and $W_t(x) = V_t(x) U_t^{-1}(x)$ is a solution of (2.6), $H(x, s \mid W_t) = U_t(s) U_t^{-1}(x)$, and from (3.9) we have

$$U_t(s) U_t^{-1}(x) = [E + \Theta(x, s \mid W_0)\{W_t(s) - W_0(s)\}]^{-1} H(x, s \mid W_0), \quad x \in X_t^-.$$

Consequently,

$$[E + \Theta(x, s \mid W_0)\{W_t(s) - W_0(s)\}] U_t(s) = H(x, s \mid W_0) U_t(x), \quad x \in X_t^-, \tag{4.3}$$

and by continuity (4.3) also holds for $x = t$. As $s \in X_t^-$ and $U_t(s)$ is non-singular, while $U_t(t) = 0$, it follows that $\Theta(t, s \mid W_0)$ is non-singular with inverse $W_0(s) - W_t(s)$, so that $\Theta(x, s \mid W_0)$ is non-singular for $x \in X_s^+$. A similar argument shows that $\Theta(t, s \mid W_0)$ is non-singular and (4.1) holds for $t \in X_s^-$.

It is to be emphasized that the non-oscillation of (2.1) on X is not a consequence of the existence of a solution $W_0(x)$ of (2.6), or the equivalent condition that there is a solution $(U_0(x);\ V_0(x))$ of (2.2) with $U_0(x)$ non-singular throughout X. Indeed, for any self-adjoint system (2.1) with coefficient matrices satisfying (2.21) the existence of a solution $(U_0(x); V_0(x))$ with $U_0(x)$ non-singular throughout X is illustrated by any solution $(U_0(x); V_0(x))$ satisfying at an initial point s the condition $U_0^*(s)V_0(s) - V_0^*(s)U_0(s) = iK_0$, where K_0 is a definite hermitian matrix. On the other hand, for the general system (2.1) that is identically normal and non-oscillatory on X the author has not settled the question as to the existence of a solution $W_0(x)$ of (2.6) throughout X.

It $W_0(x)$ is a solution of (2.6) on X, the semi-group properties

$$\begin{aligned} G(x, s \mid W_0) &= G(x, t \mid W_0)G(t, s \mid W_0)\,, \\ H(x, s \mid W_0) &= H(t, s \mid W_0)H(x, t \mid W_0)\,, \end{aligned} \qquad s, t, x \in X \tag{4.4}$$

of the solutions of (2.8), (2.9) imply for $\Theta(x, s \mid W_0)$ of (2.10) the relation

$$\Theta(x, s \mid W_0) = \Theta(t, s \mid W_0) + H(t, s \mid W_0)\Theta(x, t \mid W_0)G(t, s \mid W_0)\,. \tag{4.5}$$

Since for an identically normal system that is non-oscillatory on X we have $\Theta(x, s \mid W_0)$ non-singular for $x \neq s$, from (4.5) it follows that for $x \in X$ and distinct from both t and s the matrix

$$X(x, t, s \mid W_0) = E + H^{-1}(t, s \mid W_0)\Theta(t, s \mid W_0)G^{-1}(t, s \mid W_0)\Theta^{-1}(x, t \mid W_0) \tag{4.6}$$

is non-singular, and

$$\Theta^{-1}(x, s \mid W_0) = G^{-1}(t, s \mid W_0)\Theta^{-1}(x, t \mid W_0)X^{-1}(x, t, s \mid W_0)H^{-1}(t, s \mid W_0)\,. \tag{4.7}$$

From (4.6), (4.7) it follows that if $\Theta^{-1}(x, t \mid W_0) \to 0$ as $x \to \infty$, then also $\Theta^{-1}(x, s \mid W_0) \to 0$ as $x \to \infty$; moreover, for X_0 an arbitrary compact subinterval of X it follows from (4.7) that the convergence of $\Theta^{-1}(x, s \mid W_0)$ to 0 as $x \to \infty$ is uniform for s on X_0.

For an identically normal system that is non-oscillatory for large x a solution $(U_\infty(x); V_\infty(x))$ will be termed a *principal solution at* ∞ for (2.2) if $U_\infty(x)$ is non-singular on some subinterval (a, ∞) and for $W_\infty(x) = V_\infty(x)U_\infty^{-1}(x)$ we have $\Theta^{-1}(x, s \mid W_\infty) \to 0$ as $x \to \infty$ for at least one, (and consequently all), s on (a, ∞); the corresponding solution $W_\infty(x)$ of (2.6) will be called a *distinguished solution* at ∞ of this Riccati equation.

THEOREM 4.2. *If for an identically normal system* (2.1) *that is non-oscillatory for large x there exists a principal solution* $(U_\infty(x);$ $V_\infty(x))$ *with $U_\infty(x)$ non-singular on* $[a, \infty)$, *then*: (*a*) $U_\infty(x)$, $V_\infty(x)$ *and* $W_\infty(x) = V_\infty(x)U_\infty^{-1}(x)$ *are such that as* $t \to \infty$,

$$\begin{aligned} &W_t(s) \to W_\infty(s), \quad U_t(s)U_t^{-1}(a)U_\infty(a) \to U_\infty(s), \\ &V_t(s)U_t^{-1}(a)U_\infty(a) \to V_t(s) \end{aligned} \tag{4.8}$$

uniformly for s on an arbitrary compact subinterval of $[a, \infty)$, *where, as in Theorem* 4.1, $(U_t(x);\ V_t(x))$ *is the solution of* (2.2) *satisfying* (4.2) *and* $W_t(x) = V_t(x)U_t^{-1}(x)$; (b) *the associated distinguished solution of* (2.6) *at* ∞ *is determined uniquely and the most general principal solution of* (2.2) *at* ∞ *is* $(U_\infty(x)M;\ V_\infty(x)M)$, *where* M *is a non-singular constant matrix.*

Equation (4.1) and the remark following (4.7) imply that for a principal solution $(U_\infty(x); V_\infty(x))$ of (2.2) the associated distinguished solution $W_\infty(x) = V_\infty(x)U_\infty^{-1}(x)$ of (2.6) is such that $W_t(s) \to W_\infty(s)$ uniformly in s on an arbitrary compact subinterval of $[a, \infty)$. The second limit relation of (4.8), and the uniformity of this limit on arbitrary compact subsets, follow from the preceding limit relation and the fact that $U_t^0(x) = U_t(x)U_t^{-1}(a)U_\infty(a)$ and $U_\infty(x)$ are solutions of the differential systems

$$U_t^{0\prime} = [A(x) + B(x)W_t(x)]U_t^0\ ,$$
$$U_\infty' = [A(x) + B(x)W_\infty(x)]U_\infty\ ,$$

and $U_t^0(a) = U_\infty(a)$. In turn, the last limit relation of (4.8) and the stated property of uniformity are immediate consequences of the first two limits of (4.8) and the respective uniformity properties. Finally, the uniqueness of a distinguished solution of (2.6) at ∞, and the most general form of a principal solution for (2.2), are direct consequences of relations (4.8).

As will be shown in the next section, for a class of identically normal self-adjoint systems more inclusive than those previously studied by Hartman [3], Reid [9] and Sandor [11] the condition of non-oscillation for large x implies the existence of a principal solution of (2.2) at ∞. Such is not true for systems in general, however, as is illustrated by the simple scalar system

$$u' = v\ , \quad v' = [h''(x)/h'(x)]v, \quad 0 \leqq x < \infty\ , \tag{4.9}$$

where $h(x)$ is a function of class C'' on $[0, \infty)$ with

$$h'(x) \neq 0\ , \quad h(x_1) \neq h(x_2)\ \mathit{for}\ x_1 \neq x_2\ , \quad 0 \leqq x < \infty\ . \tag{4.10}$$

The general solution of (4.9) is $u = c_1 + c_2h(x)$, $v = c_2h'(x)$, and the associated Riccati differential equation

$$w' - [h''(x)/h'(x)]w + w^2 = 0 \tag{4.11}$$

has as solution $w(x) = [c_2h'(x)]/[c_1 + c_2h(x)]$ throughout any interval where $c_1 + c_2h(x) \neq 0$. In particular, if $w = w_0(x)$ is a solution of (4.11) on an interval $[a, \infty)$, then either $w_0(x) \equiv 0$ or $w_0(x) = h'(x)/[h(x) - c]$, where c is a constant such that $h(x) \neq c$ on this interval. If $w_0(x) \equiv 0$ then $w_0(x) = v_0(x)u_0^{-1}(x)$, where $u_0(x) \equiv k \neq 0$, $v_0(x) \equiv 0$ is a corresponding solution of (4.9), and $G(x, s \mid w_0) = h'(x)/h'(s)$, $H(x, s \mid w_0) \equiv 1$, and $\Theta^{-1}(x, s \mid w_0) = h'(s)/[h(x) - h(s)]$, so that $\Theta^{-1}(x, s \mid w_0) \to 0$ as $x \to \infty$ only if $|h(x)| \to \infty$ as $x \to \infty$. In case $w_0(x) = h'(x)/[h(x) - c]$, then $u_0(x) = k[h(x) - c]$ and $v_0(x) = kh'(x)$ with $k \neq 0$, $G(x, s \mid w_0) = (h'(x)[h(s) - c])/(h'(s)[h(x) - c])$, $H(x, s \mid w_0) = [h(s) - c]/[h(x) - c]$, and $\Theta^{-1}(x, s \mid w_0) = (h'(s)[h(x) - c])/([h(s) - c][h(x) - h(s)])$, so that $\Theta^{-1}(x, s \mid w_0) \to 0$ as $x \to \infty$ if and only is $h(x) \to c$ as $x \to \infty$.

Now if $h(x)$ is real-valued and $h'(x) \neq 0$ an $[0, \infty)$, then $h(x_1) \neq h(x_2)$ for $x_1 \neq x_2$ on this interval, and the limit of $h(x)$ as $x \to \infty$ exists, finite or infinite, so that in this case (4.9) always has a principal solution, On the other hand, there exist complex-valued $h(x)$ satisfying (4.10), and for which $h(x)$ does not tend to a limit as $x \to \infty$. Such an example is provided by $h(x) = 4(2 + \sin x)^{-1} - 2e^{-x} + i \sin^3 x$, $0 \leqq x < \infty$. If in the corresponding equation we set $u = u_1 + iu_2$, $v = v_1 + iv_2$ the equivalent system in u_1, u_2, v_1, v_2 is a system with real coefficients for which the corresponding 2×2 matrix $\Theta^{-1}(x, s \mid W_0)$ does not tend to a limit as $x \to \infty$.

5. Self-adjoint systems. Attention will now be restricted to identically normal systems (2.1) which satisfy the self-adjointness conditions (2.21), and also the following hypothesis:

$\mathfrak{H}_0$. *The matrix $B(x)$ is non-negative definite a.e. on X.*

The condition $\mathfrak{H}_0$, with x restricted to a subinterval $[c, d]$, will be denoted by $\mathfrak{H}_0[c, d]$.

THEOREM 5.1. *If an identically normal system* (2.1) *satisfying* (2.21) *and $\mathfrak{H}_0$ is non-oscillatory on X_0: (a_0, ∞), then this system possesses a principal solution at ∞. Indeed, if $a_0 < r < s < t < \infty$, $(U_{sr}(x); V_{sr}(x))$ is the solution of* (2.2) *satisfying $U_{sr}(r) = 0$, $U_{sr}(s) = E$, and $(U_{st}(s); V_{st}(x))$ is the solution of* (2.2) *satisfying $U_{st}(s) = E$, $U_{st}(t) = 0$, then $V_{sr}(s) > V_{sd}(s) > V_{st}(s)$ for $a_0 < r < s < t < d < \infty$, and consequently $V_{s\infty} = \lim_{t \to \infty} V_{st}(s)$ exists, and the solution $(U_{s\infty}(x); V_{s\infty}(x))$ of* (2.2) *satisfying $U_{s\infty}(s) = E$, $V_{s\infty}(s) = V_{s\infty}$ is a principal solution at ∞ with $U_{s\infty}(x)$ non-singular on X_0.*

For the case of a system (2.1) arising as the accessory system for a variational problem of Bolza type the result of Theorem 5.1 is given in Reid [9]. For such accessory systems the matrix $B(x)$ is of constant rank a.e. on X, whereas for the more general system the rank of $B(x)$ may not be constant a.e. on X. In particular, the more general problem includes as a very special instance systems that may be described roughly as arising through the adjunction at interfaces of a sequence of different problems, each of the accessory problem type on a corresponding interval.

The above Theorem 5.1 may be established by direct generalizations of the methods used in proving Theorem 5.1 in Reid [9], and this extension is immediate once one has established the results corresponding to Theorems 4.1, 4.2, and 4.3 of [9]. If $[c, d]$ is a compact subinterval of X let $\mathscr{D}[c, d]$ denote the class of pairs of n-dimensional vector functions $\eta(x)$, $\zeta(x)$ with $\eta(x)$ a.c. on $[c, d]$, $\zeta(x) \in \mathfrak{L}_\infty[c, d]$, the class of vector functions Lebesgue measurable and essentially bounded on $[c, d]$, and such that $L_1[\eta, \zeta] \equiv \eta' - A(x)\eta - B(x)\zeta = 0$ a.e. on this interval. The subclass of $\mathscr{D}[c, d]$ on which $\eta(c) = 0 = \eta(d)$ will be designated by $\mathscr{D}_0[c, d]$. Moreover, let $\mathfrak{H}_+[c, d]$ denote the condition that the functional

$$I[\eta, \zeta;\ c, d] = \int_c^d [\zeta^*(x)B(x)\zeta(x) + \eta^*(x)C(x)\eta(x)]dx \tag{5.1}$$

is positive definite on $\mathscr{D}_0[c, d]$, that is, $I[\eta, \zeta;\ c, d] \geqq 0$ for $(\eta, \zeta) \in \mathscr{D}_0[c, d]$, and the equality sign holds only if $B(x)\zeta(x) = 0$ a.e. and $\eta(x) \equiv 0$ on $[c, d]$. The following theorem presents a basic result concerning non-oscillation on a compact interval, and is the result for (2.1) corresponding to Theorem 4.1 of Reid [9].

THEOREM 5.2. *If* (2.1) *is an identically normal system satisfying* (2.21) *on a compact interval* $[c, d]$, *then* $\mathfrak{H}_+[c, d]$ *holds if and only if* $\mathfrak{H}_0[c, d]$ *holds, together with one of the following*:

(i) (2.1) *is non-oscillatory on* $[c, d]$;

(ii) *there exists a solution* $(U(x);\ V(x))$ *of* (2.2) *with* $U(x)$ *non-singular on* $[c, d]$ *and* $U^*(x)V(x) - V^*(x)U(x) \equiv 0$.

For systems (2.1) that arise as accessory systems for variational problems the result of Theorem 5.2 consists of the Legendre or Clebsch condition and a special oscillation theorem in the extension of the classical Sturmian theory to self-adjoint systems as initiated by M. Morse; for brief historical statements and references the reader is referred to the author's papers [8; 9] and their bibliographies. If $B(x)$ is positive definite a.e. on $[c, d]$ a proof is contained in Theorem 2.1 of Reid [8], and in the following discussion will be limited to certain aspects that differ from the special cases treated previously.

Theorem 5.2 will be established by proving the following sequence of statements: (a) $\mathfrak{H}_0[c, d]$, (ii) $\rightarrow \mathfrak{H}_+[c, d]$; (b) $\mathfrak{H}_+[c, d] \rightarrow$ (i), $\mathfrak{H}_0[c, d]$; (c) $\mathfrak{H}_0[c, d]$, (i) $\rightarrow$ (ii).

Statement (a) is an immediate consequence of the relation

$$I[\eta, \zeta;\ c, d] = \int_c^d (\zeta^* - \eta^* W)B(\zeta - W\eta)dx \geqq 0 \qquad \text{for } (\eta, \zeta) \in \mathscr{D}_0[c, d]\ , \tag{5.2}$$

where $W(x) = V(x)U^{-1}(x)$, since in view of $\mathfrak{H}_0[c, d]$ equality in (5.2) holds only if a.e. on $[c, d]$ we have $0 = B(\zeta - W\eta) = U[U^{-1}\eta]'$, so that $U^{-1}(x)\eta(x) \equiv U^{-1}(c)\eta(c) = 0$, and hence $\eta(x) \equiv 0$, $B(x)\zeta(x) = 0$ a.e. on $[c, d]$. In turn, (5.2) follows from the more general fact that if $U(x)$ and $V(x)$ are $n \times r$ a.c. matrices on $[c, d]$, and for $\alpha = 1, 2$ the vector functions $\eta_\alpha(x)$ are a.c. and $\zeta_\alpha(x) \in \mathfrak{L}_\infty[c, d]$, while there exist a.c. r-dimensional vector functions $h_\alpha(x)$ such that $\eta_\alpha(x) = U(x)h_\alpha(x)$ on $[c, d]$, then we have the identity

$$
\begin{aligned}
&\zeta_1^* B\zeta_2 + \eta_1^* C\eta_2 \\
&\quad = (\zeta_1^* - h_1^* V^*)B(\zeta_2 - Vh_2) - h_1^* V^* L_1[\eta_2, \zeta_2] - (L_1[\eta_1, \zeta_1])^* Vh_2 \\
&\qquad - h_1^*\{U^* L_2[U, V] - V^* L_1[U, V]\}h_2 - h_1^*[U^* V - V^* U]h_2' \\
&\qquad + [h_1^* U^* V h_2]' .
\end{aligned} \tag{5.3}
$$

For the proof of statement (b), it is to be noted that if (u, v) is a solution of (2.1) with $u(a) = 0 = u(b)$, where $c \leqq a < b \leqq d$, then for $\eta(x) = u(x)$, $\zeta(x) = v(x)$ on $[a, b]$ and $\eta(x) \equiv 0$, $\zeta(x) \equiv 0$ elsewhere, we have

$$I[\eta, \zeta;\ c, d] = I[u, v;\ a, b] = u^* v \,|_a^b = 0 ,$$

so that for general self-adjoint problems (2.1) condition $\mathfrak{H}_+[c, d]$ implies (i).

The fact that $\mathfrak{H}_+[c, d]$ implies $\mathfrak{H}_0[c, d]$ under the general conditions of the theorem may be proved by indirect argument. If it is not true that $B(x) \geqq 0$ a.e. on $[c, d]$, in view of the integrability of $B(x)$ on $[c, d]$, and the separability of finite dimensional Euclidean space, it follows that there exists a constant vector ζ_0 with $|\zeta_0| = 1$ and positive constants k_1, k_2 such that $X_0 = \{x \mid c \leqq x \leqq d,\ |B(x)| \leqq k_1,\ \zeta_0^* B(x)\zeta_0 < -k_2\}$ is of positive measure. If $Y(x)$ is a fundamental matrix of $Y' = A(x)Y$, and k_3 a constant such that $|Y(x)Y^{-1}(t)| \leqq k_3$ for x and t on $[c, d]$, let s be a point of outer density of X_0 belonging to (c, d), and choose a, b such that $c < a < s < b \leqq d$, and

$$(b - a)^{-1} > (k_1^2 k_3^2 / k_2) \int_c^d |C(x)|\, dx . \tag{5.4}$$

If $e(x)$ denotes the characteristic function of X_0, then there exists a continuous scalar function $g(x) \not\equiv 0$ on $[a, b]$, and such that the solution $y(x)$ of $L_1[y, \zeta_0 eg] = 0$, $y(a) = 0$, satisfies $y(b) = 0$ and $y(x) \not\equiv 0$ on $[a, b]$, indeed, $g(x)$ may be chosen of the form $g(x) = c_0 + c_1 x + \cdots + c_n x^n$ with $|c_0|^2 + \cdots + |c_n|^2 = 1$. For $\zeta_1(x) = \zeta_0 e(x)g(x)$ we have $y(x) = \int_a^x Y(x)Y^{-1}(t)B(t)\zeta_1(t)dt$, and in view of the definitive properties of k_1 and k_3 we have $|y(x)| \leqq k_1 k_3 \int_a^b |e(x)g(x)|\, dx$ for $a \leqq x \leqq b$. If $\eta(x) = y(x)$, $\zeta(x) = \zeta_1(x)$ on $[a, b]$, and $\eta(x) \equiv 0$, $\zeta(x) \equiv 0$ on $[c, a]$ and $[b, d]$, then $(\eta(x), \zeta(x)) \in \mathscr{D}_0[c, d]$ and

$$I[\eta, \zeta;\ c, d] \leqq -k_2 \int_a^b | e(x)g(x) |^2 dx + k_1^2 k_3^2 \left(\int_a^b | e(x)g(x) | \, dx \right)^2 \left(\int_c^d | C(x) | \, dx \right) .$$

As $\left(\int_a^b | e(x)g(x) | \, dx \right)^2 \leqq (b - a) \int_a^b | e(x)g(x) |^2 dx$ by the Schwarz inequality, and $\int_a^b |e(x)g(x)|^2 dx > 0$, with the aid of (5.4) it then follows that

$$I[\eta, \zeta;\ c, d] \leqq - \left(k_2 - (b - a) k_1^2 k_3^2 \int_c^d | C(x) | \, dx \right) \int_a^b | e(x)g(x) |^2 \, dx < 0 ,$$

contrary to the condition $\mathfrak{H}_+[c, d]$.

The above statement (c) may be proved by exactly the same type of argument as that used to establish the statement (f) for the proof of Theorem 2.1 in Reid [8], with the functional (5.2) replacing the $I[\eta]$ of [8], and details will be omitted here.

It is to be remarked that the result of Theorem 5.2 *is true without the assumption of identical normality; indeed, the above proofs of statements* (a) *and* (b) *do not use this condition, and* (c) *may be established without this hypothesis by using methods that have been employed for the special systems arising as accessory systems for Bolza problems, (see Bliss* [1, §89]).

With Theorem 5.2 thus established, for the general system under consideration one may prove the results corresponding to Theorems 4.2 and 4.3 of Reid [9], and then proceed as in [9] to obtain the result of Theorem 5.1. The proofs of this section are distinctly variational in character, and are in essence "classical variational proofs phrased in terms of canonical variables." For example, for accessory systems of Bolza type variational problems the identity (5.3) is in essence the well-known Clebsch transformation of the second variation, (see Bliss [1, § 23, 39], and for such systems the fact that $\mathfrak{H}_+[c, d]$ implies $\mathfrak{H}_0[c, d]$ is the "Legendre" or "Clebsch" condition.

In passing, it is to be commented that for a system (2.1) satisfying (2.21) and identically normal on a compact interval $[c, d]$ one may obtain the full extension of Theorem 2.1 of Reid [8], as well as the corresponding criteria iv_R and v_R, (see [8, p. 741]), of that paper. In particular, if $U(x)$ and $V(x)$ are $n \times n$ matrices a.c. on $[c, d]$, and we set

$$\Lambda [U, V] = U^*(x) L_2[U, V] - V^*(x) L_1[U, V] ,$$

then $\Lambda [U, V] - (\Lambda [U, V])^* \equiv (U^* V - V^* U)'$; moreover, whenever $U(x)$ is non-singular on $[c, d]$ the matrix $W(x) = V(x) U^{-1}(x)$ is such that

$$\Lambda[U, V] = U^*K[W]U + U^*(W - W^*)L_1[U, V],$$
$$U^*V - V^*U = U^*(W - W^*)U.$$

Consequently, corresponding to the statement of [8, p. 741] on the condition v_R we have: *for an identically normal system* (2.1) *satisfying* (2.21) *on* $[c, d]$ *condition* $\mathfrak{H}_+[c, d]$ *holds if and only if* $\mathfrak{H}_0[c, d]$ *holds and there is an* $n \times n$ *hermitian a.c. matrix* $W(x)$ *such that a.e. on* $[c, d]$ *the hermitian matrix* $K[W]$ *is non-positive definite.*

THEOREM 5.3. *If* (2.1) *is an identically normal system satisfying* (2.21) *and* $\mathfrak{H}_0$, *and which is non-oscillatory on an interval* $X:(a, \infty)$, *then*:

(a) *If* $W_0(x)$ *is an hermitian solution of* (2.6) *on a subinterval* $[s, \infty)$ *of* X, *and* $W(x)$ *is the solution of* (2.6) *satisfying* $W(s) = W_0(s) + \Gamma$, *then* $W(x)$ *exists on* $[s, \infty)$ *if either* $\Gamma_{\mathfrak{I}}$ *is definite, or if there are real constants* $\lambda_0 > 0$, λ_1 *such that* $\lambda_0\Gamma_{\mathfrak{R}} + \lambda_1\Gamma_{\mathfrak{I}} \geqq 0$; *in particular, if* Γ *is an hermitian matrix satisfying* $\Gamma \geqq 0$ *then* $W(x) - W_0(x) \geqq 0$ *on* $[s, \infty)$.

(b) *If* $W_\infty(x)$ *is the distinguished solution of* (2.6) *at* ∞, *then* $W_\infty(x)$ *exists and is hermitian on* X; *moreover if* $s \in X$ *and* $W(x)$ *is a solution of* (2.6) *satisfying* $W(s) = W_\infty(s) + \Gamma$, *where* Γ *is an hermitian matrix that fails to be non-negative, then* $W(x)$ *does not exist throughout* $[s, \infty)$.

For a system (2.1) satisfying (2.21) it follows that $G(x, s \mid W_0) \equiv H^*(x, s \mid W_0)$ for an hermitian solution $W_0(x)$ of (2.6), and for such a system which is non-oscillatory and satisfies $\mathfrak{H}_0$ on X we have that $\Theta(x, s \mid W_0) > 0$ for $x \in X_s^+$. If $W(x)$ is a solution of (2.6) satisfying $W(s) = W_0(s) + \Gamma$, then Lemma 2.1 implies that $W(x)$ exists on $[s, \infty)$ if and only if $E + \Theta(x, s \mid W_0)\Gamma$ is non-singular on $[s, \infty)$, and this latter condition is equivalent to the non-singularity of $\Theta^{-1}(x, s \mid W_0) + \Gamma$ on (s, ∞). If $x \in [s, \infty)$ and $[\Theta^{-1}(x, s \mid W_0) + \Gamma]\eta = 0$, then

$$\eta^*[\Theta^{-1}(x, s \mid W_0) + \Gamma_{\mathfrak{R}}]\eta = -i\eta^*\Gamma_{\mathfrak{I}}\eta,$$

and hence

$$\eta^*[\Theta^{-1}(x, s \mid W_0) + \Gamma_{\mathfrak{R}}]\eta = 0, \quad \eta^*\Gamma_{\mathfrak{I}}\eta = 0. \tag{5.5}$$

Now if $\Gamma_{\mathfrak{I}}$ is definite the second condition of (5.5) implies $\eta = 0$; on the other hand, if $\lambda_0 > 0$, λ_1 are real constants such that $\lambda_0\Gamma_{\mathfrak{R}} + \lambda_1\Gamma_{\mathfrak{I}} \geqq 0$, then from (5.5) it follows that $\eta^*\Theta^{-1}(x, s \mid W_0)\eta = 0$ and hence $\eta = 0$. Thus $\Theta^{-1}(x, s \mid W_0) + \Gamma$ is non-singular on (s, ∞) and $W(x)$ exists on

$[s, \infty)$. In particular, if Γ is an hermitian matrix satisfying $\Gamma \geqq 0$ then this latter criterion implies that $W(x)$ exists on $[s, \infty)$, and the conclusion $W(x) - W_0(x) \geqq 0$ follows from the representation formula (2.11), and the fact that if matrices Θ, Γ are such that $\Gamma \geqq 0$, $\Theta > 0$, and $E + \Theta\Gamma$ is non-singular, then

$$\Gamma[E + \Theta\Gamma]^{-1} \equiv [E + \Theta\Gamma]^{*-1}[\Gamma + \Gamma\Theta\Gamma][E + \Theta\Gamma]^{-1} \geqq 0 .$$

In view of Theorems 4.2 and 5.1, if $(U_\infty(x); V_\infty(x))$ is a principal solution of (2.2) at ∞ then $U_\infty^* V_\infty - V_\infty^* U_\infty \equiv 0$ and $U_\infty(x)$ is non-singular on X: (a, ∞), so that the corresponding distinguished solution $W_\infty(x) = V_\infty(x)U_\infty^{-1}(x)$ of (2.6) is hermitian and exists on X. Consequently, if $s \in X$ then $\Theta(x, s \mid W_\infty) > 0$ for $x \in (s, \infty)$, and hence $\Theta^{-1}(x, s \mid W_\infty) + \Gamma$ is hermitian on (s, ∞) for Γ an hermitian matrix. Moreover, since $W_\infty(x)$ is the distinguished solution of (2.6) at ∞, $\Theta^{-1}(x, s \mid W_\infty) + \Gamma \to \Gamma$ as $x \to \infty$, while $\Theta^{-1}(x, s \mid W_\infty) + \Gamma$ is positive definite for $x > s$ and sufficiently close to s. Consequently, if Γ fails to be non-negative definite there exists a value $t \in (s, \infty)$ such that $\Theta^{-1}(t, s \mid W_\infty) + \Gamma$ is singular, so that $W_\infty(x)$ is not extensible to an interval containing t, in contradiction to the existence of $W_\infty(x)$ on X.

Combining the conclusions (a) and (b) we have that if the distinguished solution $W_\infty(x)$ of (2.6) exists on an interval (a, ∞) then an hermitian solution $W(x)$ of (2. 6) exists on a subinterval $[s, \infty)$ of (a, ∞) if and only if $W(x) - W_\infty(x) \geqq 0$ for at least one value, {and consequently all values}, on $[s, \infty)$. For the case of systems (2.1) with real coefficients satisfying (2.21), and for which $B(x) > 0$ on X, this result has been proved by Sandor [11]; due to this property he has designated as "the right-hand frontier solution" the solution of (2.6) that we have called the distinguished solution at ∞.

6. Systems non-oscillatory on intervals $(-\infty, a)$ and $(-\infty, \infty)$. The behavior of (2.2) and (2.6) on an interval $(-\infty, a)$ is obviously equivalent under the reflective transformations $U^0(x) = U(-x)$, $V^0(x) = V(-x)$, $W^0(x) = W(-x)$ to the behavior of the respective equations

$$(2.2^0) \quad U^{0\prime} = -A(-x)U^0 - B(-x)V^0 , \quad V^{0\prime} = -C(-x)U^0 + D(-x)V^0 ,$$

$$(2.6^0) \quad W^{0\prime} - W^0A(-x) - D(-x)W^0 - W^0B(-x)W^0 + C(-x) = 0 ,$$

on $(-a, \infty)$. A principal solution of (2.2) at $-\infty$, and the associated distinguished solution $W_{-\infty}(x)$ of (2.6) at $-\infty$, are defined as the images under the above transformations of a principal solution of (2.2^0) at ∞ and the associated distinguished solution $W_\infty^0(x)$ of (2.6^0) at ∞. The analogues of Theorems 4.2, 5.1 and 5.3 for intervals X: $(-\infty, a)$ are immediate, and will not be presented in any further detail, with the exception of the following results, which are consequences of combined results of these theorems for (2.2), (2.6) and (2.2^0), (2.6^0). For the systems considered by Sandor, results equivalent to those of Theorem 6.1 are given in [11, § 7].

THEOREM 6.1. *If on the real line* $(-\infty, \infty)$ *the system* (2.1) *is identically normal, satisfies* (2.21) *and* $\mathfrak{H}_0$, *and is non-oscillatory, then the distinguished solutions* $W_\infty(x)$ *and* $W_{-\infty}(x)$ *of* (2.6) *are individually hermitian on* $(-\infty, \infty)$ *and such that:*

(a) *If* $(U_t(x); V_t(x))$ *is the solution of* (2.2) *determined by* (4.2) *for* $-\infty < t < \infty$, *and* $W_t(x) = V_t(x)U_t^{-1}(x)$, *then* $W_t(x) \to W_\infty(x)$ *as* $t \to \infty$, *and* $W_t(x) \to W_{-\infty}(x)$ *as* $t \to -\infty$.

(b) *If* $W(x)$ *is an hermitian solution of* (2.6) *which exists on* $(-\infty, \infty)$ *then* $W(x) - W_\infty(x) \geqq 0$ *and* $W_{-\infty}(x) - W(x) \geqq 0$ *throughout* $(-\infty, \infty)$, *while if* $W(x)$ *is an hermitian solution of* (2.6) *for which at some value* s *the matrix* $W(s) - W_\infty(s)$, $\{W_{-\infty}(s) - W(s)\}$, *fails to be nonnegative definite then* $W(x)$ *does not exist throughout the interval* $[s, \infty)$, $\{(-\infty, s]\}$.

For example, the scalar system

$$u' = v\,, \quad v' = u \tag{6.1}$$

is non-oscillatory on $(-\infty, \infty)$, and $u_t(x) = \sinh(x - t)$, $v_t(x) = \cosh(x - t)$. The corresponding Riccati equation (2.6) is

$$w' + w^2 - 1 = 0\,, \tag{6.2}$$

with respective solutions $w_t(x) = \coth(x - t)$, $w_\infty(x) \equiv -1$, and $w_{-\infty}(x) \equiv 1$.

7. **Systems with constant coefficients.** If the coefficient matrices A, B, C, D are constant, and $(U(x); V(x))$ is a solution of (2.2), then $(U(x - c); V(x - c))$ is also a solution for arbitrary real values c. Consequently, (2.1) is non-oscillatory on an interval (a, ∞) or $(-\infty, a)$ if and only if it is non-oscillatory on the whole infinite line $(-\infty, \infty)$. Moreover, if $(U_t(x); V_t(x))$ is the solution of (2.2) satisfying (4.2) then $U_s(x) \equiv U_t(x - s + t)$, $V_s(x) \equiv V_t(x - s + t)$, and the corresponding solution $W_t(x) = V_t(x)U_t^{-1}(x)$ of (2.6) exists on an interval $[c, d]$ if and only if $W_s(s) = V_s(x)U_s^{-1}(x)$ exists on $[c + s - t,\ d + s - t]$. For systems with constant coefficients the following result is a consequence of Theorem 4.2.

THEOREM 7.1. *A system* (2.2) *with constant coefficients, and which is identically normal and non-oscillatory on* $(-\infty, \infty)$, *has a principal solution at* ∞ $\{$*at* $-\infty\}$ *if and only if the solution* $(U_0(x); V_0(x))$ *of* (2.2) *for which* $U_0(0) = 0$, $V_0(0) = E$ *is such that* $W_0(x) = V_0(x)U_0^{-1}(x)$

converges to a limit $W_\infty\{W_{-\infty}\}$ *as* $x \to -\infty$ $\{x \to \infty\}$; *the corresponding distinguished solution of* (2.6) *at* ∞ $\{at\ -\infty\}$ *is* $W_\infty(x) \equiv W_\infty$ $\{W_{-\infty}(x) \equiv W_{-\infty}\}$.

In turn, Theorems 5.1 and 6.1 imply the following results for systems with constant coefficients.

THEOREM 7.2. *A system* (2.2) *with constant coefficient matrices satisfying* $A^* = D$, $C^* = C$, $B^* = B \geqq 0$, *and which is identically normal, is non-oscillatory on* $(-\infty, \infty)$ *if and only if there exists an hermitian constant matrix* W *satisfying the algebraic matrix equation*

$$(7.1) \qquad WA + A^*W + WBW - C = 0;$$

moreover, if such a system is non-oscillatory on $(-\infty, \infty)$ *then there exist hermitian matrices* W_∞ *and* $W_{-\infty}$ *which are individually solutions of* (7.1), *and are extreme solutions for* (2.6) *in the sense that if* $W(x)$ *is any hermitian solution of* (2.6) *on* $(-\infty, \infty)$ *then* $W_\infty \leqq W(x) \leqq W_{-\infty}$; *in particular, if* W *is any hermitian solution of* (7.1) *then* $W_\infty \leqq W \leqq W_{-\infty}$.

In particular, if B and C are constant matrices the system

$$(7.2) \qquad u' = Bv\,, \quad v' = Cu\,,$$

is identically normal on $(-\infty, \infty)$ if and only if B is non-singular, and the following result is an immediate consequence of the above theorem.

COROLLARY. *If* B *and* C *are constant hermitian matrices with* $B > 0$, *then* (7.2) *is non-oscillatory on* $(-\infty, \infty)$ *if and only if* $C \geqq 0$, *and whenever this latter condition holds then*

$$W_\infty = -B^{-1/2}[B^{1/2}CB^{1/2}]^{1/2}B^{-1/2}$$

and $W_{-\infty} = -W_\infty$.

It is to be remarked that this corollary provides a differential equation algorism for the nonnegative definite square root of a given nonnegative definite matrix C:

$$C^{1/2} = \lim_{x\to\infty} V_0(x)U_0^{-1}(x) = -\lim_{x\to-\infty} V_0(x)U_0^{-1}(x)\,,$$

where $(U_0(x);\ V_0(x))$ *is the solution of* $U' = V$, $V' = CU$ *satisfying* $U(0) = 0$, $V(0) = E$.

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SOME THEOREMS ON PRIME IDEALS IN ALGEBRAIC NUMBER FIELDS

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Let K be an arbitrary algebraic number field. We denote by n the degree of K, by $\boldsymbol{f}$ an arbitrary ideal of K, by $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ prime ideals of K, by $\mu(\boldsymbol{a})$ the Moebius function of the ideal $\boldsymbol{a}$ of K, by $N\boldsymbol{a}$ the norm of $\boldsymbol{a}$, by $(\boldsymbol{a}, \boldsymbol{f})$ the greatest common divisor of $\boldsymbol{a}$ and $\boldsymbol{f}$, and by $h(\boldsymbol{f})$ the number of ideal classes H mod $\boldsymbol{f}$. It is known that

$$(1)\qquad \begin{aligned} &A(x, \boldsymbol{f}) := \sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} 1 = \gamma(\boldsymbol{f})x + R(x, \boldsymbol{f}),\ R(x, \boldsymbol{f}) = O(x^{1-1/n}), \\ &\gamma(\boldsymbol{f}) = \alpha \prod_{\boldsymbol{p} | \boldsymbol{f}} \left(1 - \frac{1}{N\boldsymbol{p}}\right) \qquad (\alpha = \alpha(K) > 0) . \end{aligned}$$

According to [1], the proof of the generalized Selberg formula for ideal classes H mod $\boldsymbol{f}$ in K:

$$(2)\qquad \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} + \sum_{\substack{N\boldsymbol{p}\boldsymbol{q} \leqq x \\ \boldsymbol{p}\boldsymbol{q} \in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x)$$

can be reduced to

$$(3)\qquad \sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} \frac{\mu(\boldsymbol{a})}{N\boldsymbol{a}} \log^2 \frac{x}{N\boldsymbol{a}} = \frac{2}{\gamma(\boldsymbol{f})} \log x + O(1) ,$$

and (3) is established directly in [1]. First, we generalize (3):

THEOREM 1. *Let $r > 1$ be a rational integer; then*

$$\sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} \frac{\mu(\boldsymbol{a})}{N\boldsymbol{a}} \log^r \frac{x}{N\boldsymbol{a}} = \frac{r}{\gamma(\boldsymbol{f})} \log^{r-1} x + \sum_{t=1}^{r-2} c_t(r, \boldsymbol{f}) \log^t x + O(1) ;$$

the constants $c_t(r, \boldsymbol{f})$ resp. the constant in $O(1)$ depends on $K, r, t, \boldsymbol{f}$ resp. $K, r, \boldsymbol{f}$ only.

The formula

$$\sum_{\boldsymbol{a} | \boldsymbol{f}} \mu(\boldsymbol{a}) = \begin{cases} 1 & \text{for } \boldsymbol{f} = 1 , \\ 0 & \text{for } \boldsymbol{f} \neq 1 \end{cases}$$

yields

LEMMA 1. *Let $f(x)$ be a complex valued function $(x \geqq 1)$; then*

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$$g(x) := \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} f\left(\frac{x}{N\mathfrak{a}}\right) \quad \textit{implies} \quad f(x) = \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \mu(\mathfrak{a})\, g\left(\frac{x}{N\mathfrak{a}}\right).$$

Using the Euler summation formula, we find

$$(4) \quad \sum_{m \leqq x} \frac{1}{m} \log^{r-1} m = \frac{1}{r} \log^r x + a_r + O\left(\frac{1}{x} \log^{r-1} x\right) \quad (r \text{ integer, } >1).$$

Because of

$$\sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} N\mathfrak{a} = \sum_{m \leqq x} (A(m, \mathfrak{f}) - A(m-1, \mathfrak{f})) \frac{1}{m} \log^{r-1} m,$$

(1) and (4) imply

$$(5) \qquad \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} N\mathfrak{a} = \frac{\gamma(\mathfrak{f})}{r} \log^r x + b_r(\mathfrak{f}) + O(x^{-1/n} \log^{r-1} x) \qquad (r > 1);$$

the constants $b_r(\mathfrak{f})$ depend on $K, r, \mathfrak{f}$ only. Because of

$$\begin{aligned} &\sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n} \log^{r-1} \frac{x}{N\mathfrak{a}} \\ &\quad = \sum_{m \leqq x} (A(m, \mathfrak{f}) - A(m-1, \mathfrak{f})) \left(\frac{x}{m}\right)^{1-1/n} \log^{r-1} \frac{x}{m}, \end{aligned}$$

(1) implies

$$(6) \qquad \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n} \log^{r-1} \frac{x}{N\mathfrak{a}} = O(x).$$

By the binomial theorem and

$$\sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} \frac{1}{s+1} = \frac{1}{r},$$

(5) yields

$$(7) \qquad \begin{aligned} &\sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} \frac{x}{N\mathfrak{a}} \\ &\quad = \frac{\gamma(\mathfrak{f})}{r} \log^r x + \sum_{s=0}^{r-1} d_s(r, \mathfrak{f}) \log^s x + O(x^{-1/n} \log^{r-1} x); \end{aligned}$$

the constants $d_s(r, \mathfrak{f})$ depend on $K, s, r, \mathfrak{f}$ only.

As shown in [1],

$$(8) \qquad \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} = O(1), \qquad \sum_{\substack{N\mathfrak{a} \leqq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log \frac{x}{N\mathfrak{a}} = O(1).$$

Proof of Theorem 1. By (3), Theorem 1 is correct for $r = 2$. Suppose $r > 2$ and

$$\sum_{\substack{N\mathfrak{a} \leq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^s \frac{x}{N\mathfrak{a}} = \sum_{t=1}^{s-1} c_t(s, \mathfrak{f}) \log^t x + O(1) \qquad (1 < s < r) . \tag{9}$$

In Lemma 1, let $f(x) := x \log^{r-1} x$; then

$$\begin{aligned} g(x) &= x \sum_{\substack{N\mathfrak{a} \leq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} \frac{x}{N\mathfrak{a}} \\ &= \frac{\gamma(\mathfrak{f})}{r} x \log^r x + x \sum_{s=1}^{r-1} d_s(r, \mathfrak{f}) \log^s x + O(x^{1-1/n} \log^{r-1} x) , \end{aligned} \tag{10}$$

by (7). Lemma 1, (10), (9), (6), and (8) imply

$$\begin{aligned} x \log^{r-1} x &= \sum_{\substack{N\mathfrak{a} \leq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \mu(\mathfrak{a}) \Big(\frac{\gamma(\mathfrak{f}) x}{r N\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} + \frac{x}{N\mathfrak{a}} \sum_{s=1}^{r-1} d_s(r, \mathfrak{f}) \log^s \frac{x}{N\mathfrak{a}} \\ &\qquad + O\Big(\Big(\frac{x}{N\mathfrak{a}}\Big)^{1-1/n} \log^{r-1} \frac{x}{N\mathfrak{a}} \Big) \Big) \\ &= \frac{\gamma(\mathfrak{f}) x}{r} \sum_{\substack{N\mathfrak{a} \leq x \\ (\mathfrak{a}, \mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} + \sum_{s=2}^{r-1} d_s(r, \mathfrak{f}) \sum_{t=1}^{s-1} c_t(s, \mathfrak{f}) \log^t x + O(x) ; \end{aligned}$$

let

$$c_t(r, \mathfrak{f}) := - \frac{r}{\gamma(\mathfrak{f})} \sum_{s=t+1}^{r-1} d_s(r, \mathfrak{f})\, c_t(s, \mathfrak{f}) \qquad (t = 1, 2, \cdots, r-2) .$$

This proves Theorem 1.

The fact that

$$c_{s-1}(s, \mathfrak{f}) = \frac{s}{\gamma(\mathfrak{f})}$$

was not used in the preceding proof.

Now we derive two consequences of (2). The well-known relation (Landau (1903))

$$T(x) := \sum_{N\mathfrak{p} \leq x} \log N\mathfrak{p} = O(x)$$

implies

$$T(x, H \bmod \mathfrak{f}) := \sum_{\substack{N\mathfrak{p} \leq x \\ \mathfrak{p} \in H \bmod \mathfrak{f}}} \log N\mathfrak{p} = O(x) \tag{11}$$

By

$$\sum_{\substack{N\mathfrak{p} \leq x \\ \mathfrak{p} \in H \bmod \mathfrak{f}}} \log^2 N\mathfrak{p} = \sum_{m \leq x} (T(m, H \bmod \mathfrak{f}) - T(m-1, H \bmod \mathfrak{f})) \log m ,$$

(11) gives

$$\sum_{\substack{Np\leq x\\ p\in H\bmod f}} \log^2 Np = T(x, H\bmod f)\log x + O(x)\,. \tag{12}$$

According to Landau (1903), we have

$$s(x) := \sum_{Np\leq x} \frac{\log Np}{Np} = \log x + O(1)\,. \tag{13}$$

Using

$$\sum_{N\quad\leq x} \frac{\log^2 Np}{Np} = \sum_{m\leq x} (S(m) - S(m-1))\log m\,,$$

(13) implies

$$\sum_{Np\leq x} \frac{\log^2 Np}{Np} = \frac{1}{2}\log^2 x + O(\log x)\,. \tag{14}$$

LEMMA 2. *We have*

$$\sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log^2 Np\log Nq = \frac{\log x}{2}\sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log Np\log Nq + O(x\log x)\,.$$

Proof. Denote by $H(q)\bmod f$ the class of all ideals a of K with $aq\in H\bmod f$; then (12), (13) and the definition of $T(x, H\bmod f)$ in (11) give

$$\begin{aligned}\sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log^2 Np\log Nq &= \sum_{\substack{Nq\leq x\\ (q,f)=1}} \log Nq\left(T\left(\frac{x}{Nq}, H(q)\bmod f\right)\log\frac{x}{Nq}\right.\\ &\qquad\left. + O\left(\frac{x}{Nq}\right)\right)\\ &= \sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log Nq\log Np\,(\log x - \log Nq)\\ &\qquad + O(x\log x)\,.\end{aligned}$$

This proves Lemma 2.

THEOREM 2. *We have*

$$\begin{aligned}&\log x\sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log Np\log Nq + 2\sum_{\substack{Npqr\leq x\\ pq\in r H\bmod f}} \log Np\log Nq\log Nr\\ &= \frac{2x}{h(f)}\log^2 x + O(x\log x)\end{aligned}$$

where the constant in the remainder term depends on K and f only.

Proof. We write (2) for x/Nr and $H(r)\bmod f$ instead of x and

$H \bmod \boldsymbol{f}$, multiply by $\log N\boldsymbol{r}$, and take summation over all prime ideals $\boldsymbol{r}$ with $(\boldsymbol{r}, \boldsymbol{f}) = 1$ and $N\boldsymbol{r} \leqq x$. By (13) and (14), we find

$$\sum_{\substack{N\boldsymbol{pr} \leqq x \\ \boldsymbol{pr} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \log N\boldsymbol{r} + \sum_{\substack{N\boldsymbol{pqr} \leqq x \\ \boldsymbol{pqr} \in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r}$$
$$= \frac{x}{h(\boldsymbol{f})} \log^2 x + O(x \log x) .$$

The application of Lemma 2 completes the proof.

THEOREM 3. *If*

$$\sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for the principal class $H_0 \bmod \boldsymbol{f}$, *then*

$$\frac{1}{x} \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for all $h(\boldsymbol{f})$ *classes* $H \bmod \boldsymbol{f}$.

Proof. Suppose

$$\sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_1 \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} = O(x)$$

for a certain ideal class $H_1 \bmod \boldsymbol{f}$. Then (2) implies

$$\sum_{\substack{N\boldsymbol{pq} \leqq x \\ \boldsymbol{pq} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x) , \tag{15}$$

and Theorem 2 gives

$$\sum_{\substack{N\boldsymbol{pqr} \leqq x \\ \boldsymbol{pqr} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} = O(x \log x) . \tag{16}$$

By (15) and (13), we get

$$\begin{aligned} \sum_{\substack{N\boldsymbol{pqr} \leqq x \\ \boldsymbol{pqr} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} &\geqq \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \sum_{\substack{N\boldsymbol{qr} \leqq x/N\boldsymbol{p} \\ \boldsymbol{qr} \in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{q} \log N\boldsymbol{r} \\ &= \sum_{\substack{N\boldsymbol{p} \leq x \\ \boldsymbol{p} \in H \bmod_0 \boldsymbol{f}}} \log N\boldsymbol{p} \left(\frac{2}{h(\boldsymbol{f})} \frac{x}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O\Big(\frac{x}{N\boldsymbol{p}}\Big)\right) \\ &= \frac{2x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O(x \log x) \\ &= \frac{x \log x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p} \leqq \sqrt{x} \\ \boldsymbol{p} \in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} + O(x \log x) . \end{aligned} \tag{17}$$

(17) and (16) imply the contradiction

$$\sum_{\substack{N\mathfrak{p} \leq \sqrt{x} \\ \mathfrak{p} \in H_0 \bmod \mathfrak{f}}} \frac{\log N\mathfrak{p}}{N\mathfrak{p}} = O(1) ,$$

and Theorem 3 is proved.

The special case of Theorem 3 for the rational number field was treated in [2].

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APPROXIMATION OF FUNCTIONS ON THE INTEGERS

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How can algorithms be used to analyze nonrecursive functions? This question motivates the present work.

Let us suppose that a particular function, with natural numbers as arguments and values, is known to be completely defined but not recursive. Then by Church's thesis,[1] no algorithm gives the functional value for every argument. In some practical situation, however, where a particular sequence of arguments is of interest, it might suffice to have an "approximating algorithm" that performs as follows when applied to the successive arguments in the sequence: for each argument, the algorithm computes a number; for some arguments, this number may differ from the actual functional value, but after sufficiently many arguments have been processed, the proportion of such cases never exceeds a prescribed real number less than unity. If such an approximating algorithm exists whenever the given sequence of arguments is infinite, nonrepeating and effectively generable, then the given function is in some (conceivably useful) sense susceptible to analysis by mechanical means. Functions of this last kind are the object of our investigation; when the above notions are made precise in § 1, they are called "recursively approximable" functions.

In § 2 it is shown that uncountably many nonrecursive functions are recursively approximable; in § 3, that uncountably many functions are not recursively approximable.[2]

1. A number-theoretic notion of approximation. Given any function f, any partial function φ,[3] and any sequence $x_0, x_1, \cdots$ of natural numbers, let "err (n)" denote the number of natural numbers $i < n$ such that $f(x_i) \neq \varphi(x_i)$. If E is a real number and, for all sufficiently large n, err $(n)/n \leqq E$, then we say that φ *approximates*

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[1] Cf [**3**].

[2] An analogous notion of approximable function, involving finite sets of arguments rather than sequences, is considered in [**5**], where a function is called "m-in-n-computable" if there is an algorithm that produces at least m correct functional values for every set of n arguments. It was shown that uncountably many functions are not m-in-n computable for any $m > 0$. The existence of nonrecursive m-in-n-computable functions with $m > 0$ was left an open question; an affirmative answer, however, was soon provided by Dana Scott in an unpublished communication.

[3] By "function" we mean, unless otherwise specified, "total singulary function" (in the sense of [**1**] p. xxi). A "partial function" is any singulary function whose domain is a subset of the natural numbers.

f with error E on the given sequence.

It may happen that, for every infinite nonrepeating recursive enumeration $x_0, x_1, \cdots,$[5] there is a partial recursive function φ that approximates f with error E on $x_0, x_1, \cdots.$[6] In this case we say that f is *recursively approximable with error E.* A function recursively approximable with some error <1 is called *recursively approximable.*

2. Recursively approximable functions. Are there recursively approximable functions other than the recursive functions? The Myhill-Friedberg notion of maximal set provides an affirmative answer through Theorem 2.1.[7] By Corollary 2.2 below, every recursion function f recursively approximates the uncountably many functions which agree with f on a maximal set.[8] In fact, we establish a stronger result as follows.

We consider an extension of the notion of maximal set. For convenience, a set C is called *cohesive* if it is infinite and, for every recursively enumerable set R, either $R \cap C$ or $\bar{R} \cap C$ is finite. A set is *quasi-maximal* if for some positive natural number m, its complement is the union of m cohesive sets. Thus the maximal sets are those quasi-maximal sets for which the number m can be taken as 1. Through Theorem 2.1, the notion of quasi-maximal set provides a sufficient condition that a function f be recursively approximable. This condition is that there exist a recursive function r that agrees on some quasi-maximal set with f.

THEOREM 2.1. *Let f be any function, r any recursive function, Q any quasi-maximal set such that f and r agree on Q. Then r recursively approximates f with arbitrary positive error on every infinite nonrepeating recursive enumeration.*

Proof. Assume that $\bar{Q} = C_1 \cup \cdots \cup C_m$ where the C's are cohesive sets. Let E be any positive real number and $x_0, x_1, \cdots$ any infinite nonrepeating recursive enumeration. Choose a natural number $p \geqq (m+1)/E$ and, for each natural number $j < p$, let $X_j = \{x_i \mid i \equiv j \bmod p\}$.

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4 In order to realize an approximating algorithm in the sense of the Introduction, it would be necessary to require that φ be defined for all x_i. It will be obvious, however, that the current results would be unaffected by this additional requirement.

5 A "recursive enumeration" is any sequence $x(0), x(1), \cdots$ where x is a recursive function.

6 Terminology regarding recursive functions and recursively enumerable sets is essentially that of [**3**]. However, "recursive" is used throughout for "general recursive," and the empty set is regarded as recursively enumerable.

7 A set $\boldsymbol{M}$ is maximal if (i) $\bar{M}$ is infinite and (ii) for every recursively enumerable set R, either $R \cap \bar{M}$ or $\bar{R} \cap \bar{M}$ is finite. The existence of recursively enumerable maximal sets is established in [**2**].

8 Functions r and f are said to agree on a set X if, for all $x \in X$, $r(x) = f(x)$.

For each k $(1 \leqq k \leqq m)$ we consider two cases.

Case 1. $\{x_0, x_1, \cdots\} \cap C_k$ is finite. Let q_k be the number of its members. Then the number of numbers $i < n$ such that $f(x_i) \neq r(x_i)$ and $x_i \in C_k$ is $\leqq q_k$.

Case 2. $\{x_0, x_1, \cdots\} \cap C_k$ is infinite. Then for some $j < p$, $X_j \cap C_k$ is infinite. Because C_k is cohesive and X_j is recursively enumerable, $\bar{X}_j \cap C_k$ is finite; let q_k be the number of its members. Now $\{x_i \mid f(x_i) \neq r(x_i) \text{ and } x_i \in C_k\} \subset C_k = (X_j \cap C_k) \cup (\bar{X}_j \cap C_k) \subset X_j \cup (\bar{X}_j \cap C_k)$. Therefore the number of numbers $i < n$ such that $f(x_i) \neq r(x_i)$ and $x_i \in C_k$ is $\leqq ((n-1)/p) + 1 + q_k$.

By hypothesis, if $f(x_i) \neq r(x_i)$ then $x_i \in C_1 \cup \cdots \cup C_m$. Hence $\mathrm{err}(n) \leqq m(n-1)/p + m + q_1 + \cdots + q_m$. Therefore

$$\mathrm{err}(n)/n \leqq m/p + (p(q_1 + \cdots + q_m) + mp - m)/(np)$$

and, for all $n \geqq p(q_1 + \cdots + q_m) + mp - m$, $\mathrm{err}(n)/n \leqq E$.

COROLLARY 2.2. *For every recursive function r, there are uncountably many functions f such that r recursively approximates f with arbitrary positive error on every infinite nonrepeating recursive enumeration.*

Proof. Given any recursive function r, choose any quasi-maximal set Q. For each subset S of $\bar{Q}$, let f_S be the function such that $f_S(x) = 1 \dot{-} r(x)$ if $x \in S$, $f_S(x) = r(x)$ otherwise. The functions f_S, being in one-to-one correspondence with the subsets of $\bar{Q}$, are uncountable.

For brevity, we will call a function "maximal" if it is not recursive and it agrees on some maximal set with some recursive function, "quasi-maximal" if it is not recursive and it agrees on some quasi-maximal set with some recursive function. In Theorem 2.1, the quasi-maximal functions were shown to be recursively approximable. By means of Theorems 2.5 and 2.6, we will show that there are uncountably many quasi-maximal functions, and consequently uncountably many recursively approximable functions, that are not maximal. For this purpose, let us define the *rank of a quasi-maximal set Q* to be the minimum number m such that $\bar{Q}$ is the union of m cohesive sets. Then define the *rank of a quasi-maximal function f* to be the minimum number m such that f agrees on some quasi-maximal set of rank m with some recursive function. Thus the maximal sets (functions) are the quasi-maximal sets (functions) of rank 1.

LEMMA 2.3. *If $C_1, \cdots C_m$ are cohesive sets, then every recursively enumerable subset of $C_1 \cup \cdots \cup C_m$ is finite.*

Proof. Assume that R is an infinite recursively enumerable subset of $C_1 \cup \cdots \cup C_m$. Then there is a recursive function r such that $r(0), r(1), \cdots$ enumerates R without repetition. Let $R_j = \{r(i) \mid i \equiv j \bmod m+1\}$ $(j = 0, \cdots, m)$. Then $R_0, \cdots, R_m$ are $m+1$ disjoint infinite recursively enumerable subsets of $C_1 \cup \cdots \cup C_m$. Hence at least two distinct R's, say R_j and R_k, have an infinite intersection with the same C_i. Since $R_k \subset \bar{R}_j$, it follows that $C_i \cap R_j$ and $C_i - R_j$ are infinite, contrary to the fact that C_i is cohesive.

LEMMA 2.4. *If Q and R are quasi-maximal sets and $R - Q$ is finite, then the rank of $Q \leqq$ rank of R.*

Proof. Let m be the rank of Q, n the rank of R. There are cohesive sets $D_1, \cdots, D_n$ such that

$$\bar{R} = D_1 \cup \cdots \cup D_n \,. \tag{2.1}$$

Then

$$\bar{Q} = (D_1 - Q) \cup \cdots \cup (D_n - Q) \cup (R - Q) \,. \tag{2.2}$$

Since $\bar{Q}$ is infinite and $R - Q$ is finite, at least one $D_i - Q$ is infinite. We may assume without loss of generality that the infinite sets $D_i - Q$ are $D_1 - Q, \cdots, D_h - Q$ where $1 \leqq h \leqq n$. Hence from (2.2)

$$\bar{Q} = ((D_1 - Q) \cup F) \cup \cdots \cup ((D_h - Q) \cup F) \tag{2.3}$$

where F is finite. For each i $(1 \leqq i \leqq h)$, $D_i - Q$, being an infinite subset of the cohesive set D_i, is obviously cohesive, hence $(D_i - Q) \cup F$ is cohesive. Since Q has rank m, it follows from (2.3) that $m \leqq h \leqq n$.

THEOREM 2.5. *For every natural number $m > 1$, there is a recursively enumerable quasi-maximal set of rank m. Hence there are infinitely many quasi-maximal sets that are not maximal.*

Proof. Choose a recursively enumerable maximal set Q_1 and let e be a recursive function such that $e(0), e(1), \cdots$ enumerates Q_1 without repetition. Define by induction on m the sets Q_m and C_m $(m = 1, 2, \cdots)$ thus.

$$C_1 = \bar{Q}_1; \text{ for all } m > 1, Q_m = e(Q_{m-1}) \text{ and } C_m = Q_{m-1} - Q_m \,.^{9} \tag{2.4}$$

Clearly, each Q_m is recursively enumerable. By induction on m we establish the following properties of the Q's and C's. For all $m \geqq 1$,

$$Q_{m+1} \subset Q_m \,; \tag{2.5}$$

[9] For any function f and set X, we denote the image of X under the mapping f by "$f(X)$."

(2.6) $$C_{m+1} = e(C_m)\ ;$$

(2.7) $$C_m \text{ is cohesive;}$$

(2.8) $$\bar{Q}_m = C_1 \cup \cdots \cup C_m\ .$$

Basis. Let $m = 1$. Now $Q_2 = e(Q_1)$ and $Q_1 = e(N)$ where N is the set of all natural numbers. $Q_1 \subset N$; therefore (2.5) holds. Next, note that e is a one-to-one mapping. Hence $C_2 = e(N) - e(Q_1) = e(N - Q_1) = e(C_1)$; i.e. (2.6) holds. Because Q_1 is maximal, C_1 is cohesive; i.e. (2.7) holds. By (2.4), (2.8) holds.

Induction step. Let $m > 1$. By (2.4), $Q_{m+1} = e(Q_m)$ and $Q_m = e(Q_{m-1})$. By induction hypothesis, $Q_m \subset Q_{m-1}$. Therefore (2.5) holds. By (2.4), $C_{m+1} = e(Q_{m-1}) - e(Q_m) = e(Q_{m-1} - Q_m) = e(C_m)$; i.e. (2.6) holds. By induction hypothesis C_{m-1} is cohesive, hence infinite. Then by (2.6) C_m is infinite. Let R be any recursively enumerable set. The set $\{x \mid e(x) \in R\}$ (call it R') is recursively enumerable. In view of (2.6) and the fact that e is one-to-one, $C_m \cap R = e(C_{m-1} \cap R')$ and $C_m - R = e(C_{m-1} - R')$. Suppose that $C_m \cap R$ is infinite. Then $C_{m-1} \cap R'$ must be infinite. Then, because C_{m-1} is cohesive, $C_{m-1} - R'$ is finite, and consequently $C_m - R$ is finite. Thus (2.7) holds. Finally, in view of (2.5), $\bar{Q}_m = \bar{Q}_{m-1} \cup (Q_{m-1} - Q_m) = \bar{Q}_{m-1} \cup C_m$. Hence by induction hypothesis (2.8) holds.

Having established (2.5)–(2.8) we now show that, for all $m > 1$, Q_m has rank m. By (2.8) and (2.7), Q_m ash rank $\leqq m$. Let $D_1, \cdots, D_n$ be any cohesive sets such that $\bar{Q}_m = D_1 \cup \cdots \cup D_n$. By (2.8) each C_i has an infinite intersection with at least one D_k. Moreover, if $1 \leqq i < j \leqq m$, C_i and C_j cannot both have an infinite intersection with the same D_k. If they did, then by (2.8) $C_i \cap D_k \subset \bar{Q}_i \cap D_k$ and, by (2.4) and (2.5), $C_j \cap D_k \subset Q_{j-1} \cap D_k \subset Q_i \cap D_k$; then $Q_i \cap D_k$ and $\bar{Q}_i \cap D_k$ would both be infinite, contrary to the fact that Q_i is recursively enumerable and D_k is cohesive. Thus for each i between 1 and m there must be a distinct k between 1 and n. Therefore $n \geqq m$. We conclude that Q_m has rank m.

THEOREM 2.6. *For every natural number $m > 1$, there are uncountably many quasi-maximal functions of rank m. Hence there are uncountably many quasi-maximal functions that are not maximal.*

Proof. By Theorem 2.5 there is a recursively enumerable quasi-maximal set Q of rank m. For each of the uncountably many subsets S of $\bar{Q}$ let f_S be the function such that

$$f_S(x) = \begin{cases} 0 \text{ if } x \in Q, \\ 1 \text{ if } x \in S, \\ 2 \text{ otherwise.} \end{cases}$$

There are uncountably many functions f_S since they are in one-to-one correspondence with the sets S. If f_S were recursive, then $\bar{Q}$ would be the infinite recursively enumerable set $\{x \mid f_S(x) \neq 0\}$, contrary to Lemma 2.3. Hence each f_S is nonrecursive. Therefore, since f_S agrees on Q with the constant function 0, f_S is a quasi-maximal function of rank $\leqq m$.

Moreover, consider any quasi-maximal set R and any recursive function r such that f_S agrees on R with r. Now Q and $\{x \mid r(x) \neq 0\}$ are recursively enumerable and $\{x \mid r(x) \neq 0\} \cap Q \subset \bar{R}$. Hence $\{x \mid r(x) \neq 0\} \cap Q$, being a recursively enumerable subset of $\bar{R}$, is finite by Lemma 2.3. Hence $\{x \mid r(x) \neq 0\} - Q$, which $= \{x \mid r(x) \neq 0\} - (\{x \mid r(x) \neq 0\} \cap Q)$, is a recursively enumerable subset of $\bar{Q}$. Hence by Lemma 2.3 $\{x \mid r(x) \neq 0\} - Q$, is finite. Hence $R - Q$, which $\subset \{x \mid r(x) \neq 0\} - Q$, is finite. Hence by Lemma 2.4 R has rank $\geqq m$. Therefore f_S has rank m.

3. Functions that are not recursively approximable.

It will now be shown that not every function is recursively approximable. That is to say, there are functions f with the following property: there is an infinite nonrepeating recursive enumeration $x_0, x_1, \cdots$ such that, for every real number $E < 1$ and every partial recursive function φ, φ does not approximate f with error E on $x_0, x_1, \cdots$.

Let us call a function f *constructively nonrecursive* if there is a recursive function g such that, for all natural numbers $e, f(g(e)) \neq \{e\}(g(e))$.[10] In view of Theorems 3.1 and 3.2, the constructively nonrecursive functions form an uncountably infinite subclass of the functions that are not recursively approximable.

THEOREM 3.1. *If a function is constructively nonrecursive, then it is not recursively approximable.*

Proof. Let f be any constructively nonrecursive function and g a recursive function such that, for all e,

$$f(g(e)) \neq \{e\}\,(g(e))\,. \tag{3.1}$$

First we will exhibit a recursive binary function c such that, for all i and e,

[10] For any $n \geqq 1$ and any $e, x_1, \cdots, x_n$, "$\{e\}\,(x_1, \cdots, x_n)$" denotes the ambiguous value $\varphi(x_1, \cdots, x_n)$ of the partial recursive n-ary function φ whose Gödel number is e. (Cf. [3], p. 340.)

$$c(i, e) > i \;, \tag{3.2}$$

$$f(c(i, e)) \neq \{e\}\,(c(i, e)) \;. \tag{3.3}$$

For this purpose, let ψ be the partial recursive quaternary function defined by

$$\psi(z, i, e, x) \simeq \begin{cases} (z)_x \text{ if } x \leqq i \;,^{11} \\ \{e\}\,(x) \text{ otherwise.} \end{cases} \tag{3.4}$$

Then there is a primitive recursive ternary function a such that

$$\{a(z, i, e)\}\,(x) \simeq \psi(z, i, e, x) \;.^{12} \tag{3.5}$$

Now for any natural numbers i and e, let z be the number $\Pi_{j \leqq i} p_j^{f(j)}$.[13] Because a and g are completely defined, $g(a(z, i, e))$ is defined. Hence either $g(a(z, i, e)) \leqq i$ or $g(a(z, i, e)) > i$. But $g(a(z, i, e))$ cannot be $\leqq i$, for in that case

$$\begin{aligned} \{a(z, i, e)\}\,(g(a(z, i, e))) &\simeq [\text{by } (3.5)]\; \psi\,(z, i, e, g(a(z, i, e,))) \\ &\simeq [\text{by } (3.4)]\; (z)_{g(a(z,i,e))} \simeq f(g(a(z, i, e))) \;, \end{aligned}$$

contrary to (3.1). Hence $g(a(z, i, e)) > i$. Therefore $\mu z(g(a(z, i, e)) > i)$ is a recursive function of i and e. It now follows that (3.6) and (3.7) define b and c as recursive binary functions.

$$b(i, e) = \mu z(g(a(z, i, e)) > i) \;, \tag{3.6}$$

$$c(i, e) = g(a(b(i, e), i, e)) \;. \tag{3.7}$$

By (3.6) and (3,7), (3.2) holds. Now for any natural numbers i and e, assume that $f(c(i, e)) = \{e\}\,(c(i, e))$. Then $\{e\}(c(i, e))$ is defined and $f(g(a(b(i, e), i, e))) =$ [by (3.7)] $f(c(i, e)) = \{e\}\,(c(i, e)) =$ [by (3.2) and (3.4)] $\psi\,(b(i, e), i, e, c(i, e)) =$ [by (3.5)] $\{a(b(i, e), i, e)\}\,(c(i, e)) =$ [by (3.7)] $\{a(b(i, e), i, e)\}\,(g(a(b(i, e), i, e)))$, contrary to (3.1). Therefore (3.3) follows by contradiction.

Next, define the primitive recursive functions d and e and the recursive function x thus.

$$d(i) = \mu j((j + 1)! > i) \;; \tag{3.8}$$

$$e(i) = d(i) \dot{-} (d(d(i)))! \;; \tag{3.9}$$

$$x(i) = \begin{cases} 0 \text{ if } i = 0 \;, \\ c(x(i \dot{-} 1), e(i)) \text{ otherwise.} \end{cases} \tag{3.10}$$

By (3.2), $x(0) < x(1) < \cdots$, so that $x(0), x(1), \cdots$ is an infinite nonrepeating recursive enumeration. We now show that, for any real

[11] For the notation $(z)_x$, cf. [**3**], p. 230.

[12] Cf. [**3**], § 65, Theorem XXIII.

[13] For the notation p_j, cf. [**3**], p. 230.

number $E < 1$ and any partial recursive function φ, φ does not approximate f with error E on $x(0), x(1), \cdots$, thereby proving that f is not recursively approximable. The proof is by contradiction. Thus, assume that φ approximates f with error E on $x(0), x(1), \cdots$. Then there is a natural number N such that

$$\text{(3.11)} \qquad \text{for all } n > N,\ \text{err}\,(n)/n \leqq E\,.$$

Choose any Gödel number t of φ and let k be a natural number $> \max(N, t, 1/(1-E))$. Then for the $(k! + t + 1)! - (k! + t)!$ natural numbers i such that $(k! + t)! \leqq i < (k! + t + 1)!$, $e(i) = t$; hence by (3.10) and (3.3) $f(x(i)) \neq \{t\}(x(i))$. Therefore $(k! + t + 1)! > N$ and $\text{err}\,((k!+t+1)!)/(k!+t+1)! \geqq 1-(k!+t)!/(k!+t+1)! = 1-1/(k!+t+1) > E$, contrary to (3.11).

For Theorem 3.2, we use the following notation from [4]. For any natural number e, "W_e" denotes the set of all numbers y such that, for some x, $\{e\}(x) = y$. A set P is *productive* if and only if there is a partial recursive function ψ such that, for all e, if $W_e \subset P$ then $\psi(e) \in P - W_e$.

THEOREM 3.2. *The representing function of any productive set is constructively nonrecursive. Hence uncountably many functions are not recursively approximable.*

Proof. Given any productive set P, let f be the function such that

$$\text{(3.12)} \qquad f(x) = \begin{cases} 0 \text{ if } x \in P\,, \\ 1 \text{ otherwise.} \end{cases}$$

Myhill has shown that there is a recursive function g such that, for all natural numbers e,

$$\text{(3.13)} \qquad g(e) \in (P - W_e) \cup (W_e - P)\,.$$[14]

Moreover there is a recursive function h such that, for all natural numbers e,

$$\text{(3.14)} \qquad W_{h(e)} = \{y \mid \{e\}(y) = 0\}\,.$$

(For example, we can take for h the primitive recursive function $\Lambda x\, \mu y(y \geqq x \& \{e\}(y) = 0)$. For the Λ-notation, cf. [3], § 65.) Now let e be any natural number. By (3.12) $f(g(h(e))) = 0$ if and only if $g(h(e)) \in P$; hence by (3.13) if and only if $g(h(e)) \notin W_{h(e)}$; hence by (3.14) if and only if $\{e\}(g(h(e))) \neq 0$. Thus $g(h)$ is a recursive function such that, for all $e, f(g(h(e))) \neq \{e\}(g(h(e)))$. Therefore f is constructively

[14] Cf. [4], § 3.153.

nonrecursive.

It now follows from Theorem 3.1 that the representing functions of productive sets are not recursively approximable. Moreover, by [4], p. 47, there are uncountably many productive sets. Hence uncountably many functions are not recursively approximable.

REMARK. The proof of Theorem 3.2 can readily be generalized to show that a function f is constructively nonrecursive if there is a recursively enumerable set A and a productive set P such that $f(x) \in A$ if and only if $x \in P$.

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COMBINATORIAL FUNCTIONS AND REGRESSIVE ISOLS

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1. Introduction. It is assumed that the reader is familiar with the notions: regressive function, regressive set, regressive isol, cosimple isol, combinatorial function and its canonical extension. The first four are defined in [2], the last two in [3]. Denote the set of all numbers (nonnegative integers) by ε, the collection of all isols by Λ, the collection of all regressive isols by Λ_R and the collection of all cosimple isols by Λ_1. The following four propositions will be used.

$$(1) \quad \begin{cases} \text{Let } \tau = \rho t \text{ and } \tau^* = \rho t^*, \text{ where } t_n \text{ and } t_n^* \text{ are regressive} \\ \text{functions. Then } \tau \cong \tau^* \Longleftrightarrow t_n \cong t_n^* . \end{cases}$$

$$(2) \qquad B \leqq A \ \& \ A \in \Lambda_R \Longrightarrow B \in \Lambda_R .$$

$$(3) \quad \begin{cases} \text{Let } F(T) \text{ be the canonical extension to } \Lambda \text{ of the recursive,} \\ \text{combinatorial function } f(n). \text{ Then } T \in \Lambda_R \Longrightarrow F(T) \in \Lambda_R . \end{cases}$$

$$(4) \qquad B \leqq A \ \& \ A \in \Lambda_1 \Longrightarrow B \in \Lambda_1 .$$

The first three are Propositions 3, 9(b) and Theorem 3(a) of [2] respectively. The fourth is Theorem 56(b) of [1].

DEFINITION. Let $f(n)$ be a one-to-one function from ε into ε and let $T \in \Lambda_R - \varepsilon$. Then

$$\phi_f(T) = \operatorname{Req} \rho t_{f(n)} ,$$

where t_n is any regressive function ranging over any set in T.

Using (1) it is readily seen that ϕ_f is a well defined function from $\Lambda_R - \varepsilon$ into $\Lambda - \varepsilon$. The main result of this paper is as follows: *Let $f(n)$ be a strictly increasing, recursive, combinatorial function; let $F(X)$ be its canonical extension to Λ, and let $T \in \Lambda_R - \varepsilon$; then $\phi_f(F(T)) = T$.*

2. The operation ϕ_f.

PROPOSITION 1. *Let $f(n)$ be a strictly increasing, recursive function and let $T \in \Lambda_R - \varepsilon$. Then*

$$\phi_f(T) \leqq T \quad \textit{and} \quad \phi_f(T) \in \Lambda_R .$$

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If in addition $T \in \Lambda_1$, *then* $\phi_f(T) \in \Lambda_R \cdot \Lambda_1$.

Proof. In view of (2) and (4), it suffices to show only that $\phi_f(T) \leqq T$. Let t_n be a regressive function such that $\rho t = \tau \in T$. Put $\alpha = \rho f$ and suppose $p(x)$ is a regressing function of t_n. Define

$$p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)] \quad \text{for } x \in \delta p .$$

Then $p^*(t_n) = n$ and

$$\rho t_f \subset \{x \in \delta p^* \mid p^*(x) \in \alpha\} ,$$
$$\tau - \rho t_f \subset \{x \in \delta p^* \mid p^*(x) \notin \alpha\} .$$

Since α is recursive it follows that ρt_f is separable from $\tau - \rho t_f$. Hence $\phi_f(T) \leqq T$.

It is known (by an unpublished result of Dekker) that Λ_R is neither closed under addition nor under multiplication. We do, however, have some closure properties for isols of the type $\phi_f(T)$, where $T \in \Lambda_R - \varepsilon$ and $f(n)$ is a strictly increasing, recursive function.

PROPOSITION 2. *Let* $f(n)$ *and* $g(n)$ *be strictly increasing, recursive function and let* $T \in \Lambda_R - \varepsilon$. *Then*

(a) $\phi_f(\phi_g(T)) \in \Lambda_R - \varepsilon$,
(b) $\phi_f(T) \cdot \phi_g(T) \in \Lambda_R - \varepsilon$,
(c) $\phi_f(T) + \phi_g(T) \in \Lambda_R - \varepsilon$.

Proof. In view of Proposition 1,

$$\phi_f(\phi_g(T)) \leqq \phi_g(T) \leqq T .$$

This implies (a). To verify (a) one could also observe that $\phi_f(\phi_g(T)) = \phi_{gf}(T)$. Combining $\phi_f(T) \leqq T$ and $\phi_g(T) \leqq T$, we obtain by [1, Cor. of Thm. 77]

$$\phi_f(T) \cdot \phi_g(T) \leqq T^2 .$$

However, $T^2 \in \Lambda_R - \varepsilon$ by (3). Hence (b) follows by (2). Finally, it is readily seen that

$$\phi_f(T) + \phi_g(T) \leqq \phi_f(T) \cdot \phi_g(T) ,$$

since $\phi_f(T)$ and $\phi_g(T)$ are $\geqq 2$ (in fact, infinite). Thus (c) follows from (2) and (b).

3. The main result. We first state and prove two lemmas which might be of interest for their own sake. Let $\rho_0, \rho_1, \cdots$ be the canonical enumeration of the class Q of all finite sets defined by

$$\rho_0 = o$$

$$\rho_{x+1} = \begin{cases} (y_1, \cdots, y_k) \text{ where } y_1, \cdots, y_k \text{ are the distinct numbers} \\ \text{such that } x+1 = 2^{y_1} + \cdots + 2^{y_k} . \end{cases}$$

We denote the cardinality of ρ_x by r_x.

LEMMA 1. *Let $f(n)$ be any combinatorial function and let C_i be the function from ε into ε such that $f(n) = \sum_{i=0}^{n} c_i\binom{n}{i}$. Then*

$$f(n) = \sum_{x=0}^{2^n-1} c_{r(x)} .$$

Proof. Since every n-element set has $\binom{n}{i}$ subsets of cardinality i, we have

$$(5) \qquad f(n) = \text{card}\,\{j(x, y) \mid \rho_x \subset (0, 1, \cdots, n-1) \;\&\; y < c_{r(x)}\} .$$

It follows from the definition of ρ_x that

$$\begin{aligned} \rho_x \subset (0, 1, \cdots, n-1) &\Longleftrightarrow x \leqq 2^0 + 2^1 + \cdots + 2^{n-1} \\ &\Longleftrightarrow x \leqq 2^n - 1 . \end{aligned}$$

Combining this with (5) we obtain

$$f(n) = \text{card}\,\{j(x, y) \mid x \leqq 2^n - 1 \;\&\; y < c_{r(x)}\} = \sum_{x=0}^{2^n-1} c_{r(x)} .$$

DEFINITION. Let $a(n)$ be a one-to-one function from ε into ε. Then

$$a'(n) = l_{n0}\cdot 2^{a(0)} + \cdots + l_{nn}\cdot 2^{a(n)} ,$$

where $l_{n0}, \cdots, l_{nn}$ is the sequence of zeros and ones such that

$$n = l_{n0}\cdot 2^0 + \cdots + l_{nn}\cdot 2^n .$$

LEMMA 2. (Dekker) *Let $a(n)$ be a one-to-one function from ε into ε with range α and let $A = \text{Req}\,(\alpha)$. Then $a'(n)$ is also a one-to-one function from ε into ε. Moreover,*

$$a'(2^n) = 2^{a(n)} , \quad \rho_{a'(n)} = a(\rho_n) \textit{ and } \rho a' \in 2^A .$$

Finally, if $a(n)$ is regressive, so is $a'(n)$.

Proof. It is clear that $a'(n)$ is a one-to-one function such that $a'(2^n) = 2^{a(n)}$. We have $\rho_{a'(0)} = \rho_0 = o$ while $a(\rho_0) = a(o) = o$; for $n \geqq 1$

$$\rho_n = \{i \mid 0 \leqq i \leqq n \;\&\; l_{ni} = 1\} .$$

Hence for every number n

$$\rho_{a'(n)} = \{a(i) \mid 0 \leqq i \leqq n \ \& \ l_{ni} = 1\}$$
$$= a\{i \mid 0 \leqq i \leqq n \ \& \ l_{ni} = 1\} = a(\rho_n) \ .$$

Thus, if n ranges over ε, ρ_n ranges over the class Q of all finite sets, $\rho_{a'(n)} = a(\rho_n)$ over the class of all finite subsets of α. We conclude that $\rho a' \in 2^{A}$. Finally, assume that $a(n)$ is a regressive function. Using the three facts that

$$a'(n+1) = l_{n+1,0} \cdot 2^{a(0)} + \cdots + l_{n+1,n+1} \cdot 2^{a(n+1)} \ ,$$
$$a'(n) = l_{n0} \cdot 2^{a(0)} + \cdots + l_{nn} \cdot 2^{a(n)} \ ,$$
$$\max\{i \mid l_{ni} = 1\} \leqq \max\{i \mid l_{n+1,i} = 1\} \ ,$$

we infer that $a'(n)$ is a regressive function.

THEOREM. *Let $f(n)$ be a strictly increasing, recursive combinatorial function, let $F(X)$ be its canonical extension to Λ and let $T \in \Lambda_R - \varepsilon$. Then $\phi_f(F(T)) = T$.*

Proof. Let $f(n) = \sum_{i=0}^{n} c_i \binom{n}{i}$ be the strictly increasing, recursive, combinatorial function. Then $c_1 > 0$ since $f(n)$ is strictly increasing, and c_i is a recursive function of i, since $f(n)$ is recursive. Let $\tau \in T \in \Lambda_R - \varepsilon$ and assume that t_n is a regressive function ranging over τ. Put $g(n) = t'(n)$. By Lemma 2 we have $\rho_{g(n)} = t(\rho_n)$; thus, if n assumes successively the values $0, 1, 2, 3, 4, 5, 6, 7, \cdots$, $\rho_{g(n)}$ assumes successively the "values"

$$o, (t_0), (t_1), (t_0, t_1), (t_2), (t_0, t_2), (t_1, t_2), (t_0, t_1, t_2), \cdots \ .$$

We have by definition

$$F(T) = \operatorname{Req}\{j(x, y) \mid \rho_x \subset \tau \ \& \ y < c_{r(x)}\} \ .$$

Since $g(n)$ ranges without repetitions over $\{n \mid \rho_n \subset \tau\}$, it follows that

$$(6) \qquad F(T) = \operatorname{Req}\{j(g(x), y) \mid y < c_{r(x)}\} \ .$$

We shall use w_n to denote the function which for $0, 1, \cdots$ takes on the values of the array

$$\begin{array}{l} j(g(0), 0), \cdots, j(g(0), c_{r(0)} - 1) \\ j(g(1), 0), \cdots, j(g(1), c_{r(1)} - 1) \\ j(g(2), 0), \cdots, j(g(2), c_{r(2)} - 1) \\ \quad \vdots \qquad\qquad\quad \vdots \end{array}$$

reading from the left to the right in each row and from the top row down; it is understood that every row which starts with $j(g(k), 0)$ for some k with $c_{r(k)} = 0$ is to be deleted. From the definitions of ρ_k and $r(k)$ we see that

$$k \in (2^0, 2^1, 2^2, \cdots) \Longrightarrow r(k) = 1 \Longrightarrow c_{r(k)} = c_1 > 0 \ .$$

The function $g(n) = t'(n)$ is regressive by Lemma 2. Taking into account that c_i is a recursive function, it readily follows that w_n is a regressive function. In view of (6) we have $\rho w_n \in F(T)$ it therefore suffices to prove that $\rho w_{f(n)} \in T$. By Lemma 1

$$f(n) = \sum_{x=0}^{2^n-1} c_{r(x)} \ ,$$

hence

$$f(0) = c_{r(0)} \ , \quad f(1) = c_{r(0)} + c_{r(1)} \ , \quad f(2) = c_{r(0)} + c_{r(1)} + c_{r(2)} + c_{r(3)}, \cdots$$

and

$$w_{f(0)} = j(g(1), 0) \ , \quad w_{f(1)} = j(g(2), 0), \cdots , \quad w_{f(n)} = j(g(2^n), 0), \cdots \ .$$

We conclude that $w_{f(n)} \cong g(2^n)$. However, by Lemma 2

$$g(2^n) = t'(2^n) \cong t(n) \ .$$

Thus $w_{f(n)} \cong t_n$ and $\rho w_{f(n)} \in T$. This completes the proof.

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ON LOCALLY MEROMORPHIC FUNCTIONS WITH SINGLE-VALUED MODULI

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1. A meromorphic function of bounded characteristic in a disk is the quotient of two bounded analytic functions. This classical theorem can be extended to open Riemann surfaces W as follows. Consider the class MB of meromorphic functions w of bounded characteristic on W, defined in terms of capacity functions on subregions. Let L be the class of harmonic functions on W, regular except for logarithmic singularities with integral coefficients. Then $w \in MB$ if and only if $\log |w|$ is the difference of two positive functions in L. We shall construct these functions directly on W, without making use of uniformization.

The proof offers no essential difficulties. If $\log |w|$ is regular at the singularity of the capacity functions, then the classical reasoning carries over almost verbatim. In the general case we introduce the extended class M_e of locally meromorphic functions e^{u+iu^*}, $u \in L$, with single-valued moduli. This class seems to offer some interest in its own right.

2. The class O_{M_eB} of Riemann surfaces not admitting nonconstant M_eB-functions coincides with the class O_G of parabolic surfaces. Regarding the subclass $MB \subset M_eB$ and the strict inclusion relations $O_{HB} < O_{MB} < O_{AB}$, we refer to the pioneering work on Lindelöfian maps by M. Heins [2, 3] and M. Parreau [4], and the doctoral dissertation of K. V. R. Rao [5].

§ 1. Definitions.

3. Let W be an arbitrary open Riemann surface. Given $\zeta \in W$ let Ω, $\zeta \in \Omega$, be a relatively compact subregion of W whose boundary β_Ω consists of a finite number of analytic Jordan curves. The Green's function on Ω with pole at ζ is denoted by $g_\Omega(z, \zeta)$. For $\Omega_0 \subset \Omega$ we have $g_{\Omega_0} \leqq g_\Omega$ in Ω_0 and $\lim_{\Omega \to W} g_\Omega(z, \zeta)$ either $\equiv \infty$ or else $=$ the Green's function $g(z, \zeta)$ of W. By definition, the class O_G of parabolic Riemann surfaces consists of those W on which no $g(z, \zeta)$ exists. An equivalent definition of O_G is that there are no nonconstant nonnegative superharmonic functions on W.

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4. The capacity function $p_\Omega(z, \zeta)$ on Ω with pole at ζ is defined as the harmonic function with singularity

$$p_\Omega(z, \zeta) - \log|z - \zeta| \to 0$$

as $z \to \zeta$ and such that

$$p_\Omega(z, \zeta) = k_\Omega = \text{const. on } \beta_\Omega .$$

It is known [1] that $k_{\Omega_0} \leqq k_\Omega$ and the limit $k_\beta = \lim k_\Omega$ is thus well-defined. A necessary and sufficient condition for $W \in O_G$ is $k_\beta = \infty$.

5. Let M be the class of meromorphic functions w on W. The proximity function of w is defined [7] as

$$m(\Omega, w) = m(\Omega, \infty) = \frac{1}{2\pi}\int_{\beta_\Omega} \overset{+}{\log}|w|\,dp_\Omega^* . \tag{1}$$

If β_h is the level line $p_\Omega = h$, $-\infty \leqq h \leqq k_\Omega$, and $n(h, \infty)$ signifies the number of poles of w in $\bar{\Omega}_h$: $p_\Omega \leqq h$, counted with multiplicities, then the counting function is defined as

$$\begin{aligned} N(\Omega, w) &= N(\Omega, \infty) \\ &= \int_{-\infty}^{k_\Omega} (n(h, \infty) - n(-\infty, \infty))dh + n(-\infty, \infty)k_\Omega . \end{aligned} \tag{2}$$

The characteristic function is, by definition,

$$T(\Omega) = T(\Omega, w) = m(\Omega, w) + N(\Omega, w) .$$

The function w has at ζ the Laurent expansion

$$w(z) = c_\lambda(z - \zeta)^\lambda + c_{\lambda+1}(z - \zeta)^{\lambda+1} + \cdots , \tag{3}$$

$c_\lambda \neq 0$, and the Jensen formula reads [7, 8]

$$T(\Omega, w) = T(\Omega, w^{-1}) + \log|c_\lambda| . \tag{4}$$

6. We shall need a class M_e more comprehensive than M. We introduce:

DEFINITIONS. *The class L consists of functions u on W, harmonic except for logarithmic singularities $\lambda_i \log|z - z_i|$ at z_i, $i = 1, 2, \cdots$, with integral coefficients λ_i. The subclass of nonnegative functions in L will be denoted by LP.*

The class M_e is defined to consist of (multiple-valued) functions of the form

$$w = e^{u+iu^*} , \qquad u \in L . \tag{5}$$

The conjugate function u^* has periods around z_i and along some cycles in W. Every branch of w is locally meromorphic, the branches differing by multiplicative constants c with $|c| = 1$. The modulus $|w|$ is single-valued throughout W.

The quantities $m(\Omega, w)$, $N(\Omega, w)$, $T(\Omega, w)$, and the Jensen formula carry over to M_e without modifications [7]. We further introduce:

DEFINITION. *The class MB(or M_eB) consists of functions w in M (or M_e) with bounded characteristics,*

$$T(\Omega) = O(1) . \tag{6}$$

Explicitly, one requires the existence of a bound $C < \infty$ independent of Ω such that $T(\Omega) < C$ for all $\Omega \subset W$. That (6) is independent of ζ will be a consequence of a decomposition theorem which we proceed to establish.

§ 2. The decomposition theorem.

7. We continue considering arbitrary open Riemann surfaces W.

THEOREM. *A necessary and sufficient condition for $w \in M_eB$ on W is that*

$$\log|w| = u - v , \tag{7}$$

where $u, v \in LP$.

The proof will be given in nos. 8–18. As a corollary we observe that $w \in MB$ on W if and only if (7) holds.

8. First we shall discuss in nos. 8–11 the case $w(\zeta) = 0$ or ∞.

Suppose $w \in M_eB$. We begin by showing that $W \notin O_G$. If $w(\zeta) = \infty$, then

$$T(\Omega) \geqq N(\Omega, w) \geqq n(-\infty, \infty)k_\Omega \geqq k_\Omega .$$

From $W \in O_G$ it would follow that $k_\Omega \to \infty$ as $\Omega \to W$ and consequently $T(\Omega) \to \infty$, a contradiction. We conclude that $W \notin O_G$. If $w(\zeta) = 0$, then in Jensen's formula

$$T(\Omega, w) = T\left(\Omega, \frac{1}{w}\right) + O(1)$$

we have

$$T\left(\Omega, \frac{1}{w}\right) \geqq N\left(\Omega, \frac{1}{w}\right) \geqq n(-\infty, 0)k_\Omega \geqq k_\Omega$$

and arrive at the same conclusion $W \notin O_G$.

On the other hand, if condition (7) is true, the existence of nonnegative superharmonic functions u, v implies $W \notin O_G$. Thus either condition of the theorem gives the hyperbolicity of W, and we may henceforth assume the existence of $g(z, \zeta)$ on W if $w(\zeta) = 0$ or ∞.

9. The functions

$$\varphi(z) = e^{\lambda(g(z\,\zeta)+ig^*(z\,\zeta))}\,, \tag{8}$$

$$w_1(z) = w(z)\varphi(z) \tag{9}$$

belong to M_e. We shall show:

LEMMA. *A necessary and sufficient condition for $w \in M_eB$ is that $w_1 \in M_eB$.*

Proof. By definition,

$$T(\Omega, \varphi) = N(\Omega, \varphi) + m(\Omega, \varphi)\,. \tag{10}$$

For $\lambda > 0$ we have trivially $N(\Omega, \varphi^{-1}) \equiv 0, m(\Omega, \varphi^{-1}) \equiv 0$, hence $T(\Omega, \varphi^{-1}) \equiv 0$, and it follows from Jensen's formula that $T(\Omega, \varphi) = O(1)$. If $\lambda < 0$, then $N(\Omega, \varphi) \equiv m(\Omega, \varphi) \equiv 0$, and $T(\Omega, \varphi) \equiv 0$, hence $T(\Omega, \varphi^{-1}) = O(1)$. In both cases

$$T(\Omega, \varphi) = O(1),\ T(\Omega, \varphi^{-1}) = O(1)\,. \tag{11}$$

The inequalities

$$T(\Omega, w) \leqq T(\Omega, w_1) + T(\Omega, \varphi^{-1}) = T(\Omega, w_1) + O(1)\,,$$
$$T(\Omega, w_1) \leqq T(\Omega, w) + T(\Omega, \varphi) = T(\Omega, w) + O(1)$$

yield

$$T(\Omega, w) = T(\Omega, w_1) + O(1) \tag{12}$$

and the lemma follows.

10. The following intermediate result can now be established:

LEMMA. *A necessary and sufficient condition for*

$$\log |w| = u - v \tag{13}$$

with $u, v \in LP$ is that

$$\log |w_1| = u_1 - v_1 \tag{14}$$

with $u_1, v_1 \in LP$.

Proof. We know that

$$\log|w_1| = \log|w| + \lambda g = \log|w| + (n_0 - n_\infty)g \ , \tag{15}$$

where n_0, n_∞ are the multiplicities of the zero or pole of $w(z)$ at ζ. If (13) is true, then

$$\log|w_1| = (u + n_0 g) - (v + n_\infty g) \tag{16}$$

and (14) follows. Conversely, (14) implies

$$\log|w| = (u_1 + n_\infty g) - (v_1 + n_0 g) \ . \tag{17}$$

This proves the lemma.

11. We conclude that Theorem 7 will be proved for w with $w(\zeta) = 0$ or ∞ if we establish it for w_1. Since $w_1(\zeta) \neq 0, \infty$, the proof for w_1 will also apply to w with this property. Explicitly, we are to show that $w_1 \in M_e B$ if and only if $\log|w_1| = u_1 - v_1$, $u_1, v_1 \in LP$.

12. Let $p_{\Omega z}$ be the capacity function in Ω with pole at z. For a harmonic function h on $\bar{\Omega}$ it is known [7] that

$$h(z) = \frac{1}{2\pi}\int_{\beta_\Omega} h \, dp^*_{\Omega z} \ . \tag{18}$$

Denote by a_μ, b_ν the zeros and poles of w in W. Those in $W - \zeta$ are the zeros and poles of w_1 in W. Suppose first there is no a_μ, b_ν on β_Ω. Then the function

$$h(z) = \log|w_1(z)| + \sum_{a_\mu \in \Omega - \zeta} g_\Omega(z, a_\mu) - \sum_{b_\nu \in \Omega - \zeta} g_\Omega(z, b_\nu) \tag{19}$$

is harmonic on $\bar{\Omega}$. Throughout this paper the zeros and poles are counted with their multiplicities. We set

$$x_\Omega(z, w_1) = \frac{1}{2\pi}\int_{\beta_\Omega} \overset{+}{\log}|w_1| \, dp^*_{\Omega z} \ , \tag{20}$$

$$y_\Omega(z, w_1) = \sum_{b_\nu \in \Omega - \zeta} g_\Omega(z, b_\nu) \ , \tag{21}$$

and

$$u_\Omega(z, w_1) = x_\Omega(z, w_1) + y_\Omega(z, w_1) \ . \tag{22}$$

Then

$$\log|w_1(z)| = u_\Omega(z, w_1) - u_\Omega(z, w_1^{-1}) \ . \tag{23}$$

Since all terms are continuous in a_μ, b_ν, the equation remains valid if there are zeros or poles of w on β_Ω.

We observe that

$$x_\Omega(\zeta, w_1) = m(\Omega, w_1) \,, \tag{24}$$

$$y_\Omega(\zeta, w_1) = N(\Omega, w_1) \,. \tag{25}$$

Here we shall only make use of the consequence

$$u_\Omega(\zeta, w_1) = T(\Omega, w_1) \,. \tag{26}$$

13. We next show:

LEMMA. *For* $\Omega_0 \subset \Omega$,

$$u_{\Omega_0}(z, w_1) \leqq u_\Omega(z, w_1) \,, \tag{27}$$

$$u_{\Omega_0}(z, w_1^{-1}) \leqq u_\Omega(z, w_1^{-1}) \,. \tag{27$'$}$$

Proof. By (23),

$$\overset{+}{\log} \mid w_1(z) \mid \leqq u_\Omega(z, w_1) \tag{28}$$

for every Ω. It follows that

$$\begin{aligned} x_{\Omega_0}(z, w_1) &\leqq \frac{1}{2\pi} \int_{\beta_{\Omega_0}} u_\Omega(t, w_1) dp^*_{\Omega_0 z} \\ &= \frac{1}{2\pi} \int_{\beta_{\Omega_0}} (u_\Omega(t, w_1) - y_{\Omega_0}(t, w_1)) dp^*_{\Omega_0 z} \\ &= u_\Omega(z, w_1) - y_{\Omega_0}(z, w_1) \,, \end{aligned}$$

because this difference is regular harmonic in Ω_0. We have reached statement (27),

$$x_{\Omega_0}(z, w_1) + y_{\Omega_0}(z, w_1) \leqq u_\Omega(z, w_1) \,,$$

and inequality (27)′ follows in the same fashion.

14. From (26) and (27) we infer that $T(\Omega, w_1)$ increases with Ω. We can set

$$T(W, w_1) = \lim_{\Omega \to W} T(\Omega, w_1) \tag{29}$$

and use alternatively the notations $T(\Omega) = 0(1)$ and $T(W) < \infty$.

15. The convergence of u_Ω can now be established:

LEMMA. *If* $T(W, w_1) < \infty$, *then the functions*

$$u(z, w_1) = \lim_{\Omega \to W} u_\Omega(z, w_1) \,, \tag{30}$$

(30) $$u(z, w_1^{-1}) = \lim_{\Omega\to W} u_\Omega(z, w_1^{-1})$$

are positive harmonic on W except for logarithmic poles of $u(z, w_1)$ at the $b_\nu \in W - \zeta$ and those of $u(z, w_1^{-1})$ at the $a_\mu \in W - \zeta$.

Proof. By Harnack's principle the limit in (30) is either identically infinite or else harmonic on $W - \{b_\nu\}$. That the latter alternative occurs is a consequence of

$$\lim_{\Omega\to W} u_\Omega(\zeta, w_1) = T(W, w_1) .$$

The statement for $u_\Omega(z, w_1^{-1})$ follows similarly from $u_\Omega(\zeta, w_1^{-1}) = T(\Omega, w_1^{-1}) = T(\Omega, w_1) + O(1)$.

16. On combining the lemma with (23) we see that $w_1 \in M_e B$ has the asserted representation

(31) $$\log |w_1(z)| = u(z, w_1) - u(z, w_1^{-1})$$

with the u-functions in LP. It remains to establish the converse.

17. Suppose

(32) $$\log |w_1(z)| = u_1(z) - v_1(z)$$

where $u_1, v_1 \in LP$. The positive logarithmic poles of $u_\Omega(z, w_1)$ are those of $\log |w_1(z)|$ in Ω, hence among those of $u_1(z)$. Consequently $u_1(z) - u_\Omega(z, w_1)$ is superharmonic in Ω and its minimum on $\bar{\Omega}$ is reached on β_Ω, where $u_1(z) - u_\Omega(z, w_1) = u_1(z) - \overset{+}{\log} |w_1(z)| \geqq 0$. One infers that $u_1(z) \geqq u_\Omega(z, w_1)$ in $\bar{\Omega}$. At ζ this means

(33) $$T(\Omega, w_1) = u_\Omega(\zeta, w_1) \leqq u_1(\zeta) .$$

If $u_1(\zeta) < \infty$, the proof is complete.

18. If $u_1(\zeta) = \infty$, then

(34) $$u_1(z) + \lambda_1 \log |z - \zeta|$$

is harmonic at ζ for some positive integer λ_1. We set

(35) $$w_2 = w_1 \cdot e^{-\lambda_1(g+ig^*)} \in M_e ,$$

where $g = g(z, \zeta)$, and obtain

(36) $$\log |w_2| = \log |w_1| - \lambda_1 g = (u_1 - \lambda_1 g) - v_1 .$$

The function $u_1 - \lambda_1 g_\Omega$ with $g_\Omega = g_\Omega(z, \zeta)$ is superharmonic on Ω, hence its minimum on $\bar{\Omega}$ is taken on β_Ω, where

(37) $$u_1 - \lambda_1 g_\Omega = u_1 \geqq 0 .$$

From $u_1 \geqq \lambda_1 g_\Omega$ on Ω it follows that

(38) $$u_1 - \lambda_1 g = \lim_{\Omega \to W} (u_1 - \lambda_1 g_\Omega) \geqq 0$$

on W. On setting

(39) $$u_2 = u_1 - \lambda_1 g,\ v_2 = v_1$$

one gets

(40) $$\log | w_2 | = u_2 - v_2$$

with $u_2, v_2 \in LP$.

The positive logarithmic poles of $u_\Omega(z, w_2)$ are those of $\log | w_2 |$ on Ω, hence among those of u_2. The minimum of the superharmonic function $u_2(z) - u_\Omega(z, w_2)$ on $\bar{\Omega}$ is taken on β_Ω, where it is

$$\min_{\beta_\Omega} (u_2 - \overset{+}{\log} | w_2 |) \geqq 0 .$$

One infers that

(41) $$T(\Omega, w_2) = u_\Omega(\zeta, w_2) \leqq u_2(\zeta) < \infty ,$$

that is, $T(\Omega, w_2) = O(1)$. The reasoning leading to (12) yields

(42) $$T(\Omega, w_1) = T(\Omega, w_2) + O(1) ,$$

and consequently $T(\Omega, w_1) = O(1)$.

We have shown that (32) implies $T(W, w_1) < \infty$. The proof of Theorem 7 is complete.

19. As an immediate consequence we see that the property $T(\Omega, w) = O(1)$ and thus the class $M_e B$ is independent of ζ.

§ 3. Extremal decompositions.

20. Consider an arbitrary $w \in M_e$. In contrast with no. 12 we now make no restrictive assumptions on $w(\zeta)$ and form

(43) $$x_\Omega(z, w) = \frac{1}{2\pi} \int_{\beta_\Omega} \overset{+}{\log} | w | \, dp^*_{\Omega z} ,$$

(44) $$y_\Omega(z, w) = \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) ,$$

(45) $$u_\Omega(z, w) = x_\Omega(z, w) + y_\Omega(z, w) .$$

It is seen as in no. 13 that u_Ω increases with Ω and that

$$u(z, w) = \lim_{\Omega \to W} u_\Omega(z, w) \tag{46}$$

is either identically infinite or else positive harmonic on W except for logarithmic poles b_ν. The same is true of

$$u(z, w^{-1}) = \lim_{\Omega \to W} u_\Omega(z, w^{-1}) \tag{47}$$

with singularities a_μ.

The functions (46) and (47) will now be shown to be extremal in all decompositions (7):

THEOREM. *If there is a decomposition*

$$\log | w(z) | = u_1(z) - u_2(z) \tag{48}$$

with $u_1, u_2 \in LP$, *then also*

$$\log | w(z) | = u(z, w) - u(z, w^{-1}) \tag{49}$$

and

$$\begin{aligned} u(z, w) &\leqq u_1(z) \\ u(z, w^{-1}) &\leqq u_2(z) \ . \end{aligned} \tag{50}$$

Proof. One observes that the positive logarithmic poles of $u_\Omega(z, w)$ are those of $\log | w(z) |$ in Ω, hence among those of $u_1(z)$ in Ω. The superharmonic function $u_1(z) - u_\Omega(z, w)$ in Ω dominates

$$\min_{\beta\Omega} (u_1(z) - \overset{+}{\log} | w(z) |) \geqq 0$$

and we find that $u_1(z) - u(z, w) = \lim_{\Omega \to W}(u_1(z) - u_\Omega(z, w)) \geqq 0$ in W. Similarly, the superharmonic function $u_2(z) - u_\Omega(z, w^{-1}) \geqq 0$ on Ω, and $u_2(z) \geqq u(z, w^{-1})$ on W. By virtue of Harnack's principle, equality (49) then follows on letting $\Omega \to W$ in

$$\log | w(z) | = u_\Omega(z, w) - u_\Omega(z, w^{-1}) \ . \tag{51}$$

21. The extremal functions $u(z, w), u(z, w^{-1})$ can in turn be decomposed:

THEOREM. *A function* w *on* W *belongs to* $M_e B$ *if and only if*

$$\log | w | = (x(z, w) + y(z, w)) - (x(z, w^{-1}) + y(z, w^{-1})) \ , \tag{52}$$

where the functions $x \geqq 0$ *are regular harmonic and the functions* $y \geqq 0$ *have the representations*

$$(53)\qquad \begin{aligned} y(z, w) &= \Sigma\, g(z, b_\nu) \\ y(z, w^{-1}) &= \Sigma\, g(z, a_\mu)\ . \end{aligned}$$

Here the sums are extended over all poles b_ν and all zeros a_μ of w on W respectively, each counted with its multiplicity.

22. Suppose indeed that $w \in M_e B$. It is evident from the maximum principle that

$$(54)\qquad y_{\Omega_0}(z, w) \leqq y_\Omega(z, w)$$

for $\Omega_0 \subset \Omega$. We know that

$$(55)\qquad \log |\, w \,| = u_1 - u_2\ ,$$

$u_1, u_2 \in LP$, and the superharmonic function $u_1(z) - y_\Omega(z, w)$ on Ω cannot exceed $\min_{\beta_\Omega} u_1 \geqq 0$. Hence $y_\Omega(z, w) \leqq u_1(z)$ on Ω and, by Harnack's principle,

$$(56)\qquad y(z, w) = \lim_{\Omega \to W} y_\Omega(z, w)$$

is positive harmonic on W except for logarithmic poles b_ν. Analogous reasoning shows that

$$(57)\qquad y(z, w^{-1}) = \lim_{\Omega \to W} y_\Omega(z, w^{-1})$$

is positive harmonic on $W - \{a_\mu\}$.

23. To prove (53) we must show that

$$(58)\qquad \lim_{\Omega \to W} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) = \sum_{b_\nu \in W} g(z, b_\nu)$$

and similarly for $\Sigma\, g(z, a_\mu)$. First,

$$(59)\qquad \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) \leqq \sum_{b_\nu \in \Omega} g(z, b_\nu) \leqq \sum_{b_\nu \in W} g(z, b_\nu)\ ,$$

and we have

$$(60)\qquad \overline{\lim_{\Omega \to W}} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu) \leqq \sum_{b_\nu \in W} g(z, b_\nu)\ .$$

Second, for $\Omega_0 \subset \Omega$,

$$(61)\qquad \sum_{b_\nu \in \Omega_0} g(z, b_\nu) = \lim_{\Omega \to W} \sum_{b_\nu \in \Omega_0} g_\Omega(z, b_\nu) \leqq \underline{\lim}_{\Omega \to W} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu)$$

and a fortiori

$$(62)\qquad \sum_{b_\nu \in W} g(z, b_\nu) = \lim_{\Omega_0 \to W} \sum_{b_\nu \in \Omega_0} g(z, b_\nu) \leqq \underline{\lim}_{\Omega \to W} \sum_{b_\nu \in \Omega} g_\Omega(z, b_\nu)\ .$$

Statement (58) follows.

24. The convergence of $x_\Omega(z, w)$ is obtained at once from

$$x_\Omega(z, w) = u_\Omega(z, w) - y_\Omega(z, w) , \tag{63}$$

and the limiting function is

$$x(z, w) = u(z, w) - y(z, w) . \tag{64}$$

The limit $x(z, w^{-1})$ of $x_\Omega(z, w^{-1})$ is obtained in the same way. Both limits are obviously positive and regular harmonic on W.

Necessity of (52) for $w \in M_e B$ has thus been established. Sufficiency is a corollary of the main Theorem 7.

§ 4. Consequences.

25. If only the x-terms in (52) are considered, the following corollary of Theorem 21 is obtained:

THEOREM. *If $w \in M_e B$ on W, then*

$$\lim_{\Omega \to W} \int_{\beta\Omega} |\log|w|| \, dp_\Omega^* < \infty \tag{65}$$

for any ζ.

Here p_Ω signifies, as before, the capaity function on Ω with pole at ζ. For the proof we have

$$\begin{aligned} \int_{\beta\Omega} |\log|w|| \, dp_\Omega^* &= \int_{\beta\Omega} \overset{+}{\log}|w| \, dp_\Omega^* + \int_{\beta\Omega} \overset{+}{\log}\left|\frac{1}{w}\right| dp_\Omega^* \\ &= 2\pi(x_\Omega(\zeta, w) + x_\Omega(\zeta, w^{-1})) , \end{aligned} \tag{66}$$

and this quantity tends to

$$2\pi(x(\zeta, w) + x(\zeta, w^{-1})) < \infty . \tag{67}$$

The limit (65) thus exists.

26. A consideration of the y-terms in (52) gives:

THEOREM. *Suppose $w \in M_e B$. Then the sum $\Sigma g(z, z_i)$, with z_i ranging over all poles and zeros of w, is harmonic on $W - \{a_\mu\} - \{b_\nu\}$.*

In fact,

$$\sum_{z_i \in W} g(z, z_i) = \lim_{\Omega \to W} \sum_{z_i \in \Omega} g(z, z_i) \tag{68}$$
$$= \lim_{\Omega \to W} \left(\sum_{a_\mu \in \Omega} g(z, a_\mu) + \sum_{b_\nu \in \Omega} g(z, b_\nu)\right)$$
$$= \sum_{a_\mu \in W} g(z, a_\mu) + \sum_{b_\nu \in W} g(z, b_\nu) .$$

27. For a sufficient condition the first terms of both x- and y-parts in (52) must be taken into account:

THEOREM. *If for some* $\zeta \in W$

$$\int_{\beta_\Omega} \overset{+}{\log} | w | \, dp_\Omega^* = O(1) \tag{69}$$

and

$$\sum_{b_\nu \in W} g(z, b_\nu) < \infty \quad in \quad W - \{b_\nu\} , \tag{70}$$

then $w \in M_e B$ *and hence*

$$\lim_{\Omega \to W} \int_{\beta_\Omega} | \log | w || \, dp_\Omega^* < \infty \tag{71}$$

and

$$\sum_{a_\mu \in W} g(z, a_\mu) < \infty \quad on \quad W - \{a_\mu\} \tag{72}$$

as well.

Indeed, the characteristic

$$T(\Omega) = u_\Omega(\zeta, w) = x_\Omega(\zeta, w) + y_\Omega(\zeta, w)$$
$$= \frac{1}{2\pi} \int_{\beta_\Omega} \overset{+}{\log} | w | \, dp_\Omega^* + \sum_{b_\nu \in \Omega} g_\Omega(\zeta, b_\nu)$$

is $O(1)$ if (69), (70) hold. Properties (71), (72) then follow from $w \in M_e B$.

Another sufficient condition for $w \in M_e B$ is, of course, that $\int_{\beta_\Omega} \overset{+}{\log} | w^{-1} | \, dp_\Omega$ is bounded and $\Sigma g(\zeta, a_\mu) < \infty$ in $W - \{a_\mu\}$.

28. For "entire" functions in $M_e B$ the conditions simplify. Let $E_e B$ be the class of such functions, characterized by $w(z) \neq \infty$ on W.

THEOREM. *A necessary and sufficient condition for* $w \in E_e B$ *on* W *is that*

$$\int_{\beta_\Omega} \overset{+}{\log} | w | \, dp_\Omega = O(1) . \tag{73}$$

The proof is evident.

29. Consider the class H of regular harmonic functions h on W and let HP be the subclass of nonnegative functions. Set $\overset{+}{h} = \max(0, h)$.

THEOREM. *A harmonic function h on W has a decomposition*

$$h = u_1 - u_2\,, \qquad u_1, u_2 \in HP \tag{74}$$

if and only if, for some ζ,

$$\int_{\beta_\Omega} \overset{+}{h}\, dp_\Omega^* = O(1)\,, \tag{75}$$

or, equivalently,

$$\lim_{\Omega \to W} \int_{\beta_\Omega} |h|\, dp_\Omega^* < \infty\,. \tag{76}$$

Proof. The multiple-valued function $w = e^{h+ih^*}$ is in M_e, and $w \neq 0, \infty$ on W. If (74) is given, then $\log|w| = u_1 - u_2$ and $w \in M_eB$. This implies

$$\lim_{\Omega \to W} \int_{\beta_\Omega} |\log|w||\, dp_\Omega^* = \lim_{\Omega \to W} \int_{\beta_\Omega} |h|\, dp_\Omega^* < \infty$$

and consequently $\int_{\beta_\Omega} \overset{+}{h}\, dp_\Omega^* = O(1)$. Conversely, suppose the latter condition holds,

$$\int_{\beta_\Omega} \overset{+}{\log}\,|w|\, dp_\Omega^* = O(1)\,.$$

Then $w \in M_eB$ and

$$h = \log|w| = x(z, w) - x(z, w^{-1})\,,$$

the y-terms vanishing because of the absence of zeros and poles of w.

It is known that functions u harmonic in the interior W of a compact bordered Riemann surface and with property (76) have a Poisson-Stieltjes representation (e.g., Rodin [6]). For further interesting results see Rao [5].

30. It is clear that theorems on $\log|w|$ can also be expressed directly in terms of $|w|$. Theorem 7, e.g., takes the following form:

THEOREM. *$w \in M_eB$ if and only if*

$$|w| = \left| \frac{\eta(z, w)}{\eta(z, w^{-1})} \right| , \tag{77}$$

where $\eta \in M_eB$ *and* $|\eta| < 1$ *on* W.

Proof. Suppose $w \in M_eB$, hence

$$\log |w| = u(z, w) - u(z, w^{-1}) , \tag{78}$$

$u \in LP$. Set

$$\eta(z, w) = \exp[-u(z, w^{-1}) - iu(z, w^{-1})^*] , \tag{79}$$

and (77) follows. Conversely, if (77) is given, then

$$\log |w| = \log |\eta(z, w)| - \log |\eta(z, w^{-1})| \tag{80}$$

is a difference of two functions in LP, and we have $w \in M_eB$.

31. The counterpart of Theorem 21 is as follows:

THEOREM. $w \in M_eB$ *if and only if*

$$|w| = \left| \frac{\varphi(z, w)\psi(z, w)}{\varphi(z, w^{-1})\psi(z, w^{-1})} \right| , \tag{81}$$

where $\varphi, \psi \in M_eB$ *and* $\varphi \neq 0$ *on* W, $|\varphi| < 1, |\psi| < 1$.

If $w \in M_eB$, choose

$$\begin{aligned} \varphi(z, w) &= \exp[-x(z, w^{-1}) - ix(z, w^{-1})^*] , \\ \psi(z, w) &= \exp[-y(z, w^{-1}) - iy(z, w^{-1})^*] , \end{aligned} \tag{82}$$

and we have (81). Conversely, (81) gives $\log |w| = u_1 - u_2$ with u_1, $u_2 \in LP$, hence $w \in M_eB$.

32. We introduce the classes O_{MB} and O_{M_eB} of Riemann surfaces on which there are no nonconstant functions in MB and M_eB respectively. Similarly, let O_{EB} and O_{E_eB} be the subclasses determined by entire functions $w(z) \neq \infty$ on W in MB and M_eB. The problem here is to arrange these four classes in the general classification scheme of Riemann surfaces [1].

The inclusion relations

$$\begin{aligned} O_{M_eB} \subset O_{MB} \subset O_{EB} , \\ O_{M_eB} \subset O_{E_eB} \subset O_{EB} \end{aligned} \tag{83}$$

are immediately verified.

33. The smallest class in (83) is easily identified:

THEOREM. *All functions in* M_eB *on* W *reduce to constants if and only if* W *is parabolic,*

$$O_G = O_{M_eB} \,. \tag{84}$$

Proof. If $W \notin O_G$, there is a Green's function $g(z, \zeta)$, and

$$w = e^{-g-ig^*} \in M_eB \,. \tag{85}$$

In fact, g is bounded above in any $W - \Omega$, hence $m(\Omega, w) = O(1)$, and $N(\Omega, w) = 0$ gives $T(\Omega) = O(1)$. Conversely, if there is a nonconstant $w \in M_eB$ on W, then $\log |w| = u_1 - u_2$ where at least one $u_i \in LP$ is nonconstant superharmonic. This means that $W \notin O_G$. The same proof gives $O_G = O_{E_eB}$.

34. By the preceding theorem, every M_e-function on a parabolic W has unbounded characteristic. Even more can be said of M-functions on the larger class O_{MB} by comparing $T(\Omega)$ with k_Ω (no. 4):

THEOREM. *On* $W \in O_{MB}$, *the characteristic* $T(\Omega)$ *of any* $w \in M$ *tends so rapidly to infinity that*

$$\varliminf_{\Omega \to W} \frac{T(\Omega)}{k_\Omega} \geqq 1 \,. \tag{86}$$

Proof. Let $w(\zeta) = a$. The counting function of w for a is, by denfinition,

$$N(\Omega, a) = \int_{-\infty}^{k_\Omega} (n(h, a) - n(-\infty, a))dh + n(-\infty, a)k_\Omega \,,$$

where $n(h, a)$ is the number of a-points of w in the set $\bar{\Omega}_h$: $p_\Omega \leqq h \leqq k_\Omega$. We obtain from the first fundamental theorem [7] that

$$T(\Omega) + O(1) \geqq N(\Omega, a) \geqq n(-\infty, a)k_\Omega \,, \tag{87}$$

and (86) follows.

Thus (86) is obviously a property of every $w \in M, w \notin MB$, on every W.

35. We also observe:

THEOREM. *A function* $w \in M$ *on* $W \in O_{MB}$ *cannot omit a set of values of positive capacity.*

More accurately, the counting function $N(\Omega, a)$ of $w \in M$ on O_{MB} is unbounded on any set E of positive capacity. To see this we distribute mass $d\mu(a) > 0$ at $a \in E$, with $\int_E d\mu = 1$, and integrate Jensen's formula

$$\log|w(\zeta) - a| = \frac{1}{2\pi}\int_{\beta_\Omega} \log|w - a|\, dp_\Omega^* + N(\Omega, \infty) - N(\Omega, a) \tag{88}$$

$(w(\zeta) \neq \infty)$ over E with respect to $d\mu(a)$. We obtain Frostman's formula on W:

$$N(\Omega, \infty) - \frac{1}{2\pi}\int_{\beta_\Omega} u(w) dp_\Omega^* = \int_E N(\Omega, a) d\mu(a) - u(w(\zeta)) \,, \tag{89}$$

where $u(w) = \int_E \log|w - a|^{-1} d\mu(a)$. For equilibrium distribution $d\mu$ it is known from the classical theory that $u(w) = -\overset{+}{\log}|w| + O(1)$, and a fortiori $\int_{\beta_\Omega} u(w) dp_\Omega^* = -2\pi\, m(\Omega, \infty) + O(1)$, where $O(1)$ depends on E only. Substitution into (89) gives

$$T(\Omega) = \int_E N(\Omega, a) d\mu(a) + O(1). \tag{90}$$

This proves our assertion.

36. A comprehensive study of the role played by O_{MB} in the classification theory of Riemann surfaces is contained in the doctoral dissertation of K. V. R. Rao [5].

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SEMIGROUPS AND THEIR SUBSEMIGROUP LATTICES

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1. Introduction. Let S be a semigroup of order at least 2, and $L(S)$ be the system of all subsemigroups of S. Generally $L(S)$, including the empty subset, is a lattice with respect to inclusion. $L(S)$ is called the subsemigroup lattice of S. A semigroup S contains at least one nonempty subsemigroup besides S itself. In the previous paper [4], as the first step towards the investigation of the structure of S with a given type of $L(S)$, we determined all the Γ-semigroups,[1] namely, the semigroups S's in which $L(S)$'s are chains. In the present paper we shall define Γ^*-semigroups as generalization of Γ-semigroups and shall obtain all the types of Γ^*-semigroups except for infinite simple Γ^*-groups.

Since all the semigroups of order 2 are Γ^*-semigroups, we shall treat non-trivial Γ^*-semigroups, namely, those of order $\geqq 3$ in the discussion below. First, in § 2 we shall prove that Γ^*-semigroups of order $\geqq 3$ are unipotent, i.e., having a unique idempotent, and that they are periodic; and hence a Γ^*-semigroup is determined by a group and a Z-semigroup, i.e., a unipotent semigroup with zero. Accordingly, in § 3 we shall determine all the types of Γ^*-Z-semigroups which will have to be of order <5; in § 4 we shall treat solvable Γ^*-groups and prove that finite Γ^*-groups or non-simple Γ^*-groups are solvable; finally in § 5, unipotent Γ^*-semigroups which are neither groups nor Z-semigroups will be discussed. It is interesting that there are no infinite unipotent Γ^*-semigroups except groups.

For convenience, the results from the paper [4] are stated as follows:

LEMMA 1.1. *A semigroup is a Γ-semigroup if and only if it has one of the following types.*[2] *Except for* (1.3) *they are all cyclic semigroups, i.e., semigroups generated by an element d. We show defining relations below.*

(1.1) *Z-semigroups*:

(1.1.1) $\qquad d^2 = d^3 \qquad$ (*order* 2)

(1.1.2) $\qquad d^3 = d^4 \qquad$ (*order* 3)

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[1] The author called them Γ-monoids in [4].

[2] As the trivial case, a semigroup of order 1 is also regarded as a Γ-semigroup. This remark will be needed for the definition of a Γ^*-semigroup.

(1.2) *Cyclic groups* $G(p^m)$ *of a prime power order*: $d = d^{p^m+1}$
(1.3) *Quasicyclic groups* [1]: $G(p^\infty)$, *i.e.*,

$$G(p^\infty) = \sum_{k=1}^{\infty} G(p^k)$$

where $Q(p) \subset G(p^2) \subset \cdots \subset G(p^k) \subset \cdots$, *p being a prime.*

(1.4) *Unipotent semigroups of order n, the kernel (the least ideal) of which is a group* $G(p^m)$:

(1.4.1)	*if* $p = 2$	$d^2 = d^{2^m+2}$	(*order* $n = 2^m + 1$)
(1.4.2)	*if* $p \neq 2$		
(1.4.2.1)		$d^2 = d^{p^m+2}$	(*order* $n = p^m + 1$)
(1.4.2.2)		$d^3 = d^{p^m+3}$	(*order* $n = p^m + 2$)

2. Preliminaries.

DEFINITION. A semigroup S is called a Γ^*-semigroup if every subsemigroup different from S is a Γ-semigroup.

S is a Γ^*-semigroup if and only if the subsemigroup lattice $L(S)$ is a lattice satisfying

(2.1) Any subset which cantains the greatest element 1 is a subsemilattice with respect to join, equivalently to

(2.1′) Let x, y be any elements of a lattice. Then

$$x \cup y = x \text{ or } y \text{ or } 1 .$$

Notation. If X and Y are subsets of S, $X \mid Y$ means either $X \subseteqq Y$ or $X \supseteqq Y$; $X \parallel Y$ means that X and Y are incomparable, that is, neither is contained in the other. $((X, Y, \cdots))$ denotes the subsemigroup generated by $X, Y, \cdots$. In particular, $((x))$ denotes the subsemigroup generated by an element x, $((x, y))$ the subsemigroup generated by elements x and y, while $\{x_1, x_2, \cdots\}$ is the set composed of $x_1, x_2, \cdots$.

S is a Γ^*-semigroup if and only if any two subsemigroups A and B satisfy the following condition: $A \parallel B$ implies $S = ((A, B))$. Of course a Γ-semigroup is a Γ^*-semigroup. Since the homomorphic image of a Γ-semigroup is also a Γ-semigroup, we get easily

LEMMA 2.1. *A homomorphic image of a* Γ^**-semigroup is a* Γ^**-semigroup.*

LEMMA 2.2. *A* Γ^**-semigroup is periodic.*

Proof. Suppose there is an element x of infinite order. S contains an infinite cyclic subsemigroup $\{x^i; i = 1, 2, \cdots\}$. Hence we can consider a proper subsemigroup[3] T of S.

$$T = \{x^{2i}; i = 1, 2, \cdots\}$$

which contains two incomparable subsemigroups T_1 and T_2:

$$T_1 = \{x^{4i}; i = 1, 2, \cdots\}\ , \qquad T_2 = \{x^{6i}; i = 1, 2, \cdots\}\ .$$

This contradicts the assumption of S.

By Lemma 2.2, we have seen that a Γ^*-semigroup has at least one idempotent. However, we have

THEOREM 2.1. *A Γ^*-semigroup of order >2 is unipotent.*

Proof. Suppose that a Γ^*-semigroup S of order >2 contains at least two idempotents, say, e, f. First, since e is a right identity of Se, and f is a left identity of fS, we see easily that if $Se = fS$, then $e = f$. Second, we shall say that either both of Se and Sf or both of eS and fS are proper subsemigroups. Suppose either of Se and Sf is equal to S, say, $Se = S$. Then, by the above fact, $fS \subset S$, and so we have to show $eS \subset S$. Let us assume $Se = eS = S$. There is a proper subsemigroup $\{e, f\}$ of order 2 because $ef = fe = f$; but $\{e, f\}$ is not a Γ-semigroup since e and f are both idempotents. This is a contradiction. Therefore $eS \subset S$.

Next, assume that both eS and fS are proper subsemigroups of S. Since eS and fS are Γ-semigroups with left identities, they are groups by Lemma 1.1. We shall prove that $\{e, f\}$ is a proper subsemigroup which is not a Γ-semigroup, and then the contradiction will be derived. For proof, the idempotency of ef and fe is shown as follows:

$$(ef)(ef) = (efe)f = (ef)f = e(ff) = ef$$
$$(fe)(fe) = (fef)e = (fe)e = f(ee) = fe$$

because e and f are two-sided identities of the groups eS and fS respectively. Since $ef \in eS$ and $fe \in fS$, we have

$$ef = e\ , \qquad fe = f$$

whence $\{e, f\}$ is a subsemigroup. We can have the same result, when $Se \subset S$ and $Sf \subset S$. Thus the proof of the theorem has been completed.

LEMMA 2.3. *The index of an element a of a Γ^*-semigroup S cannot exceed* 3.

[3] By "a proper subsemigroup T of S" we mean "a subsemigroup T which is different from S."

Proof. Let a have index greater than 1. Then $((a)) - \{a\}$ is a Γ-semigroup, so $((a^2)) \mid ((a^3))$. Hence there is a positive integer n such that either

$$a^2 = a^{3n} \quad \text{or} \quad a^3 = a^{2n} .$$

This shows that a has index 2 or 3.

3. Γ^*-Z-Semigroups. In this section we shall determine the types of Γ^*-Z-semigroups, i.e., unipotent Γ^*-semigroup with zero 0.

Let S be a Γ^*-Z-semigroup with 0. Since S is periodic, every element of S is nilpotent, that is, some power of the element is 0. By Lemma 2.3,

$$x^3 = 0 \quad \text{for all} \quad x \in S .$$

LEMMA 3.1. *$x = xy$ implies $x = 0$; $x = yx$ implies $x = 0$.*

Proof. $x = xy = xy^2 = xy^3 = 0$; the proof of the second part is obtained in a similar way.

LEMMA 3.2. *If $x^2 = 0$, then $xy = yx = 0$ for all y.*

Proof. We may assume $x \neq 0$, let $y \neq 0$. If $((x)) \mid ((xy))$, $xy = 0$ because of Lemma 3.1. If $((x)) \parallel ((xy))$, then $S = ((x, xy))$ and so $y = xu$ for some u.

$$xy = x^2u = 0 .$$

The proof of $yx = 0$ is similar.

To determine the types of Γ^*-Z-semigroups, we consider the possible three cases:

Case I. $x^2 = 0$ for all $x \neq 0$.

Case II. There exists only one nonzero element x such that $x^3 = 0$, $x^2 \neq 0$.

Case III. There exist at least two nonzero elements x and y such that $x^3 = 0$, $x^2 \neq 0$, $y^3 = 0$, $y^2 \neq 0$.

THEOREM 3.1. *S is a non-trivial Γ^*-Z-semigroup if and only if S is isomorphic or anti-isomorphic to one of the following:*

Case I. *$S = \{0, a, b\}$ where $xy = 0$ for all $x, y \in S$.*

Case II. $S = \{0, a, a^2\}$ *where* $a^3 = 0$. *This is a Γ-semigroup which is isomorphic to* (1.1.2).

Case III. $S = \{0, a, b, c\}$ *where* $a^2 = b^2 = c$, $a^2x = xa^2 = 0$ *for all* $x \in S$.
Subcase III_1 $ab = ba = c$
Subcase III_2 $ab = c, ba = 0$
Subcase III_3 $ab = ba = 0$

Proof.

Case I. Let a and b be distinct nonzero elements of S. Since $((a)) \| ((b))$, $S = ((a, b))$. By Lemma 3.2, we have $ab = ba = 0$. Hence

$$S = ((a, b)) = \{0, a, b\} .$$

Case II. Let a be an element with index 3. Suppose that there is $b \in S - ((a))$. In the present case we know $b^2 = 0$. By Lemma 3.2, $ab = ba = 0$, whence $A = \{0, a^2, b\}$ is a subsemigroup which does not contain a, and hence A is a Γ-semigroup. On the other hand, since $b \neq a^2$, we have $((a^2)) \| ((b))$. It is impossible in a Γ^*-semigroup S. Therefore $S = ((a))$.

Case III. Let a and b be distinct nonzero elements, both of which have index 3. Since $(a^2)^2 = (b^2)^2 = 0$, Lemma 3.2 gives us

$$a^2b = ba^2 = b^2a = ab^2 = 0 \quad \text{and so } a^2b^2 = b^2a^2 = 0 . \tag{3.1}$$

Using (3.1) and Lemma 3.2 repeatedly, since $(aba)^2 = aba^2ba = 0$, we have

$$(ab)^2 = (aba)b = 0 \tag{3.2}$$

and hence

$$aba = 0 . \tag{3.3}$$

Similarly we get

$$bab = 0 . \tag{3.3'}$$

Now we have two subsemigroups $T = ((a^2, b^2))$ and $U = ((ab, a^2))$:

$$T = ((a^2, b^2)) = \{0, a^2, b^2\} \not\ni a$$

where we see $a \neq b^2$, otherwise, $a = b^2$ would imply $a^2 = 0$; also

$$U = ((ab, a^2)) = \{0, ab, a^2\} \not\ni b .$$

Accordingly both T and U are Γ-semigroups and so

$$((a^2)) \mid ((b^2)) \quad \text{and} \quad ((ab)) \mid ((a^2)) .$$

The first implies (3.4); the second implies (3.5)

$$a^2 = b^2 \tag{3.4}$$

$$ab = a^2 \quad \text{or} \quad 0 . \tag{3.5}$$

Similarly we have

$$ba = a^2 \quad \text{or} \quad 0 . \tag{3.5$'$}$$

Clearly $((a)) \| ((b))$. By (3.1) through (3.5$'$),

$$S = ((a, b)) = \{0, a, b, a^2\}$$

which consists of exactly four elements. Thus we have obtained the three types for Case III. It is easy to show that the systems thus obtained are Γ^*-Z-semigroups.

4. **Γ^*-groups.** By Lemma 2.2, a group G is a Γ^*-semigroup if and only if it is a Γ^*-group, i.e, every proper gubgroup of G is a Γ-group. By Lemma 1.1, every Γ-group is of type $G(p^k)$, $k \leqq \infty$. In this chapter we determine all solvable Γ^*-groups. We also show that every finite Γ^*-group is solvable. The question whether infinite simple Γ^*-groups can exist remains open.

LEMMA 4.1. *Let G be a non-abelian solvable Γ^*-group which is not also a Γ-group. Then G contains a proper normal subgroup $N \neq 1$ and an element a not in N, such that*

$$N \| ((a)), \quad \textit{so that} \quad G = ((N, a)) \tag{4.1}$$

$$a^q \in N \quad \textit{for a prime number } q . \tag{4.2}$$

Proof. Since G is solvable, it contains a proper normal subgroup N such that G/N is abelian. $N \neq 1$ since G is not abelian. Since N is a proper subgroup of G, it is a Γ-group. Since G is not itself a Γ-group, there exist a and b in G such that $((a)) \| ((b))$, and then we have $G = ((a, b))$. If $N \| ((b))$, then (4.1) holds with b instead of a. To prove (4.1) suppose $N \| ((b))$. If $N \supseteqq ((b))$, then $N \not\supseteqq ((a))$, since N is a Γ-group; and $((a)) \| ((b))$, and $N \not\subseteqq ((a))$ since otherwise $((b)) \subseteqq N \subseteqq ((a))$. Hence $N \| ((a))$ in this case. If $N \subseteqq ((b))$, then, since G/N is abelian, $aba^{-1}b^{-1} \in N \subseteqq ((b))$, so $aba^{-1} \in ((b))b \subseteqq ((b))$. Since $G = ((a, b))$, we conclude that $N' = ((b))$ is a normal subgroup of G, and (4.1) holds with N' instead of N. Hence N and a exist such that (4.1) holds. Let k be the least positive integer such that $a^k \in N$, and let $k = k'q$ with q a prime. Let $a' = a^{k'}$. Then (4.1) and (4.2) hold with a' instead of a.

THEOREM 4.1. *Let G be a solvable Γ^*-group which is not a Γ-group. Then one of the following holds*:

(4.3) *G is a group of order pq, p and q primes excluding the cyclic group of order p^2.*

(4.4) *G is the quaternion group of order* 8.

Proof. First let us take the case G abelian. If G were directly indecomposable, it would be isomorphic with $G(p^k)$ for some $k \leqq \infty$ (cf. Theorem 10, p. 22, [2]), and so would be a Γ-group. Hence G is directly decomposable: $G = G_1 \times G_2$ where $G_1 \neq 1$, $G_2 \neq 1$. Let a_i be an element of G_i of prime order p_i $(i = 1, 2)$. Then $((a_1)) \,||\, (a_2))$, so $G = ((a_1, a_2)) = ((a_1)) \times ((a_2))$. Thus G has type (4.3).

Let G be non-abelian. By Lemma 4.1, G contains a proper normal subgroup $N \neq 1$, and an element a not in N such that $N \,||\, ((a))$ and $a^q \in N$ for some prime q. Since N is a proper subgroup of G, it is isomorphic with $G(p^k)$ for some prime p and some $k \leqq \infty$. Hence a^q has prime power order p^n, say.

If $q \neq p$, then $a_1 = a^{p^n} \notin N$, and $a_1^q = 1$. If b is any element of N of order p, we have $((a_1)) \,||\, ((b))$ and hence $G = ((a_1, b))$. Since $a_1 N a_1^{-1} \subseteqq N$, and every subgroup of N is characteristic, $a_1((b))a_1^{-1} \subseteqq ((b))$. Hence G is an extension of the cyclic group $((b))$ of order p by the cyclic group $((a_1))$ of order q.

We may now assume $q = p$. Since $N \nsubseteqq ((a))$, there exists b in N such that $b^p = a^p$. Let $c = a^p = b^p$. Since c commutes with a and b, and $G = ((a, b))$, c belongs to the center C of G. If $c = 1$, then, as in the above statements, G is an extension of the cyclic group $((b))$ of order p by the cyclic group $((a))$ of order p. Hence we may assume that the order of c is p^n with $n > 0$.

Since $((b))$ is invariant under a, we have $aba^{-1} = b^r$ for some positive integer $r > 1$. Then

$$c = b^p = ab^pa^{-1} = (aba^{-1})^p = b^{rp} = c^r ,$$

whence $r = 1 + sp^n$ for some integer s. Hence

$$aba^{-1} = b^r = bd \quad \text{or} \quad b^{-1}aba^{-1} = d \neq 1$$

where $d = b^{sp^n} = c^{sp^{n-1}}$ is an element of C of order p. As in the familiar way,

$$(ab^{-1})^p = d^{p(p-1)/2}a^pb^{-p} = d^{p(p-1)/2} .$$

If p is odd, we conclude that $(ab^{-1})^p = 1$. Let $a_1 = ab^{-1}$. Then $a_1^p = 1$ and this case is reduced to the previous case $c = 1$. We are left with the case $p = 2$. Setting $a_1 = ab^{-1}$, we have $a_1^2 = d$. Let b_1 be an element of N such that $b_1^2 = d$. Then $G = ((a_1, b_1))$, and $((b_1))$ is invariant under a_1. Since $b_1^4 = 1$, and G is not abelian, we must have

$$a_1 b_1 a_1^{-1} = b_1^3 .$$

Together with $a_1^4 = b_1^4 = 1$, this shows that G is the quaternion group of order 8. Thus this theorem has been proved.

THEOREM 4.2. *A finite Γ^*-group is solvable.*

Proof. For Γ-groups, the theorem is obvious. Let G be a finite Γ^*-group which is not a Γ-group. If G is of order p^m of a prime power, then this theorem holds, since G has a normal subgroup of order p^{m-1} by the familiar theorem of p-groups. So we may assume that the order of G has at least two distinct prime divisors.

First we shall prove that G has a proper normal subgroup. Let M be a Sylow subgroup of G, and consider the normalizer H of M. If $H = G$, then M is normal; if $M \subseteqq H \subset G$, then H is a Γ-group, a cyclic group. By Burnside's theorem ([8], p. 169), G has a proper normal subgroup N such that $G = NH$, $N \cap H = 1$.

Since N is a proper subgroup, it is a Γ-group, say, $G(p^{\alpha_1})$. Then, suppose the order of the factor group G/N is

$$p^{\alpha_2} q^{\beta} r^{\gamma} \cdots, \quad \alpha_2 \geqq 0, \beta \geqq 1, \quad \gamma \geqq 0, \cdots \tag{4.5}$$

which has a prime divisor $q \neq p$. Since G/N has a subgroup of order q, G has a proper subgroup of order $p^{\alpha_1} q$, which contains two incomparable subgroups, unless

$$\alpha_2 = 0, \beta = 1 . \tag{4.6}$$

Thus we have proved that the index of N is a prime q.

THEOREM 4.3. *A non-simple Γ^*-group is solvable.*

Proof. Let G be a non-simple Γ^*-group and N be a proper normal subgroup of G. We may assume that G/N contains a proper subgroup $\bar{H}$ of prime order p, since G/N is a Γ^*-group and so G/N is periodic. Consider a coset xN which is a generator of $\bar{H}$ and take an element $a \in xN$. Then $H = ((a))$ is a group of order p, and there is a subgroup K of G such that $K/N \cong \bar{H}$. Clearly $K = NH$. On the other hand, since $N || H$, we have $G = ((N, H)) = NH = K$. Accordingly, $G/N \cong \bar{H}$, which is prime order. Thus the proof has been completed.

Consequently, (4.3) and (4.4) of Theorem 4.1 give us all the types of finite or non-simple Γ^*-groups which are not Γ-groups.

5. Unipotent Γ^*-semigroups.

1. In this chapter we shall discuss unipotent Γ^*-semigroups S's which are neither groups nor Z-semigroups. By Lemma 2.2 and Theorem 2.1 we see that a Γ^*-semigroup S of order >2 is a unipotent inversible semigroup. By "inversible" we mean "for any element a of S there is an element b such that $ab = e$ where e is a unique idempotent." According to [5], [6], a unipotent inversible semigroup which contains properly a group is determined by a group G (or kernel, i.e., least ideal), and a Z-semigroup D (the difference semigroup of S modulo G), and certain mapping of the bases of D into G: $a \to ea$.

First of all we shall prove that the kernel is finite.

LEMMA 5.1. *Let S be a unipotent inversible semigroup with the kernel G of type $G(p^k)$, k being infinite or finite, and let d be an element of S which is not in G such that ed generates $G(p^m)$, $m < k$, and $d^{l-1} \notin G(p^k)$, $d^l \in G(p^k)$. Then there is a subsemigroup H of order $p^{m+1} + l - 1$ of S which contains two incomparable subsemigroups: $G(p^{m+1})$ and $\{d^i; i \geqq 1\}$.*

Proof. Let $a = ed$. As is easily seen (cf. [5]), we have

$$a = ed = de,\ d^i = a^i,\ i \geqq l \tag{5.1}$$

$$xd = dx = xa = ax \quad \text{for every} \quad x \in G\,. \tag{5.2}$$

Especially for $x \in G(p^{m+1})$, $xd = dx \in G(p^{m+1})$. Therefore the set union $H = G(p^{m+1}) \cup \{d^i; l-1 \geqq i \geqq 1\}$ is a subsemigroup of S; and the two subsemigroups $G(p^{m+1})$ and $\{d^i; i \geqq 1\}$ are incomparable, because $\{d^i; i \geqq l\} \subseteqq G(p^m)$.

THEOREM 5.1. *Let S be a unipotent inversible semigroup which is neither a group nor a Z-semigroup. If S is a Γ^*-semigroup, then S is finite.*

Proof. The proper subgroup G is a Γ-group $G(p^\infty)$ or $G(p^n)$, and the difference semigroup $D = (S\colon G)$ of S modulo G in Rees' sense [3] is a Γ^*-Z-semigroup of order $\leqq 4$ by theorems in § 3. There is an element z_1 outside G such that $z_1^2 \in G$, for example, we may take a nonzero annihilator as z_1 (cf. [6]); and let m be a positive integer such that ez_1 generates a subgroup $G(p^m)$. If S is infinite, then G is of the type $G(p^\infty)$ and so S has a proper subsemigroup of order $p^{m+1} + 1$, which contains two incomparable subsemigroups by Lemma 5.1. This contradicts the definition of Γ^*-semigroups of S. Thus the theorem has been proved.

Hereafter we shall determine the desired semigroups S in each case according as the order of D.

2. The case with D of order 2.

Let $G(p^n)$ denote the kernel of S, and let d be a unique element outside $G(p^n)$. Of course $d^2 \in G(p^n)$. $G(p^k)$ denotes the subgroup generated by $a = ed$. If $k = n$, then, by (5.1), we have

$$S = \{d^i; i \geqq 1\} , \qquad G(p^n) = \{d^i; i \geqq 2\}$$

that is, S is a Γ-semigroup of type (1.4.1) or (1.4.2.1).

If $k < n$, then by Lemma 5.1 there is a subsemigroup $H = G(p^{k+1}) \cup \{d\}$ of order $p^{k+1} + 1$ which contains two incomparable subsemigroups, so that $S = H$ and hence we have $k = n - 1$. In other words, a is a generator of $G(p^{n-1})$; this a determines S and there is a unique S to within isomorphism, independent of choice of generator a (cf. [6]). Conversely, a semigroup S thus obtained is easily seen to be a Γ^*-semigroup. In fact, by (5.1) we see that a proper subsemigroup incomparable to $G(p^n)$ is nothing but

$$G(p^{n-1}) \cup \{d\} = ((d)) .$$

3. The case with D of type Case I of order 3.

Let $S = G(p^n) \cup \{d_1, d_2\}$ where $d_1d_2, d_1^2, d_2^2, d_2d_1 \in G(p^n)$. S is determined by the two elements a_1, a_2, i.e.,

$$a_1 = ed_1 , \qquad a_2 = ed_2$$

where a_1 and a_2 can be taken independently arbitrarily. The proper subsemigroups $G(p^n) \cup \{d_1\}$ and $G(p^n) \cup \{d_2\}$ are Γ-semigroups of type (1.4.1) or (1.4.2.1). We have already known that a_1 and a_1 are the generators of $G(p^n)$, and

$$G(p^n) \cup \{d_1\} = ((d_1)) , \qquad G(p^n) \cup \{d_2\} = ((d_2)) .$$

We can easily prove that there are two possible distinct types

$$a_1 = a_2 , \qquad a_1 \neq a_2$$

in all cases except for the case $p = 2$ and $n = 1$. They are immediately seen to be Γ^*-semigroups.

4. The case with D of type Case II of order 3.

Let d be a generator of D: $D = \{0, d, d^2\}$, $d^3 = 0$, and let $S =$

$G(p^n) \cup \{d, d^2\}$. We shall prove that $a = ed$ generates $G(p^n)$. Suppose that an element a generates $G(p^k)$, $k < n$. Then, since $ed^2 = (ed)^2$ and $(d^2)^2 \in G(p^n)$, ed^2 generates a subgroup $G(p^m)$, $m \leqq k$, and a subsemigroup $K = G(p^{m+1}) \cup \{d^2\}$ contains two incomparable subsemigroups by Lemma 5.1. K is a proper subsemigroup of S because

$$p^{m+1} + 1 < p^n + 2 .$$

This contradicts the assumption of Γ^*-semigroup of S. Hence it has been proved that $G(p^n)$ is generated by ed. Accordingly we get $G(p^n) = \{d^i; i \geqq 3\}$ by (5.1), whence S is generated by d. In the same way as the Case with D of order 2, we see that arditrary different generators of $G(p^n)$ give some isomorphic S's.

The remaining thing to do is to testify the subsemigroup lattice of such semigroups.

If $p \neq 2$, then ed^2 generates $G(p^n)$, and only a proper subsemigroup between S and $G(p^n)$ is

$$((d^2)) = G(p^n) \cup \{d^2\} \quad \text{by (5.1)}$$

and so S is a Γ-semigroup of type (1.4.2.2).

If $p = 2$, then ed^2 generates $G(2^{n-1})$ and so, by Lemma 5.1, we have a proper subsemigroup

$$G(2^n) \cup \{d^2\}$$

which contains two incomparable $G(2^n)$ and $((d^2))$. Therefore, S is not a Γ^*-semigroup.

5. The case with D of order 4.

Let $S = G(p^n) \cup \{d_1, d_2, d_3\}$ where $d_1 = d_2^2 = d_3^2$. D has any one of the types of Case III with elements denoted by d_1, d_2, d_3 instead of a, b, c, respectively. Since the proper subsemigroups $G(p^n) \cup \{d_1, d_2\}$ and $G(p^n) \cup \{d_1, d_3\}$ are both Γ-semigroups of type (1.4.2.2), we have by (5.1)

$$G(p^n) \cup \{d_1, d_2\} = ((d_2)) , \qquad G(p^n) \cup \{d_1, d_3\} = ((d_3))$$

where $p \neq 2$, and $a_2 = ed_2$ and $a_3 = ed_3$ are both generators of $G(p^n)$. One the other hand, there are relations between a_2 and a_3 as follows: (We called these relations the primary equations for D in [6], § 3.)

$$a_2^2 = a_3^2 \qquad \text{in Case III}_3 ,$$
$$a_2 = a_3 \qquad \text{in Cases III}_1 \text{ and III}_2 .$$

We see easily that $a_2^2 = a_3^2$ in $G(p^n)$ implies $a_2 = a_3$ because $p \neq 2$. Thus for $G(p^n)$ and each D, there is a unique S to within isomorphism.

As far as the subsemigroups containing $G(p^n)$ are concerned, besides $((d_2))$ and $((d_3))$, there is $((d_1))$ and we have

$$((d_1)) = ((d_2)) \cap ((d_3))$$

because $p \neq 2$. Accordingly it can be seen that S is a Γ^*-semigroup. Thus we have

THEOREM 5.2. *When $G(p^n)$ is given, all the possible unipotent Γ^*-semigroups S whose kernel is $G(p^n)$ and which are not Γ-semigroups are determined as shown below. Let e be the unique idempotent of S, and let $D = (S: G(p^n))$. We remark $G(p^0) = 1$, $G(p^{-1}) = $ empty.*

(5.3.1) *In the case D of order 2, $S = G(p^n) \cup \{d\}$, $n \neq 0$, $ed \in G(p^{n-1}) - G(p^{n-2})$*

(5.3.2) *In the case D of order 3, D is of Case I and $S = G(p^n) \cup \{d_1, d_2\}$, $n \neq 0$*

(5.3.2.1) $\qquad ed_1 = ed_2 \in G(p^n) - G(p^{n-1})$

(5.3.2.2) $\qquad p^n \neq 2, ed_1 \neq ed_2,$ *and* $ed_1, ed_2 \in G(p^n) - G(p^{n-1})$

(5.3.3) *In the case D of order 4, $S = G(p^n) \cup \{d_1, d_2, d_3\}$, $d_2^2 = d_3^2 = d_1$, $n \neq 0$, $p \neq 2$*

$$\left.\begin{array}{ll} (5.3.3.1) & D \text{ of type Case III}_1 \\ (5.3.3.2) & D \text{ of type Case III}_2 \\ (5.3.3.3) & D \text{ of type Case III}_3 \end{array}\right\} ed_2 = ed_3 \in G(p^n) - G(p^{n-1}) .$$

After all, under the given $G(p^n)$, if $p \neq 2$, then there are six types of S; if $p = 2$ and $n \neq 1$, then three types of S; if $p = 2$ and $n = 1$, then two types of S.

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EXISTENCE AND ASYMPTOTIC BEHAVIOR OF PROPER SOLUTIONS OF A CLASS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

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1. This paper deals with proper solutions of the second-order nonlinear differential equation

$$y'' = yF(y, x)\ , \tag{1.1}$$

where (i) $F(u, x)$ is continuous in u and x for $0 \leqq u < +\infty$ and $x \geqq x_0$,

(ii) $F(u, x) > 0$ for $u > 0$ and $x \geqq x_0$,

(iii) $F(u, x) < F(v, x)$ for each $x \geqq x_0$ and $0 < u < v < +\infty$.

By a proper solution we understand a real-valued solution y of (1.1) which is of class $C^2[a, \infty)$, where $x_0 \leqq a < +\infty$. An example of equations of this type is the Emden-Fower equation [2, chapter 7]

$$y'' = x^\lambda y^n\ . \tag{1.2}$$

Our interest is in the existence and asymptotic behavior of *positive proper solutions* of (1.1). Since $F(y, x) > 0$ for $y > 0$, all positive solutions of this equation are convex. They are therefore of two types: (1) those which are monotonically decreasing and tending to nonnegative limits as $x \to +\infty$, and (2) those which are ultimately increasing and becoming unbounded as x becomes infinite.

In this section we shall consider proper solutions which are of type (1), i.e., solutions which are confined to the semi-infinite strip $S = \{(x, y)\colon 0 \leqq y \leqq K, a \leqq x < +\infty\}$. We observe that in view of properties (i) and (iii) the function $yF(y, x)$ satisfies a Lipschitz condition

$$|\, uF(u, x) - vF(v, x)\,| \leqq H|\, u - v\,| \tag{1.3}$$

in every closed rectangle $R = \{(x, y)\colon 0 \leqq y \leqq K, a \leqq x \leqq b\}$, where $H = H(K, a, b)$. Before taking up the existence of such solutions, we first derive the following lemmas.

LEMMA 1.1. *Let $u(x)$ be a nonnegative solution of* (1.1) *passing*

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through two points (a, A) *and* (b, B), *where* $a < b$ *and* $A, B > 0$. *Then the solution is unique.*

Proof. Suppose that $v(x)$ is a second nonnegative solution such that $u(a) = v(a) = A$ and $u(b) = v(b) = B$. We first assume that (a, A) and (b, B) are two consecutive points of intersection of u and v and that $u(x) > v(x)$ for $a < x < b$. Using (1.1) and property (iii) we find that

$$(1.4) \qquad \int_a^b (u''v - uv'')dx = \int_a^b uv[F(u, x) - F(v, x)]dx > 0 .$$

Since

$$(1.5) \qquad \int_a^b (u''v - uv'')dx = B[u'(b) - v'\{(b)] - A[u'(a) - v'(a)] ,$$

and since $u'(a) > v'(a)$ while $u'(b) < v'(b)$, the right-hand side of (1.5) is clearly negative which contradicts (1.4). If u and v should have other points of intersection on (a, b) we can partition the interval $[a, b]$ into several segments whose end points are the abscissas of the consecutive points of intersection of u and v. The same argument leads to a contradiction in each case. This proves the assertion.

LEMMA 1.2. *Let* $u(x)$ *be a nonnegative solution of* (1.1) *passing through* (a, A) *such that* $\lim_{x\to b} u'(b) = 0$, *where* b *may be finite or infinite. Then* $u(x)$ *is unique.*

The proof is identical with that of Lemma 1.1 since the right-hand side of (1.5) will also be negative under the present assumptions.

The next lemma guarantees the existence of solutions passing through two points, provided the abscissas of these points are sufficiently near each other.

LEMMA 1.3. *Let* (a, A) *and* (b, B) *be two points such that*

$$a < b, \qquad A, B > 0$$

$(b - a)$ *is small enough so that*

$$(1.6) \qquad H(b - a)^2 < \rho < 1$$

and

$$(1.7) \qquad L(x) > \int_a^b g(x, t)L(t)F(L, t)dt ,$$

where

$$L(x) = \frac{A(b-x) + B(x-a)}{(b-a)}, \tag{1.8}$$

and

$$g(x,\, t) = \begin{cases} \dfrac{(b-x)(t-a)}{(b-a)}, & t \leqq x \\ \dfrac{(b-t)(x-a)}{(b-a)}, & x \leqq t\,. \end{cases} \tag{1.9}$$

Then there exists exactly one positive solution $y \in C^2[a, b]$ *of* (1.1) *which passes through these points.*

Proof. In view of Lemma 1.1, a solution, if it exists, is necessarily unique. To establish the existence we replace the boundary value problem by the equivalent integral equation

$$y(x) = L(x) - \int_a^b g(x,\, t)y(t)F(y,\, t)dt\,, \tag{1.10}$$

where $L(x)$ and $g(x, t)$ are given by (1.8) and (1.9) respectively. To solve (1.10) by successive approximations, we introduce a sequence $\{y_k(x)\}$ of twice differentiable convex functions passing through (a, A) and (b, B) defined by

$$\begin{cases} y_0(x) = L(x) \\ y_{k+1}(x) = L(x) - \displaystyle\int_a^b g(x,\, t)y_k(t)F(y_k,\, t)dt \\ k = 0, 1, 2, \cdots\,. \end{cases} \tag{1.11}$$

Since both $g(x, t)$ and $L(x)$ are positive in (a, b), (1.7) shows that $0 < y_1(x) < L(x)$. If we assume that $0 < y_k(x) < L(x)$, then (1.7) and property (iii) implies

$$\begin{aligned} L(x) > y_{k+1}(x) &= L(x) - \int_a^b g(x,\, t)y_k(t)F(y_k,\, t)dt \\ &> L(x) - \int_a^b g(x,\, t)L(t)F(L,\, t)dt = y_1(x) > 0\,. \end{aligned}$$

It follows by induction that $0 < y_k(x) \leqq L(x) \leqq \max(A, B)$ for all k. The sequence $\{y_k(x)\}$ is thus positive and uniformly bounded.

Let $K = \max(A, B)$ and $M = \sup F(K, x)$, then

$$\begin{aligned} |\, y_1(x) - y_0(x)\,| &= \int_a^b g(x,\, t)L(t)F(L,\, t)dt \\ &\leqq KM \int_a^b g(x,\, t)dt \\ &\leqq KM(b-a)^2\,. \end{aligned}$$

If R denotes the closed rectangle defined by $0 \leqq y \leqq K$ and $a \leqq x \leqq b$, then by (1.3),

$$|uF(u, x) - vF(v, x)| \leqq H|u - v|$$

for all points of R. Moreover, (1.11) shows that

$$|y_{k+1}(x) - y_k(x)| \leqq H\int_a^b g(x, t)|y_k(t) - y_{k-1}(t)|dt$$

so that we have, by induction,

$$(1.12) \qquad |y_{k+1}(x) - y_k(x)| \leqq (KM)H^k(b - a)^{2(k+1)} .$$

We thus obtain the estimate

$$(1.13) \qquad |y_n(x)| \leqq K + H^{-1}KM\sum_1^n [H(b - a^2)]^{k+1}$$

which, in view of (1.6), implies the uniform convergence of $\{y_n(x)\}$. This proves the lemma.

As pointed out before, a positive proper solution of (1.1) is either monotonically decreasing or monotonically increasing. As the following theorem shows there always exists exactly one solution of the former type which passes through a given point (a, A).

THEOREM 1.1. *For any given point* (a, A) *where* $A > 0$, *there exists exactly one positive proper solution* y *of the class* $C^2[a, \infty)$ *which passes through* (a, A) *and is monotonically decreasing in* $[a, \infty)$.

To prove this result we consider the variational problem of minimizing the functional

$$(1.14) \qquad J(y) = \int_a^\infty [(y')^2 + 2h(y, x)]dx ,$$

where

$$(1.15) \qquad h(y, x) = \int_0^y tF(t, x)dt ,$$

within the class Ω of all nonnegative functions $y \in D^1[a, \infty)$ such that $y(a) = A$ and that the integral (1.14) exists. Since (1.1) is the Euler-Lagrange equation of problem (1.14), the solution y of (1.14) will be a solution of (1.1), provided, of course, y exists and is of class $C^2[a, \infty)$.

Since the functional $J(y)$ is positive-definite, $J(y)$ has the trivial lower bound 0. We next remark that we may restrict our attention to positive functions $y \in \Omega$ which are convex in $[a, \infty)$. To show this, we assume that the positive function y is concave in an interval (c, d), i.e.,

$$y(x) \geqq \frac{y(c)(d-x) + y(d)(x-c)}{(d-c)} \equiv L(x) .$$

In view of hypothesis (iii) and the definition of $h(y, x)$, we then have

$$h(L, x) \leqq h(y, x), \qquad c \leqq x \leqq d ,$$

and, by a variational argument,

$$\int_c^d [L'(x)]^2 dx < \int_c^d [y'(x)]^2 dx \tag{1.16}$$

unless $y(x) \equiv L(x)$ in (c, d). Hence, if y^* denotes the function obtained from y by substituting $L(x)$ for $y(x)$ in (c, d),

$$J(y^*) < J(y) .$$

Also, we need only consider positive convex functions y which are nonincreasing in $[a, \infty)$, since, as (1.16) shows, the functional $J(y)$ becomes infinite for convex increasing functions. Finally, the problem $J(y) = \min$ is not vacuous, since the function v defined by

$$v(x) = \begin{cases} A\left(\dfrac{b-x}{b-a}\right), & a \leqq x \leqq b \\ 0 , & b \leqq x \end{cases}$$

is in Ω and evidently $J(v) < C < +\infty$.

The proof of the theorem depends on the validity of an analogous result for a finite interval $[a, b]$ and the performing of a suitable passage to the limit $b \to \infty$. The result in question is the following:

LEMMA 1.4. *There exists a unique positive solution $u(x)$ of equation* (1.1) *which passes through the two points* (a, A) *and* (b, B), *where $b > a$ and $A, B > 0$. If v denotes any other positive function of $D^1[a, b]$ for which $v(a) = A$, $v(b) = B$, and if $J(y; b)$ denotes the functional*

$$J(y; b) = \int_a^b [(y')^2 + 2h(y, x)] dx , \tag{1.14'}$$

then

$$J(u; b) < J(v; b) \tag{1.17}$$

unless $v(x) \equiv u(x)$ in $[a, b]$.

We first assume that the interval $[a, b]$ is short enough so that conditions (1.6) and (1.7) are satisfied. Lemma 1.3 will then guarantee the existence of the unique positive solution u of (1.1) through the two points, and all we have to prove is inequality (1.17). To do so, we note that the solution $w(x)$ of the linear differential system

$$\begin{cases} w'' = p(x)w, & p(x) > 0 \\ w(a) = A \\ w(b) = B \end{cases} \tag{1.18}$$

satisfies the inequality

$$\int_a^b [(w')^2 + p(x)w^2]dx < \int_a^b [(v')^2 + p(x)v^2]dx \,, \tag{1.19}$$

where v is any other function of $D^1[a, b]$ which satisfies the same boundary conditions and does not coincide with $w(x)$. Inequality (1.19) is an obvious consequence of the identity

$$\begin{aligned} &\int_a^b [(v' - w')^2 + p(x)(v - w)^2]dx \\ &= \int_a^b [(v')^2 + p(x)v^2]dx - \int_a^b [(w')^2 + p(x)w^2]dx \end{aligned}$$

which is obtained by expanding the left-hand side and observing that, in view of (1.18) and the boundary conditions,

$$\int_a^b v'w'dx = [vw']_a^b - \int_a^b vw''dx = [vw']_a^b - \int_a^b pvwdx$$

and

$$[vw']_a^b = [ww']_a^b = \int_a^b (w'^2 + ww'')dx = \int_a^b (w'^2 + pw^2)dx \,.$$

Setting, in particular, $p(x) = F(u, x)$, we have $w(x) = u(x)$ and thus, by (1.19)

$$\int_a^b [(u')^2 + u^2F(u, x)]dx < \int_a^b [(v')^2 + v^2F(u, x)]dx \,. \tag{1.20}$$

Since $F(s, x)$ is a nondecreasing function of s for $s > 0$, the function $h(u, x)$ defined by (1.15) is convex in u. Hence, for nonnegative u and v,

$$2[h(u, x) - h(v, x)] \leqq (u^2 - v^2)F(u, x) \,.$$

Combining this with (1.20), we obtain

$$\int_a^b [(u')^2 + 2h(u, x)]dx < \int_a^b [(v')^2 + 2h(v, x)]dx$$

unless u and v coincide. This establishes (1.17) in the case in which the interval $[a, b]$ is short enough so as to satisfy conditions (1.6) and (1.7).

If b is an arbitrary value in (a, ∞), it is sufficient to consider the problem

$$
(1.21) \qquad \begin{cases} J(y; b) = \int_a^b [(y')^2 + 2h(y, x)]dx = \min \\ y(a) = A, \qquad y(b) = B \end{cases}
$$

in the class Ω_b of nonnegative convex functions $y \in D^1[a, b]$. We thus may assume

$$
0 \leqq y(x) \leqq \max(A, B) = K, \qquad a \leqq x \leqq b .
$$

Now we divide the interval $[a, b]$ into a finite number of subintervals $[a_k, a_{k+1}]$ $(a = a_0 < a_1 \cdots < a_m = b)$ in each of which the assumptions of Lemma 1.3 are satisfied. If $y(a_k) = A_k$, where $y \in \Omega_b$, the conditions restricting the length of these subintervals will be

$$
(1.22) \qquad H(a_{k+1} - a_k)^2 < \rho < 1
$$

and

$$
(1.23) \qquad \int_{a_k}^{a_{k+1}} g_k(x, t)L_k(t)F(L_k, t)dt < L_k(x) ,
$$

where

$$
(1.24) \qquad L_k(x) = \frac{A_k(a_{k+1} - x) + A_{k+1}(x - a_k)}{(a_{k+1} - a_k)} ,
$$

and

$$
(1.25) \qquad g_k(x, t) = \begin{cases} \dfrac{(a_{k+1} - x)(t - a_k)}{(a_{k+1} - a_k)} , & t \leqq x \\ \dfrac{(a_{k+1} - t)(x - a_k)}{(a_{k+1} - a_k)} , & x \leqq t . \end{cases}
$$

Since $A_k \leqq \max(A, B) = K$, we have $F[L_k(t), t] < F(K, t)$. Hence, if $M = \max F(K, x)$ in $[a, b]$, condition (1.23) will be satisfied if

$$
M \int_{a_k}^{a_{k+1}} g_k(x, t)L_k(t)dt < L_k(x) .
$$

In view of (1.24), this will be true if both the inequalities

$$
M \int_{a_k}^{a_{k+1}} g_k(x, t)(a_{k+1} - t)dt < (a_{k+1} - x)
$$

and

$$
(1.26) \qquad M \int_{a_k}^{a_{k+1}} g_k(x, t)(t - a_k)dt < (x - a_k)
$$

hold. Since these inequalities are equivalent, it is sufficient to consider one of them. A computation shows that

$$\int_{a_k}^{a_{k+1}} g_k(x, t)(t - a_k)dt = \frac{1}{6}(x - a_k)(a_{k+1} - x)(x + a_{k+1} - 2a_k) ,$$

and (1.26) will therefore follow if

$$\frac{M}{6}(a_{k+1} - x)(x + a_{k+1} - 2a_k) < 1 .$$

Since

$$(a_{k+1} - x) \leqq (a_{k+1} - a_k)$$

and $(x + a_{k+1} - 2a_k) = (x - a_k) + (a_{k+1} - a_k) \leqq 2(a_{k+1} - a_k)$, the length of the interval is thus restricted by the condition

$$M(a_{k+1} - a_k) < 2$$

and inequality (1.22). Since $H = H(K, a, b)$, this shows that a finite partition of the type indicated is indeed possible.

In each of these subintervals we now replace $y, y \in \Omega_b$, by the solution of (1.1) having the same values at the ends of the interval. If the new function so obtained is y^*, it follows from the result just proved that

$$J(y^*; b) < J(y; b) .$$

In the treatment of the minimum problem (1.21) it is therefore sufficient to consider curves y consisting of a finite number of arcs each of which is a solution of (1.1). Moreover, the abscissas of the points where two adjacent arcs meet may be taken to be the same for all functions of a sequence $\{y_n\}$ minimizing the functional $J(y; b)$.

Since in each of the subintervals $[a_k, a_{k+1}]$ the functions y_n are solutions of (1.1), elementary considerations show that we can select a subsequence $\{y_{n'}\}$ which converges in each subinterval $[a_k, a_{k+1}]$ to a solution $y_{(k)}$ of (1.1) and that, moreover, $y_{(k)}(a_{k+1}) - y_{(k+1)}(a_{k+1})$. The function y defined by $y(x) = y_{(k)}(x)$ for $a_k \leqq x \leqq a_{k+1}$ is therefore of class $D^1[a, b]$, and it is thus a solution of the minimum problem (1.21).

To show that $y(x)$ coincides in all these intervals with the *same* solution of (1.1), we have to show that y' is continuous at the points a_k. To do so, we choose a positive ε such that

$$(a_{k-1} + \varepsilon) < a_k, \ (a + \varepsilon) < a_{k+1}$$

and ε is small enough so that Lemma 1.3 applies to the interval $[a_k - \varepsilon, a_k + \varepsilon]$. There will then exist a solution u of (1.1) for which $u(a_k - \varepsilon) = y(a_k - \varepsilon)$, $u(a_k + \varepsilon) = y(a_k + \varepsilon)$ and, as shown above, we have the inequality

$$\int_{a_k-\varepsilon}^{a_k+\varepsilon}[u'^2 + 2h(u, x)]dx < \int_{a_k-\varepsilon}^{a_k+\varepsilon}[y'^2 + 2h(y, x)]dx ,$$

unless $y(x) \equiv u(x)$ in this interval. Hence if y' is discontinuous at $x = a_k$, it is possible to replace y by another function which yields a smaller value of $J(y; b)$. But this contradicts the minimum property of y, and we have thus proved that y' must be continuous throughout $[a, b]$. This completes the proof of Lemma 1.4.

We are now in a position to complete the proof of Theorem 1.1. As pointed out above, it is sufficient to consider positive admissible functions $y \in \Omega$ which are convex and decreasing in $[a, \infty)$. If y is any such function, we choose a value b in (a, ∞) and define a function $u \in \omega$, $\omega \subset \Omega$, as follows: $u(x) = y(x)$ in $[b, \infty)$ and $u(x) = y_b(x)$, where $y_b(x)$ denotes the solution of (1.1)—whose existence is established in Lemma 1.4—which satisfies $y_b(a) = A$ and $y_b(b) = y(b)$. In view of Lemma 1.4, we have

$$J(y_b) < J(y) ,$$

and it is clear that $0 \leqq y_b(x) \leqq A$ in $[a, \infty)$.

We now take a sequence $\{y_n\}$ in Ω for which

$$\lim_{n\to\infty} J(y_n) = \inf J(y) , \tag{1.27}$$

and we choose a sequence of values $b_m (a < b_1 < b_2 < \cdots)$ for which $\lim b_m = +\infty$. For each of these values b_m we construct the corresponding function $y_{n,b_m} \in \omega$. As just shown, we have

$$J(y_{n,b_{m+1}}) \leqq J(y_{n,b_m}) .$$

Hence, the diagonal sequence $J(y_{n,b_n})$ cannot have a larger limit than the sequence $J(y_n)$, and (1.27) shows that $J(y_{n,b_n})$ is likewise a minimizing sequence.

Since $0 \leqq y_{n,b_n} \leqq A$, and since y_{n,b_n} is a solution of (1.1) in $[a, b_N]$ if $n \geqq N$, an elementary argument shows that this sequence contains a limit function y which is a solution of (1.1) in $[a, b_N]$. But N is arbitrary, and y is thus a solution of (1.1) throughout $[a, \infty)$, the function y—being necessarily convex—must be decreasing for $a \leqq x < +\infty$. This completes the proof of Theorem 1.1.

Such a solution separates those solutions which are convex and increasing to $+\infty$ from those which are decreasing and becoming ultimately negative.

We add here a property of the positive decreasing solutions whose existence is established in Theorem 1.1.

LEMMA 1.5. *If y is a decreasing, positive proper solution of* (1.1), *then*

(1.28) $$\lim_{x\to\infty} xy'(x) = 0 \ .$$

Since $(xy' - y)' = xy'' = xyF(y, x) > 0$, the negative quantity $\phi(x) \equiv xy' - y$ is increasing for $x > a$. Let $\lim y(x) = c$, $c \geqq 0$, then clearly $\phi(x) \leqq -c$. If $\lim \phi(x) = -c$, the lemma is proved. If $\lim \phi(x) = -(c + A)$, where $A > 0$, we have $xy' - y \leqq -(A + c)$ for x in (a, ∞), i.e.,

$$\frac{y - A - c}{x} \leqq 0, \qquad a < x < +\infty \ .$$

This, however, implies a contradiction, since the expression $x^{-1}(y - A - c)$ is negative for large x and tends to zero for $x \to +\infty$. This completes the proof.

THEOREM 1.2. *Equation* (1.1) *has solutions which ultimately decrease monotonically to positive constants if, and only if, there is some $\beta > 0$ such that*

(1.29) $$\int^{\infty} xF(\beta, x)dx < +\infty \ .$$

Proof. If y is such a solution, it is easily confirmed that

$$y(x) = y(b) + y'(b)(x - b) + \int_x^b (t - x)y(t)F(y, t)dt \ .$$

Since $y(b) > 0$ and $y'(b) < 0$, it follows that

$$y(a) \geqq y(x) \geqq \int_x^b (t - x)y(t)F(y, t)dt \geqq \alpha \int_x^b (t - x)F(\alpha, t)dt \ ,$$

where $\lim y(x) = \alpha > 0$. This shows that condition (1.29) is necessary.

To show sufficiency, we consider the integral equation

(1.30) $$y(x) = \alpha + \int_x^{\infty} (t - x)y(t)F(y, t)dt \ ,$$

and suppose that β is a positive constant such that (1.29) holds. Then we can find a point $a \geqq x_0$ such that for all $x \geqq a$, we have

$$\int^{\infty} (t - x)F(\beta, t)dt < \frac{1}{2} \ .$$

We define a sequence of functions $\{y_k(x)\}$ by

(1.31) $$\begin{cases} y_0(x) = \alpha \\ y_{k+1}(x) = \alpha + \int_x^{\infty} (t - x)y_k(t)F(y_k, t)dt \\ k = 0, 1, 2, \cdots . \end{cases}$$

If we choose α such that $0 < \alpha < \beta/2$, we see that $0 < \alpha < y_k(x)$ and $y_0(x) = \alpha < \beta$. By assuming $y_k(x) < \beta$ we find that

$$\begin{aligned} y_{k+1}(x) &= \alpha + \int_x^\infty (t - x)y_k(t)F(y_k, t)dt \\ &< \frac{\beta}{2} + \beta\int_x^\infty (t - x)F(\beta, t)dt < \beta\ . \end{aligned}$$

Hence induction shows that $0 < \alpha \leqq y_k(x) < \beta$ for all k. Moreover, if x_1 and x_2 are any two points such that $a \leqq x_1 < x_2 < \infty$, then, from (1.31), we have

$$\begin{aligned} |\, y_{k+1}(x_2) - y_{k+1}(x_1) \,| &\leqq |\, x_2 - x_1 \,| \left\{\int_{x_1}^{x_2} y_kF(y_k, t)dt + \int_{x_1}^\infty y_kF(y_k, t)dt\right\} \\ &= |\, x_2 - x_1 \,| \int_{x_1}^\infty y_k(t)F(y_k, t)dt\ . \end{aligned}$$

In view of the uniform boundedness of $\{y_k\}$ and (1.29), it follows that the sequence is equi-continuous also. Since $F(u, x) < F(v, x)$ whenever $0 < u < v < \infty$, it follows from the assumption $y_{k+1} > y_k$ that

$$y_{k+2}(x) - y_{k+1}(x) = \int_x^\infty (t - x)[y_{k+1}F(y_{k+1}, t) - y_kF(y_k, t)]dt > 0\ .$$

This, together with the fact that $y_1 > y_0$, shows that $\{y_k(x)\}$ is a monotonically increasing sequence. We can therefore find a uniformly converging subsequence whose limit function $y(x)$ is the solution of equation (1.30).

It remains to show that the solution of (1.31) so obtained is indeed of class $C^2[a, \infty)$ and satisfies (1.1). To this end, we observe that, for $h > 0$,

$$\begin{aligned} \left|\frac{y(x + h) - y(x)}{h} + \int_x^\infty y(t)F(y, t)dt\right| &\leqq \int_x^{x+h} \left|\frac{x - t}{h}\right| y(t)F(y, t)dt \\ &\leqq \int_x^{x+h} y(t)F(y, t)dt\ . \end{aligned}$$

A corresponding inequality holds for $h < 0$. The solution y of (1.31) being continuous in $[a, \infty)$, it follows that

$$y'(x) = -\int_x^\infty y(t)F(y, t)dt\ .$$

In a similar manner, we can show that $y'' = yF(y, x)$, and the conclusion follows.

COROLLARY. *Equation* (1.1) *has proper solutions which ultimately decrease monotonically to zero if, and only if, for each* $\beta > 0$

$$\int^{\infty} xF(\beta, x)dx = +\infty \ . \tag{1.32}$$

Proof. We note that Theorem 1.1 assures the existence of a positive solution of (1.1) which is asymptotically equivalent to either a positive constant or zero, and that Theorem 1.2 gives a condition which is both necessary and sufficient for the former to hold, it follows that (1.32) is both necessary and sufficient for a solution to decrease to zero. The necessity can also be shown directly by the following simple argument.

If $y(x) \to 0$ as $x \to \infty$, we can choose a value $a \geqq x_0$ such that $y(x) < \lambda$ if $x > a$ and $y(a) = \lambda$, where λ is a positive constant. Writing (1.1) in the form

$$\begin{aligned}\lambda = y(a) &= y(b) + y'(b)(a-b) + \int_a^b (t-a)y(t)F(y, t)dt \\ &\leqq y(b) + y'(b)(a-b) + \lambda \int_a^b (t-a)F(\lambda, t)dt \ ,\end{aligned}$$

where b is a number in (a, ∞). By Lemma 1.5, we can make $|y'(b)(a-b)|$ arbitrarily small by taking b large enough. Since $y(b) \to 0$ for $b \to \infty$, we can thus choose a b such that

$$|y(b) + y'(b)(a-b)| < \frac{\lambda}{2} .$$

Hence,

$$\frac{1}{2} < \int_a^b tF(\lambda, t)dt < \int_a^{\infty} tF(\lambda, t)dt \ .$$

Since a can be taken arbitrarily large, the result follows.

2. In this section we consider positive proper solutions of (1.1) which are convex and increasing. We begin with a necessary condition for the existence of such a solution, which is valid if hypothesis (iii) is replaced by the nonlinearity condition (iv) $u^{-2\varepsilon}F(u, x)$ is a strictly increasing function of u for each $x \geqq x_0$ and some positive constant ε.

THEOREM 2.1. *If $F(y, x)$ satisfies hypothesis* (iv) *instead of* (iii), *and if* (1.1) *has positive, convex increasing proper solutions, then*

$$\int^{\infty} [x^{-2\varepsilon}F(\beta x, x)]^{1/2+\varepsilon}dx < +\infty \tag{2.1}$$

for some $\beta > 0$.

Proof. Let y be a positive, convex increasing proper solution of (1.1), then $y(x) > \beta x$ for $\beta > 0$ and some $x \geqq x_0$. Let

$$w(x) = y(x)y'(x) \tag{2.2}$$

so that by (iv)

$$\begin{aligned} w' &= (y')^2 + y^2 F(y, x) \\ &= (y')^2 + y^2 \frac{F(y, x)}{F(\beta x, x)} F(\beta x, x) \\ &> (y')^2 + y^{2+2\varepsilon} G(x) \\ &= yy'[y'y^{-1} + G(x)y^{1+2\varepsilon}(y')^{-1}] , \end{aligned} \tag{2.3}$$

where $G(x) = (\beta x)^{-2\varepsilon} F(\beta x, x)$. If we set $r = (1 + \varepsilon)/(2 + \varepsilon)$ and $s = (2 + \varepsilon)^{-1}$, then, $r, s > 0$ and $r + s = 1$. With the help of the inequality [4, p. 37]

$$rA + sB > A^r B^s , \tag{2.4}$$

where we have set

$$y'y^{-1} = rA$$

and

$$G(x)y^{1+2\varepsilon}(y')^{-1} = sB ,$$

we find that

$$w'w^{-\alpha-1} > \rho[x^{-2\varepsilon} F(\beta x, x)]^{1/2+\varepsilon} , \tag{2.5}$$

where $\rho =$ constant and $0 < \alpha = \varepsilon(2 + \varepsilon)^{-1} < 1$. We now define

$$h(x) = \rho \int_{x0}^{x} [x^{-2\varepsilon} F(\beta x, x)]^{1/2+\varepsilon} dx , \tag{2.6}$$

and

$$H(x) = \frac{1}{\alpha}[w(x)]^{-\alpha} + h(x) ,$$

then (2.5) becomes

$$H'(x) < 0 .$$

The positive function H is thus necessarily decreasing for sufficiently large x and must ultimately tend to some finite limit $\lambda^2 \geqq 0$. Since $w^{-\alpha}$ is bounded for all $x \geqq x_0$, we conclude that $h(x)$ must ultimately be bounded also. This proves our assertion.

In the case of the special equation

$$y'' = Q(x)y^{2n+1} , \tag{2.7}$$

where Q is a nonnegative continuous function in $[x_0, \infty)$, Theorem 2.1 reduces to

COROLLARY 3.1. *A necessary condition for equation* (2.7) *to have positive convex increasing proper solutions is that*

$$\int^{\infty} [Q(x)]^{1/n+2} dx < +\infty \ . \tag{2.8}$$

With slight changes, the technique used in the proof of Theorem 2.1 will yield the following more general result:

THEOREM 2.2. *If $F(y, x)$ satisfies hypothesis* (iv) *instead of* (iii), *and if equation* (1.1) *has positive, convex increasing proper solutions, then there is some constant $\beta > 0$ such that*

$$\int^{\infty} x^{2s-\delta-1}[F(\beta x, x)]^s dx < +\infty \ , \tag{2.9}$$

where δ and s are any two positive constants which satisfy

$$\begin{cases} 0 < s < 1 \\ \delta + 2s \leqq 1 \\ \delta + 1 \leqq 2s(1+\varepsilon) \ . \end{cases} \tag{2.10}$$

Proof. If y is a positive, convex increasing proper solution of (1.1), then there is some $\beta > 0$ such that

$$y(x) > \beta x \ , \qquad y'(x) > \beta \ , \qquad \text{for all } x > x_0 \ . \tag{2.11}$$

From (2.3) and inequality (2.4), we see that

$$\begin{aligned} \frac{w'}{w} &> y'y^{-1} + G(x)y^{1+2\varepsilon}(y')^{-1} \\ &> \left(\frac{1}{r}\right)^r \left(\frac{1}{s}\right)^s [G(x)]^s (y')^{r-s} y^{s(1+2\varepsilon)-r} \ . \end{aligned}$$

Hence, for any $\delta > 0$,

$$w^{-1-\delta}w' > k[G(x)]^s (y')^{1-2s-\delta} y^{2s(1+\varepsilon)-\delta-1} \ , \tag{2.12}$$

where $k =$ constant. If moreover, s and δ are so chosen as to satisfy condition (2.10), then the exponents of y and y' in the inequality (2.12) above are both nonnegative. Combining this inequality with (2.11), and using the fact that $G(x) = (\beta x)^{-2\varepsilon} F(\beta x, x)$, we obtain

$$w^{-1-\delta}w' > \rho x^{2s-\delta-1}[F(\beta x, x)]^s \ , \tag{2.13}$$

for all $x \geqq x_0$, and $\rho =$ constant.

As in Theorem 2.1, we now define

$$h^*(x) = \rho\int_{x_0}^{x} x^{2s-\delta-1}[F(\beta x, x)]^s dx$$

and

$$H^*(x) = \frac{1}{\delta}[w(x)]^{-\delta} + h^*(x) .$$

It follows from (2.13) that

$$\frac{d}{dx}H^*(x) < 0 ,$$

and we thus conclude, as in Theorem 2.1, that $h^*(x)$ is necessarily bounded. This completes the proof.

It is easily confirmed that for $\delta = \varepsilon(2+\varepsilon)^{-1}$ and $s = (2+\varepsilon)^{-1}$, condition (2.10) is satisfied, and (2.9) reduces to (2.1) so that Theorem 2.1 is indeed a special case of Theorem 2.2. If we apply Theorem 2.2 to equation (2.7), we obtain the following extension of Corollary 2.1:

COROLLARY 2.2. *If δ and s are any two positive constants for which condition* (2.10) *holds, and if equation* (2.7) *has positive, convex increasing proper solutions, then*

$$\int^{\infty} x^{\lambda}[Q(x)]^s dx < +\infty , \tag{2.14}$$

where $\lambda = 2s(n+1) - \delta - 1$.

We will now consider the problem of existence of positive increasing proper solutions of (1.1) having specified asymptotic forms. The simplest case is that of finding a solution y such that $y(x) \sim \alpha x$, where $\alpha > 0$.

THEOREM 2.3. *Equation* (1.1) *has positive proper solutions y of the form*

$$y(x) \sim \alpha x , \qquad \alpha > 0 , \tag{2.15}$$

if, and only if, there exists a positive constant β such that

$$\int^{\infty} xF(\beta x, x)dx < +\infty . \tag{2.16}$$

We write $y(x) = xu(t)$, where $t = 1/x$. The function $u(t)$ will then have a constant limit if t decreases to zero. Making the necessary substitutions in equation (1.1) we obtain

$$\frac{d^2u}{dt^2} = ut^{-4}F\left(\frac{u}{t}, \frac{1}{t}\right) = uG(u, t) \tag{2.17}$$

for $u(t)$. Since

$$\int_0^a tG(\beta, t)dt = \int_0^a F\left(\frac{\beta}{t}, \frac{1}{t}\right)\frac{dt}{t^3} = \int_{1/a}^\infty xF(\beta x, x)dx ,$$

Theorem 2.3 will be a consequence of the following result (which we formulate in terms of x, y and F rather t, u and G):

THEOREM 2.4. *If $F(y, x)$ is continuous for $0 < x < b$ and otherwise satisfies hypotheses* (i), (ii) *and* (iii), *then equation* (1.1) *will have solutions which are continuous in some interval $[0, a)$ $(0 < a < b)$ and decrease to a positive constant as x decrease to zero if, and only if, there exists a constant $\beta > 0$ such that*

$$\int_0^a xF(\beta, x)dx < +\infty . \tag{2.18}$$

Theorem 2.4 is in many respects analogous the Theorem 1.2, and its proof depends likewise on our solving a suitable integral equation. The integral equation in question is

$$y(x) = A + Bx - \int_0^a g(x, t)y(t)F(y, t)dt , \tag{2.19}$$

where $g(x, t)$ is the Green's function

$$g(x, t) = \begin{cases} x , & x \leqq t \leqq a , \\ t , & 0 \leqq t \leqq x . \end{cases}$$

To show that condition (2.18) is necessary for the existence of a solution y with the required properties, we note that $y(x)$ must satisfy the integral equation

$$y(x) = A_1 + B_1x - \int_\varepsilon^a g(x, t)y(t)F(y, t)dt , \tag{2.20}$$

where $0 < \varepsilon < a$, and A_1 and B_1 are determined from the conditions

$$y(\varepsilon) = A_1 + B_1\varepsilon$$
$$y'(a) = B_1 - \int_\varepsilon^a y(t)F(y, t)dt .$$

Since $y'(a) > 0$, B_1 must be positive. In view of the fact that

$$y(x) \geqq \lim_{x\to 0} y(x) = A > 0 ,$$

it thus follows from (2.20) that

$$A\int_\varepsilon^x tF(A, t)dt \leqq A_1 + B_1 a \ ,$$

and this implies (2.18).

To show that (2.18) is also sufficient, we solve the integral equation (2.19) by the iteration

$$(2.21) \qquad \begin{cases} y_0(x) = A \\ y_{k+1}(x) = A + Bx - \int_0^a g(x, t) y_k(t) F(y_k, t) dt \\ k = 0, 1, 2, \cdots , \end{cases}$$

where $A = \beta/2$, $B = \beta/2a$, and the value a is chosen so that

$$\int_0^a xF(\beta, x) \leqq \frac{1}{2} \, .$$

The possibility of choosing such a value of a follows from (2.18). If $0 \leqq y_k(x) \leqq \beta$, we have

$$\int_0^a g(x, t) y_k(t) F(y_k, t) dt \leqq \beta \int_0^a tF(\beta, t) dt \leqq \frac{\beta}{2}$$

and thus, by (2.21),

$$y_{k+1}(x) \geqq A + Bx - \frac{\beta}{2} = \frac{\beta x}{2a} \geqq 0$$

Moreover,

$$y_{k+1}(x) \leqq A + Bx = \frac{\beta}{2}\left(1 + \frac{x}{a}\right) \leqq \beta \, .$$

It follows that $0 \leqq y_{k+1}(x) \leqq \beta$. Since $y_0(x) = \beta/2$, all functions $y_k(x)$ of the sequence (2.21) satisfy these inequalities.

The rest of the convergence proof for the iteration (2.21) is exactly the same as the corresponding argument used in the proof of Lemma 1.3.

COROLLARY 2.4. *Under the hypotheses of Theorem* 2.4, *equation* (1.1) *will have solutions which are continuous in* $[0, a)$ *and decrease to zero for* $x \to 0$ *if, and only if, there exists a positive constant* β *such that*

$$(2.22) \qquad \int_0^a xF(\beta x, x) dx < +\infty \, .$$

With the help of the transformation $y(x) = xu(t)$, where $t = 1/x$, and equation (2.17), we have

$$\int_{1/a}^{\infty} tG(\beta, t)dt = \int_{1/a}^{\infty} F\left(\frac{\beta}{t}, \frac{1}{t}\right)\frac{dt}{t^3}$$
$$= \int_0^a xF(\beta x, x)dx .$$

Hence, if $u(t)$ is any positive solution of (2.17) which decreases monotonically to a positive constant as $t \to \infty$, then $y(x) = xu(x)$ will be the desired solution of (1.1) in $[0, a)$. By Theorem 1.2, a necessary and sufficient condition for (2.17) to have such solutions is that

$$\int^{\infty} tG(\beta, t)dt < +\infty ,$$

for some $\beta > 0$, and the result follows from (2.23).

We will now consider the following more general question: Let v be a given positive convex increasing function of class $C^2[a, \infty)$. The problem is to determine whether equation (1.1) has positive proper solutions which are asymptotically equivalent to v. To answer this question we introduce a Liouville type transformation

$$\begin{cases} y = uv \\ x = x(t) , \end{cases} \tag{2.24}$$

where the new independent variable t is defined by

$$t = \int_x^{\infty} [v(s)]^{-2} ds . \tag{2.25}$$

Under this transformation, the interval $[a, \infty)$ is mapped onto $(0, b]$, and a computation shows that u must satisfy the equation

$$\frac{d^2u}{dt^2} = u\left[\dot{x}^2 F(uv, t) - \frac{1}{2}\{x, t\}\right] = uG(u, t) , \tag{2.26}$$

where $\{x, t\}$ denotes the Schwarzian differential operator

$$\{x, t\} = \frac{d}{dt}\left(\frac{\ddot{x}}{\dot{x}}\right) - \frac{1}{2}\left(\frac{\ddot{x}}{\dot{x}}\right)^2 .$$

In order that $y(x) \sim cv(x)$, $u(x)$ must therefore be a positive solution of (2.26) which decreases to a positive constant for $t \to 0$.

We observe that if the given function v were convex decreasing rather than convex increasing, the problem of determining whether (1.1) has proper solutions of this type can be treated in the same way. However, the new variable t in the Liouville transformation will now be given by

$$t = \int_a^x [v(s)]^{-2} ds ,$$

where v is now a positive, convex decreasing function of class $C^2[a, \infty)$. Since the procedure is the same in either case, we need only consider the convex increasing case.

To simplify matters we shall further restrict ourselves to those convex functions $v(x)$ for which the positive continuous function $p(x)$ defined by

$$p(x) = \frac{v''(x)}{v(x)} \tag{2.27}$$

is such that $[F(uv, x) - p(x)]$ is ultimately of one sign. That is to say, we assume that either (1) $G(u, x) < 0$ for all $u > 0$ and $0 < t \leqq a < b$, or (2) $G(\beta, t) > 0$ for some $\beta > 0$ and all sufficiently small t.

If case (1) holds, then the Atkinson-Nehari criterion [6, Theorem I] shows that a necessary and sufficient condition for the existence of a positive solution $u(t)$ which decreases to a positive constant as t decreases to zero, is that

$$0 \leqq -\int_0^a tG(\mu, t)dt < +\infty \tag{2.28}$$

for some constant $\mu > 0$.

On the other hand, if (2) holds, then by Theorem 2.4, the corresponding necessary and sufficient condition is the existence of some positive constant β for which

$$\int_0^a tG(\beta, t)dt < +\infty \ . \tag{2.29}$$

Expressed in terms of x and $v(x)$, both (2.28) and (2.29) may be combined into a single condition:

$$\int^\infty v^2(x)\int_x^\infty \frac{ds}{v^2(s)}\left| F(\beta v, x) - \frac{v''}{v}\right| dx < +\infty \ . \tag{2.30}$$

If we regard (2.27) as a linear homogeneous equation with $p(x)$ given, and that u and v are two linearly independent positive solutions whose Wronskian is negative, then one solution must be convex increasing and the other is convex decreasing. Moreover, if v denotes the increasing solution, then

$$u(x) = v(x)\int_x^\infty [v(s)]^{-2}ds$$

so that (2.30) may be written as

$$\int^\infty u(x)v(x)\,|\,F(\beta v, x) - p(x)\,|\,dx < +\infty \ .$$

We can now state the following result:

THEOREM 2.5. *Let $p(x)$ be a positive continuous function in $[x_0, \infty)$ and u and v be two linearly independent positive solutions of* (2.27). *If, moreover, $[F(\mu v, x) - p(x)]$ is either negative for all $\mu > 0$ or positive for some $\mu > 0$, then a necessary and sufficient condition for equation* (1.1) *to have positive, convex proper solutions y of the form*

$$y(x) \sim cv(x)\ , \qquad c > 0\ , \tag{2.31}$$

is that there is some $\beta > 0$ such that

$$\int^{\infty} u(x)v(x)\,|\,F(\beta v, x) - p(x)\,|\,dx < +\infty\ . \tag{2.32}$$

COROLLARY 2.51. *If $F(ux^{\alpha}, x) - \alpha(\alpha - 1)x^{-2}$ is ultimately of one sign, where $\alpha > 1$, then a necessary and sufficient condition for equation* (1.1) *to have positive proper solutions of the form*

$$y(x) \sim cx^{\alpha}\ , \qquad c > 0\ , \qquad \alpha > 1\ , \tag{2.33}$$

is that, for some $\beta > 0$,

$$\int^{\infty} x\,|\,F(\beta x^{\alpha}, x) - \alpha(\alpha - 1)x^{-2}\,|\,dx < +\infty\ . \tag{2.34}$$

Proof. If we let $p(x) = \alpha(\alpha - 1)x^{-2}$, then $u(x) = x^{1-\alpha}$ and $v(x) = x^{\alpha}$, and the result follows from (2.32).

COROLLARY 2.52. *If $F(ue^{\alpha x}, x) - \alpha^2$ is ultimately of one sign, where $\alpha > 0$, then equation* (1.1) *has positive proper solutions of the form*

$$y(x) \sim ce^{\alpha x}\ , \qquad \alpha, c > 0\ , \tag{2.35}$$

if, and only if, there exists some constant $\beta > 0$ such that

$$\int^{\infty} |\,F(\beta e^{\alpha x}, x) - \alpha^2\,|\,dx < +\infty\ . \tag{2.36}$$

As pointed out before, the Emden-Fowler equation

$$y'' = x^{\lambda}y^{n}\ , \qquad n > 1\ , \tag{1.2}$$

is a particular example of equation (1.1) with $F(y, x) = x^{\lambda}y^{n-1}$. We can therefore apply the results obtained here to investigate the existence and asymptotic behavior of proper solutions of this equation.

From Theorem 1.2, we see that a necessary and sufficient condition for equation (1.2) to have positive proper solutions which ultimately decrease to positive constants is that

$$\int^{\infty} x^{\lambda+1} dx < +\infty .$$

It follows that we must have

$$\lambda + 2 < 0 . \tag{2.37}$$

From Theorem 2.3 we find that equation (1.2) has positive proper solutions y of the form $y(x) \sim cx$, $c > 0$, if, and only if

$$\int^{\infty} x^{\lambda+n} dx < +\infty .$$

Hence, we obtain the condition

$$\lambda + n + 1 < 0 . \tag{2.38}$$

Corollary 2.51 shows that a necessary and sufficient condition for (1.2) to have positive proper solutions of the form $y(x) \sim cx^{\alpha}$, $\alpha > 1$, is that

$$\int^{\infty} x \,|\, \beta^{n-1} x^{\alpha(n-1)+\lambda} - \alpha(\alpha-1)x^{-2} \,|\, dx < +\infty .$$

This condition will be satisfied if, and only if $\beta^{n-1} = \alpha(\alpha - 1)$ and $\alpha(n-1) + \lambda = -2$. Thus, the required condition in this case will be

$$\alpha = -\frac{\lambda+2}{n-1} > 1 . \tag{2.39}$$

From Corollary 2.52, it is easy to see that equation (1.2) cannot have any proper solution which is exponential. Finally, suppose that $u(x)$ is *any* positive, convex increasing proper solution of the Emden-Fowler equation, then, by Corollary 2.1, it is necessary that

$$\int^{\infty} [Q(x)]^{2/n+3} dx = \int^{\infty} x^{2\lambda/n+3} dx < +\infty .$$

In other words, we must have

$$2\lambda + n + 3 < 0 . \tag{2.40}$$

Applying this inequality to the special equation

$$y'' = x^{-2} y^{n} , \qquad n > 1 ,$$

we find that it cannot have any proper solution which is convex and increasing. Moreover,

$$\int^{\infty} x^{\lambda+1} dx = \int^{\infty} x^{-4/n+3} dx = +\infty$$

so that, by the Corollary of Theorem 1.2, this equation has a decreasing proper solution through every point (a, A), $A > 0$, which decreases to zero as $x \to \infty$. (cf. [2], Chapter 7, Theorems 1 to 5).

An elementary example of the Emden-Fowler equation for which an explicit solution is known is the equation

$$y'' = 2x^{-6}y^3 .$$

It is easily confirmed that x^2 is a solution of this equation. If we set $\alpha = 2$ and $\beta = 1$ in (2.34) we find that the integral vanishes so that the condition of Corollary 2.51 is indeed satisfied.

If we assume moreover that $v(x)$ is a proper solution of (1.1), then (2.30) may be used to determine the possible existence of a second proper solution y distinct from v such that their ratio is asymptotically constant. Without loss of generality we may assume that $y(x) \geqq v(x)$ for each $x \geqq x_0$. A necessary and sufficient condition for the existence of such solutions is the boundedness of

$$\int^{\infty} v^2(x) \int \frac{dx}{v^2(x)} [F(\beta v, x) - F(v, x)]dx$$

for some $\beta > 1$.

A condition for the difference of two proper solutions to be asymptotically constant may be obtained as follows: Let $w(x)$ be a positive proper solution of (1.1), and we let a second proper solution y be of the form

$$y(x) = u(x) + w(x) ,$$

where $u \in C^2[a, \infty)$ and $u(x) \sim k$, $k > 0$. Differentiation shows that u must satisfy the equation

$$\begin{cases} u'' = G(u, x) , \\ G(u, x) = uF(u + w, x) + [F(u + w, x) - F(w, x)] . \end{cases}$$

In view of Theorem 1.2, this equation will have proper solutions which ultimately decrease to positive constants if, and only if, there exists some $\beta > 0$ such that

$$\int^{\infty} xG(\beta, x)dx < +\infty .$$

THEOREM 2.6. *Let $w(x)$ be a positive proper solution of* (1.1).

1. *A necessary and sufficient condition for the existence of a second positive proper solution y such that $y(x)/w(x) \sim k$, $k > 0$, is that*

$$\int^{\infty} w^2(x) \int^{x} \frac{ds}{w^2(s)} [F(\beta w, x) - F(w, x)]dx < +\infty$$

for some $\beta > 1$.

2. *A necessary and sufficient condition for the existence of a second solution* y *such that* $y(x) - w(x) \sim c$, $c > 0$, *is that for some* $\mu > 0$

$$\int^{\infty} x[(\mu + w)F(\mu + w, x) - F(w, x)]dx < +\infty .$$

All results obtained thus far concern the asymptotic behavior of proper solutions, but the question of positive convex solutions having finite asymptotes is also of interest. As the following result shows, equation (1.1) always has such discontinuous solutions.

THEOREM 2.7. *If* F *satisfies hypothesis* (iv) *instead of* (iii), *and if* A *is an arbitrary real number and* a *and* δ *are positive; then there exists a solution* y *of* (1.1) *with* $y(a) = A$, *which is not continuous in* $(a, a + \delta)$.

Proof. Since $y(x)$ is convex, the value of $y(a + \delta)$ can be made arbitrarily large by a sufficiently large choice of $y'(a)$. We may accordingly assume that $y(a + \delta) > 1$. Let c be the point in $(a, a + \delta)$ where $y(c) = 1$, and we recall that, for $y > 1$, $F(y, x) > y^{2\varepsilon}F(1, x)$. It follows from (1.1) that

$$[y'(x)]^2 = \alpha^2 + 2\int_a^x y(t)F(y, t)y'(t)dt , \tag{2.41}$$

where $y'(a) = \alpha$, and $a < x < a + \delta$. If $0 < \rho \leqq F(1, x)$ for $x \in [a, a + \delta]$ and $x > c$, we then have

$$\begin{aligned}
2\int_a^x yy'F(y, t)dt &> 2\int_c^x yy'F(y, t)dt \\
&\geqq 2\int_0^x y^{1+2\varepsilon}y'F(1, t)dt \\
&\geqq 2\rho\int_c^x y^{1+2\varepsilon}y'dt \\
&= \frac{\rho}{1+\varepsilon}\{[y(x)]^{2+2\varepsilon} - 1\} .
\end{aligned}$$

For $x \in [a, c]$, this holds trivially. Choosing α^2 large enough so that $\alpha^2 > \rho(1 + \varepsilon)^{-1}$, we conclude from (2.41) that

$$[y'(x)]^2 \geqq \alpha^2 + \frac{\rho}{1+\varepsilon}[y^{2+2\varepsilon} - 1] ,$$

or, with $\beta^2 = \alpha^2 - \rho(1 + \varepsilon)^{-1}$ and $\lambda^{2+2\varepsilon} = \rho(1 + \varepsilon)^{-1}$,

$$[y'(x)]^2 \geqq \beta^2 + (\lambda y)^{2+2\varepsilon} .$$

If $y(x)$ is continuous in $[a, a+\delta]$, $y'(x)$ necessarily remains positive. Hence

$$\int_0^\infty \frac{dt}{\sqrt{\beta^2 + (\lambda t)^{2+2\varepsilon}}} = \int_{y(a)}^{y(b)} \frac{dt}{\sqrt{\beta^2 + (\lambda t)^{2+2\varepsilon}}} \geqq (b-a)\,,$$

where $b = a + \delta$, and this reduces to

$$(b-a) \leqq \beta^{-\varepsilon/1+\varepsilon} \int_0^\infty \frac{dt}{\sqrt{1 + (\lambda t)^{2+2\varepsilon}}}\,.$$

Since the integral exists, this provides a bound for the right end point of the interval of continuity. In view of the fact that $\beta^2 = \alpha^2 - \rho(1+\varepsilon)^{-1}$, it is also obvious that $(b-a)$ can be made arbitrarily small by a sufficiently large choice of $\alpha = y'(a)$. This completes the proof.

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FREE EXTENSIONS OF BOOLEAN ALGEBRAS

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Introduction. This paper is concerned with the problem of imbedding a Boolean algebra B into an α-complete Boolean algebra B^* in such a way that certain homomorphisms of B can be extended to B^*. We investigate two such imbeddings which arose naturally from the consideration of the work of Rieger and Sikorski in [5] and [7]. In [5] Rieger proved the existence of a certain class of free Boolean algebras and investigated their representability by α-fields of sets. Rieger's theorem on the existence of "the free α-complete Boolean algebra on m generators" is equivalent to the following statement: Every free Boolean algebra B can be imbedded in an α-complete Boolean algebra B^* such that every homomorphism of B into an α-complete Boolean algebra C can be extended to an α-homomorphism of B^* into C. The question now arises: Does this result hold for an arbitrary Boolean algebra B which is not necessarily free? If such an imbedding exists, we call B^* the *free α-extension of B.*

In [7], Sikorski gave a characterization of all the σ-regular extensions of a Boolean algebra B. To obtain this characterization, he first proved that B can be imbedded as a σ-regular subalgebra of a σ-complete Boolean algebra B^* such that every σ-homomorphism of B into a σ-complete Boolean algebra C can be extended to a σ-homomorphism of B^* into C. We call B^* the *free σ-regular extension of B.*

In § 2 of this paper we prove that the free α-extension B_α of B exists uniquely for every Boolean algebra B and every infinite cardinal number α. In § 3 we investigate the representability of B_α by an α-field of sets. We first prove that B_α is isomorphic to an α-field of sets if and only if it is α-representable. A corollary to this result is that the free σ-extension B_σ of an arbitrary Boolean algebra B is isomorphic to a σ-field of sets. The problem of characterizing those Boolean algebras B for which B_α is α-representable for $\alpha \geqq 2^{\aleph_0}$ is also discussed. In § 4 we investigate the α-regular extensions of Boolean algebras for an arbitrary cardinal number α. Sikorski's results on the σ-regular extensions depend on the Loomis-Sikorski theorem which does not hold for uncountable cardinal numbers. We use our results on the free α-extension B_α of B to prove the existence of the free α-regular extension and to give a characterization of the α-regular

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extensions of B.

Our result on the existence of the free α-regular extension of B is a special case of a more general result of Kerstan [3], but it is obtained here by a different method. We also learned through a written communication from Professor Sikorski that he also proved the same result and his proof will appear in [10]. Sikorski's proof is similar to ours; however he works with the free α-complete Boolean algebras while we work with the free α-extension of B (see Theorem 4.1 below). The characterization of the α-regular extensions of B given in Theorem 4.2 does not appear in [3] or [10]; the free α-extensions of Boolean algebras have not been considered in either of these papers.

1. Preliminaries. Throughout this paper, the product (=greatest lower bound) of a set $\{x_t: t \in T\}$ of elements of a Boolean algebra B will be denoted, whenever it exists, by $\prod_{t\in T} x_t$. If A is a subalgebra of B and $x_t \in A$ for every $t \in T$, then the set $\{x_t: t \in T\}$ may have two products, one taken in A and the other in B; we denote these products, whenever they exist, by $\prod^A_{t\in T} x_t$ and $\prod^B_{t\in T} x_t$ respectively. The complement of an element x of B will be denoted by $\bar{x}$, and the symbol "0" will designate the zero element of B.

Definitions of the more familiar Boolean concepts which are not given in this section can be found in [9] or [2]. A homomorphism h of a Boolean algebra A into a Boolean algebra B is an *α-homomorphism* if it preserves α-sums (hence α-products) whenever they exist in A. Equivalently ([9], § 22), h is an α-homomorphism of A into B if and only if $\prod_{x\in S} h(x) = 0$ for every subset S of A such that $|S| \leqq \alpha$ and $\prod_{x\in S} x = 0$. h is an *α-isomorphism* if it is a one-to-one α-homomorphism. h is a *complete homomorphism* (*complete isomorphism*) if it is an α-homomorphism (α-isomorphism) for every infinite cardinal number α. A subalgebra A of a Boolean algebra B is *α-regular* if the injection mapping of A into B is an α-isomorphism. Equivalently, A is an α-regular subalgebra of B if and only if $\prod^B_{x\in S} x = 0$ for every subset S of A such that $|S| \leqq \alpha$ and $\prod^A_{x\in S} x = 0$. A is a *regular* subalgebra of B if it is α-regular for every infinite cardinal number α.

A Boolean algebra B is *free on m generators* (m is an arbitrary cardinal number) if it is generated by a subset E with cardinality m and with the property that every mapping of E into a Boolean algebra C can be extended to a homomorphism of B into C. All free Boolean algebras on m generators are isomorphic ([9], § 14) and will be denoted throughout this paper by A_m. An α-complete Boolean algebra B is a *free α-complete Boolean algebra on m generators* if it is α-generated by a subset E with cardinality m and with the property that every mapping of E into an α-complete Boolean algebra C can be extended to an α-homomorphism of B into C. All free α-complete Boolean algebras on m generators are isomorphic [5] and will be denoted here by A_m^α.

For every Boolean algebra B and every infinite cardinal number α, there exists an α-complete Boolean algebra B^* and a complete isomorphism h of B into B^* such that $h(B)$ α-generates B^* ([9], § 36). This Boolean algebra B^*, which is unique up to isomorphisms, is called the *normal α-completion* (also minimal α-extension) *of* B and will be denoted here by B^α. If B^* is complete and h is a complete isomorphism of B into B^* such that B^* is completely generated by $h(B)$, then B^* is called the *normal completion* (also, minimal extension) *of* B and will be denoted by B^∞. When dealing with the normal α-completion (completion) of B, we shall usually identify B with $h(B)$ and thus consider B as a regular subalgebra of both B^α and B^∞.

The *Stone space* (=Boolean space) of a Boolean algebra B is the compact, Hausdorff, totally disconnected space whose open-and-closed subsets, ordered by set inclusion, form a Boolean algebra isomorphic to B. For every Boolean algebra B and every infinite cardinal number α, $S(B)$ will denote the Stone space of B, $F_0(B)$ the Boolean algebra of open-and-closed subsets of $S(B)$, and $F_\alpha(B)$ the smallest α-field of subsets of $S(B)$ containing $F_0(B)$.

A Boolean algebra B is called *α-representable* if it is isomorphic to an α-regular subalgebra of a quotient algebra F/I, where F is an α-field of sets and I is an α-ideal of F. If B is α-complete, then this definition reduces to: B is α-representable if and only if it is an α-homomorph of an α-field of sets. There are α-complete (even complete) Boolean algebras which are not α-representable for $\alpha \geqq 2^{\aleph_0}$ ([9], § 29). However, for the case $\alpha = \aleph_0$, we have the Loomis-Sikorski theorem ([9], § 29): *Every Boolean algebra is σ-representable.*

2. Free α-extensions.

DEFINITION 2.1. An α-complete Boolean algebra B^* is called a *free α-extension* of the Boolean algebra B if B^* is α-generated by a subalgebra B_0 isomorphic to B such that every homomorphism of B_0 into an α-complete Boolean algebra C can be extended to an α-homomorphism of B^* into C.

We shall show in this section that for every Boolean algebra B and every infinite cardinal number α, the free α-extension of B exists and is unique up to isomorphisms. We denote the free α-extension of B by B_α, and we shall consider B as a subalgebra of B_α, thus identifying it with the subalgebra B_0 of Definition 2.1.

The following lemma follows immediately from Definition 2.1.

LEMMA 2.1. *Let A_m be the free Boolean algebra on m generators and A_m^α the free α-complete Boolean algebra on m generators. Then A_m^α is the free α-extension of A_m.*

LEMMA 2.2. *Let I be an ideal of A_m and I^* the smallest α-ideal of A_m^α containing I. Then $I^* \cap A_m = I$.*

Proof. Let h be the canonical homomorphism of A_m onto A_m/I and let i be an isomorphism of A_m/I onto $F_0(A_m/I)$. Then the homomorphism ih can be extended to an α-homomorphism h^* of A_m^α into $F_\alpha(A_m/I)$. And the kernel of h^* is an α-ideal which contains I^* and intersects A_m in I. Hence $I^* \cap A_m = I$.

THEOREM 2.1. *For every Boolean algebra B and every infinite cardinal number α, the free α-extension of B exists and is unique up to isomorphisms.*

Proof. Let $|B| = m$. Then there exists an ideal I of A_m such that A_m/I is isomorphic to B. Let I^* be the smallest α-ideal of A_m^α containing I. We shall show that A_m^α/I^* is a free α-extension of B.

Lemma 2.2 shows that the subalgebra A_m/I^* of A_m^α/I^* is isomorphic to B. And since A_m^α is α-generated by A_m, it follows that A_m/I^* α-generates A_m^α/I^*. Thus it only remains to show that homomorphisms of A_m/I^* can be extended to A_m^α/I^*. Let h be a homomorphism of A_m/I^* into an α-complete Boolean algebra C. Let f be the canonical α-homomorphism of A_m^α onto A_m^α/I^* and denote the restriction of f to A_m by f'. Then the homomorphism $g = hf'$ has an extension g^* which is an α-homomorphism of A_m^α into C. Since both I^* and the kernel of g^* are α-complete ideals containing I, it follows that I^* is contained in the kernel of g^*. We now define the mapping h^* by:

$$h^*(f(x)) = g^*(x),\ x \in A_m^\alpha .$$

Then h^* is the desired extension of h; hence A_m^α/I^* is a free α-extension of B.

The uniqueness of the free α-extension of B follows from the standard argument used to show that all free Boolean algebras on the same number of generators are isomorphic. Indeed, suppose that B has two free α-extensions B_1 and B_2. Let i be an isomorphism of the subalgebra B of B_1 onto the subalgebra B of B_2. Then i can be extended to an α-homomorphism of B_1 onto B_2 and the isomorphism i^{-1} can be extended to an α-homomorphism i_2 of B_2 onto B_1. Let $B^* = \{x \in B_1 \colon i_2(i_1(x)) = x\}$. Then B^* is an α-complete, α-regular subalgebra of B_1 containing B. Hence $B^* = B_1$, and i_1 is an isomorphism of B_1 onto B_2.

LEMMA 2.3. *Let h be a homomorphism of a Boolean algebra B into an α-complete Boolean algebra C. Then the extension of h to an α-homomorphism of B_α into C is unique.*

Proof. Suppose h has two extensions h_1 and h_2. Let $B^* = \{x \in B_\alpha: h_1(x) = h_2(x)\}$. Then B^* is an α-complete, α-regular subalgebra of B_α containing B. Hence $B^* = B_\alpha$, and $h_1 = h_2$.

A slight modification of the proof of Theorem 2.1 yields the following result.

LEMMA 2.4. *If I is an ideal of B, then the free α-extension of B/I is isomorphic to B_α/I^*, where I^*, is the smallest α-ideal of B_α containing I.*

LEMMA 2.5. *If A is a subalgebra of B, then A_α is isomorphic to the α-complete, α-regular subalgebra A^* of B_α α-generated by A.*

Proof. We only need to show that if h is a homomorphism of A into an α-complete Boolean algebra C, then h can be extended to an α-homomorphism of A^* into C. Thus, we imbed C into its normal completion C^∞. Then, by a known result ([9], § 33.1), h can be extended to a homomorphism h_1 of B into C^∞. Furthermore, h_1 can be extended to an α-homomorphism h_2 of B_α into C^∞. Let h^* be the restriction of h_2 to A^*. Then, since A^* is an α-regular subalgebra of B_α, h^* is an α-homomorphism of A^* into C^∞, and the proof will be complete once we show that $h^*(A^*)$ is contained entirely in C. Since both $h^*(A^*)$ and C are α-complete, α-regular subalgebras of C^∞, their intersection $h^*(A^*) \cap C$ is also an α-complete, α-regular subalgebra of C^∞. And since $h^*(A^*)$ is α-generated by $h(A)$, it follows that $h^*(A^*) = h^*(A^*) \cap C$. Hence $h^*(A^*) \subset C$, and the proof is now complete.

3. **Representability by α-field of sets.** In investigating the representability problem of the free α-extensions of Boolean algebras, the following two natural questions arise: When is the free α-extension B_α of a Boolean algebra B isomorphic to an α-field of sets? And, when is B_α α-representable? The following theorem shows that these two questions are equivalent.

THEOREM 3.1. *For every Boolean algebra B and every infinite cardinal number α, there is an α-homomorphism j^* of B_α onto $F_\alpha(B)$ whose restriction to B is the canonical imbedding of B in $F_0(B)$. Moreover, j^* is one-to-one if and only if B_α is α-representable.*

Proof.[1] Let j be the canonical isomorphism of B onto $F_0(B)$ and

[1] This proof, which is considerably shorter than the one intended, is due to the referee.

extend j to an α-homomorphism j^* of B_α into $F_\alpha(B)$. Since $F_\alpha(B)$ is α-generated by $F_0(B)$, j^* is onto. Since B is a subalgebra of B_α, there is a continuous mapping λ of $S(B_\alpha)$ onto $S(B)$ such that for every $x \in B$, $\lambda^{-1}(j(x)) = i(x)$, where i is the canonical isomorphism of B_α onto $F_0(B_\alpha)$. Let k denote the homomorphism $E \to \lambda^{-1}(E)$, mapping the subsets of $S(B)$ to subsets of $S(B_\alpha)$. Then k is an α-isomorphism which maps $F_0(B)$ into $F_0(B_\alpha)$, since $k(j(x)) = i(x)$ for every $x \in B$. Consequently, k maps $F_\alpha(B)$ into $F_\alpha(B_\alpha)$. If B_α is α-representable, then $F_0(B_\alpha)$ is an α-retract of $F_\alpha(B_\alpha)$; that is, there is an α-homomorphism h of $F_\alpha(B_\alpha)$ onto $F_0(B_\alpha)$ whose restriction to B_α is the identity mapping. Then $i^{-1}hkj^*$ is an α-homomorphism of B_α onto itself which is an extension of the identity mapping on B. Thus, it follows from Lemma 2.3 that $i^{-1}hkj^*(x) = x$ for all $x \in B_\alpha$. Thus j^* is an α-isomorphism.

Since every Boolean algebra is σ-representable (the Loomis-Sikorski Theorem), the last theorem yields the following corollary which answers the representability question for the free σ-extensions of Boolean algebras.

COROLLARY 3.1. *For every Boolean algebra B, B_σ is isomorphic to the σ-field of sets $F_\sigma(B)$.*

The next theorem gives a strong necessary condition that a Boolean algebra B must satisfy in order for B_α to be α-representable when $\alpha \geqq 2^{\aleph_0}$.

LEMMA 3.1. *If B_α is α-representable, then so is every subalgebra and every homomorphic image of B.*

Proof. Let h be a homomorphism of B onto a Boolean algebra C. Imbed C into its normal α-completion C^α and extend h to an α-homomorphism of B_α onto C^α. Since B_α is α-representable, so is C^α. And since C is an α-regular subalgebra of C^α, C itself is α-representable. On the other hand, if A is a subalgebra of B, then it follows from Lemma 2.5 that A_α is α-representable. Hence A is α-representable.

DEFINITION 3.1. A Boolean algebra B is called *super-atomic* if every subalgebra and every homomorphic image of B is atomic.

THEOREM 3.2. *Let B be a Boolean algebra and $\alpha \geqq 2^{\aleph_0}$. If B_α is α-representable, then B is super-atomic.*

Proof. We shall first show that if B_α is α-representable, $\alpha \geqq 2^{\aleph_0}$, then B is atomic. Suppose B is not atomic. Then B has an element

x such that the principal ideal (x), when considered as a Boolean algebra, is atomless. Now the Boolean algebra (x) is isomorphic to a subalgebra A of B. For let P be a prime ideal of (x) and let $P^* = \{\bar{x}: x \in P\}$. Then it is not difficult to show that $A = P \cup P^*$ is a subalgebra of B isomorphic to the Boolean algebra (x). Since A is atomless, it has a subalgebra A' isomorphic to the free Boolean algebra on $\aleph_0$ generators ([1], § 1.7). And since B_α is α-representable, Lemmas 2.5 and 3.1 show that A' is α-representable also. This contradicts the fact that the free Boolean algebra on $\aleph_0$ generators is not α-representable if $\alpha \geqq 2^{\aleph_0}$. Thus we conclude that B is atomic.

The proof of the theorem now follows immediately. If C is a subalgebra of B, then, by Lemma 2.5, C_α is α-representable. Hence C is atomic. On the other hand, if C is a homomorphic image of B, then C_α is an α-homomorph of B_α. Thus C_α is α-representable, hence C is atomic.

Super-atomic Boolean algebras were discussed briefly in [4] and more recently in more detail by G. W. Day [1]. In particular, Day proved ([1], Theorem 16) the converse of Theorem 3.2. Day also gave the following characterization of super-atomic Boolean algebras: *A Boolean algebra B is super-atomic if and only if every subalgebra of B is atomic if and only if every homomorph of B is atomic.* A characterization of super-atomic Boolean algebras with ordered basis is given by Theorem 3.3 of [4].

Combining Day's result ([1], Theorem 16) with Theorem 3.2, we obtain:

THEOREM 3.3. *Let B be a Boolean algebra and $\alpha \geqq 2^{\aleph_0}$. Then B_α is α-representable if and only if B is super-atomic.*

If B is not super-atomic and $\alpha \geqq 2^{\aleph_0}$, then $F_\alpha(B)$ is not isomorphic to B_α; however; we shall now show that $F_\alpha(B)$ is the free α-extension of B "over the class of α-representable Boolean algebras." An α-complete, α-representable Boolean algebra B^* is called the *free α-representable extension of B* if B^* is α-generated by a subalgebra B_0 isomorphic to B such that every homomorphism of B_0 into an α-complete, α-representable Boolean algebra C can be extended to an α-homomorphism of B^* into C. We need the following result of Sikorski ([9], 31.1):

LEMMA 3.2. *Let A_m be the free Boolean algebra on m generators and α an infinite cardinal number. Then $F_\alpha(A_m)$ is the free α-representable extension of A_m.*

A slight modification of the proof of Theorem 2.1 shows the following:

THEOREM 3.4. *For every Boolean algebra B and every infinite cardinal number α, the free α-representable extension of B exists and is unique up to isomorphisms.*

The following theorem can be proved by an argument similar to the one used in the proof of Theorem 3.1.

THEOREM 3.5. *For every Boolean algebra B and every infinite cardinal number α, $F_\alpha(B)$ is the free α-representable extension of B.*

4. Free α-regular extensions.

DEFINITION 4.1. An α-complete Boolean algebra B^* is called an *α-regular extension* of the Boolean algebra B if B^* is α-generated by an α-regular subalgebra B_0 isomorphic to B. If, in addition, every α-complete homomorphism of B_0 into an α-complete Boolean algebra C can be extended to an α-homomorphism of B^* into C, then B^* is called a free *α-regular extension* of B.

σ-regular extensions of Boolean algebras were investigated by Sikorski [7]. In this section we investigate the α-regular extensions of Boolean algebras for an arbitrary infinite cardinal number α. We denote the free α-regular extension of B by B_α^* (its existence and uniqueness are proved in Theorem 4.1). Also, for every Boolean algebra B and every infinite cardinal number α, we define the two ideals I_α and J_α as follows: I_α is the smallest α-ideal of B_α containing all elements u such that $u = \prod_{t\in T}^{B_\alpha} x_t$, where $|T| \leqq \alpha$, each $x_t \in B$, and $\prod_{t\in T}^{B} x_t = 0$. The elements u will be called the generators of I_α. J_α is the smallest α-ideal of $F_\alpha(B)$ containing all the nowhere dense α-closed subsets of the Stone space of B. (A subset E of a topological space X is called *α-closed* if E is the intersection of at most α open-and-closed subsets of X.)

LEMMA 4.1. *Let B be a Boolean algebra and I an α-ideal of B_α such that*: (a) $I \supset I_\alpha$, (b) $I \cap B = (0)$. *Then B_α/I is an α-regular extension of B.*

Proof. Let h be the canonical α-homomorphism of B_α onto B_α/I and observe that h is an isomorphism of B onto the subalgebra B/I. Suppose that $|T| \leqq \alpha$ and, for each $t \in T$, $h(x_t) \in B/I$ such that $\prod_{t\in T}^{B/I} h(x_t) = 0$. Then

$$\prod_{t\in T}^{B_\alpha/I} h(x_t) = h\Big(\prod_{t\in T}^{B_\alpha} x_t\Big) = 0 ,$$

where the last equality follows from the fact that $\prod_{t\in T}^{B\alpha} x_t \in I_\alpha$ and condition (a) of the hypothesis. Thus B/I is an α-regular subalgebra of B_α/I. Furthermore, since B_α is α-generated by B and h is an α-homomorphism, it follows that B/I α-generates B_α/I. Hence B_α/I is an α-regular extension of B.

THEOREM 4.1. *Let B be a Boolean algebra and α an infinite cardinal number. Then the free α-regular extension B_α^* of B exists and is unique up to isomorphisms. Moreover, B_α^* is isomorphic to B_α/I_α.*

Proof. We shall first show that $I_\alpha \cap B = (0)$. Let B^α be the normal α-completion of B; thus B^α is α-generated by a regular subalgebra B_1 isomorphic to B. Let i be an isomorphism of B onto B_1 and observe that i is a complete isomorphism of B into B^α. Extend i to an α-homomorphism i^* of B_α into B^α and let u be a generator of I_α. Then $u = \prod_{t\in T}^{B\alpha} x_t$, where $|T| \leqq \alpha$, and $\prod_{t\in T}^{B} x_t = 0$. And

$$i^*(u) = \prod_{t\in T}^{B\alpha} i^*(x_t) = \prod_{t\in T}^{B\alpha} i(x_t) = \prod_{t\in T}^{B_1} i(x_t) = i\Big(\prod_{t\in T}^{B} x_t\Big) = 0 .$$

It follows from this that I_α is contained in the kernel J of i^*. And since $J \cap B = (0)$, we have $I_\alpha \cap B = (0)$ also.

Now, it follows from Lemma 4.1 that B_α/I_α is an α-regular extension of B. Let h be an α-homomorphism of B/I_α into an α-complete Boolean algebra C. We wish to extend h to B_α/I_α. Let f be the canonical α-homomorphism of B_α onto B_α/I_α and let $g = hf_1$, where f_1 is the restriction of f to B. Then g can be extended to an α-homomorphism g^* of B_α into C. And, if $u = \prod_{t\in T}^{B\alpha} x_t$ is a generator of I_α, then

$$g^*(u) = \prod_{t\in T} g^*(x_t) = \prod_{t\in T} g(x_t) = \prod_{t\in T} hf(x_t) = hf\Big(\prod_{t\in T}^{B} x_t\Big) = 0 .$$

Therefore I_α is contained in the kernel of g^*. We now define the mapping h^* by

$$h^*(f(x)) = g^*(x) , \qquad x \in B_\alpha .$$

Then h^* is the desired extension of h, and B_α/I_α is a free α-regular extension of B. The uniqueness of B_α^* can be proved easily by an argument similar to the one used in proving that B_α is unique. (See the proof of Theorem 2.1.)

COROLLARY 4.1. (Sikorski). *For every Boolean algebra B, B_σ^* is isomorphic to $F_\sigma(B)/J_\sigma$.*

Proof. By Corollary 3.1, B_σ is isomorphic to $F_\sigma(B)$, and the ideal I_σ, when considered as a σ-ideal of $F_\sigma(B)$, coincides with J_σ. Thus the conclusion follows from Theorem 4.1.

THEOREM 4.2. *An α-complete Boolean algebra B^* is an α-regular extension of B if and only if B^* is isomorphic to B_α/I, where I is an α-ideal of B_α satisfying the following two conditions*: (a) $I \supset I_\alpha$; (b) $I \cap B = (0)$.

Proof. Suppose B^* is an α-regular extension of B. Then B^* is α-generated by an α-regular subalgebra B_0 isomorphic to B. Let i be an isomorphism of B/I_α onto B_0. Then i is an α-isomorphism of B/I_α into B^*, hence it can be extended to an α-homomorphism i^* of B_α/I_α onto B^*. Let $I = \{x \in B_\alpha : i^*([x]_{I_\alpha}) = 0\}$. Then B^* is isomorphic to B_α/I and I satisfies conditions (a) and (b). The converse was proved in Lemma 4.1.

Theorem 4.2 and Corollary 3.1 yield the following result of Sikorski [7].

COROLLARY 4.2. *A σ-complete Boolean algebra B^* is a σ-regular extension of B if and only if B^* is isomorphic to $F_\sigma(B)/I$, where I is a σ-ideal of $F_\sigma(B)$ satisfying the conditions*: (a) $I \supset J_\sigma$; (b) $I \cap F_0(B) = \phi$.

The following result is well known ([9], § 35 and 23.2).

THEOREM 4.3. *The normal completion B^∞ of a Boolean algebra B is isomorphic to B^*_∞. That is, B^∞ has the property that every complete homomorphism of B into a complete Boolean algebra C can be extended to a complete homomorphism of B^∞ into C.*

Using Theorem 3.5 and arguments similar to the ones used in the proofs of Theorems 4.1 and 4.2, we obtain the following two theorems which also can be proved by using Sikorski's methods for the σ-case (see [9], § 36).

THEOREM 4.4. *For every Boolean algebra B and every infinite cardinal number α, B^*_α is isomorphic to $F_\alpha(B)/J_\alpha$ if and only if B^*_α is α-representable.*

THEOREM 4.5. *Let B be a Boolean algebra for which B^*_α is α-representable. Then an α-complete Boolean algelra B^* is an α-regular extension of B if and only if B^* is isomorphic to $F_\alpha(B)/I$, where I is an α-ideal of $F_\alpha(B)$ satisfying the conditions*: (a) $I \supset J_\alpha$; (b) $I \cap F_0(B) = \phi$.

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