Pacific Journal of Mathematics

SOME DEGENERATE CAUCHY PROBLEMS WITH OPERATOR COEFFICIENTS

ROBERT WAYNE CARROLL

Vol. 13, No. 2 April 1963

SOME DEGENERATE CAUCHY PROBLEMS WITH OPERATOR COEFFICIENTS

ROBERT CARROLL

1. Motivated in part by connections with problems in transonic gas dynamics there has been considerable interest in equations of the form

$$u_{tt} - K(t)u_{xx} + bu_x + eu_t + du - h = 0 \tag{1.1}$$

where d, b, e and h are functions of (x, t) (see here Bers [4] for a bibliography and discussion). In particular there arises the Cauchy problem for (1.1) in the hyperbolic region with data given on the parabolic line $t = 0$ (see in particular Protter [20], Conti [9], Bers [3], Berezin [2], Hellwig [12; 13], Frankl [10], Weinstein [25], Krasnov [15; 16], Carroll [8], Germain and Bader [11], and Barancev [1]). Protter assumes that $K(t)$ is a monotone increasing function of t, $K(0) = 0$, and shows that the Cauchy problem for (1.1) with initial data $u(x, 0)$ and $u_t(x, 0)$ prescribed on a finite x-interval, is correctly set (under suitable regularity assumptions) if $tb(x, t)/\sqrt{K(t)} \to 0$ as $t \to 0$. Thus in particular if $b \equiv 0$ the condition is automatically true. Krasnov considers generalized solutions and the equation

$$u_{tt} - \Sigma \frac{\partial}{\partial x_i}\left(a_{ik}\frac{\partial u}{\partial x_k}\right) + \Sigma b_i \frac{\partial u}{\partial x_i} + e\frac{\partial u}{\partial t} + du = h\,. \tag{1.2}$$

Again the presence of first order terms b_i complicates the matter and (as with Protter for $K(t) \sim t^{\alpha}$) it is assumed that $b_i = O(t^{\alpha/2-1}\beta(t))$ where $\beta(t) \to 0$ (additional assumptions are also made). Krasnov supposes $\Sigma a_{ik}\xi_i\xi_k \geqq ct^{\alpha}\Sigma\xi_i^2$ with $h/t^{\frac{\alpha-1+\delta_0}{2}} \in L^2$ ($\delta_0 > 0$ is a number for which bounds are determined in the proof) and finds solutions u such that $u_t/t^{\frac{\alpha+1+\delta_0}{2}} \in L^2$ and $u_{x_i}/t^{\frac{1+\delta_0}{2}} \in L^2$. Thus the growth of h appears to play an important role in determining a solution in this more general equation (1.2). Slightly more general degeneracies for $\Sigma a_{ik}\xi_i\xi_k$ are mentioned by Krasnov but always in some comparison to a power of t.

It is one of the aims of the present paper to give a more precise estimate of the allowable degeneracy in relation to the growth of h and to give estimates for the solution. In particular we will not require that $K(t)$ be monotone. For simplicity we omit here first order terms in $\partial u/\partial x_i$; this will be dealt with, in an abstract framework, in a subsequent article. A summary of some of the present work was

Received March 26, 1962.

given in [8]. We remark that an operational treatment of the type of degenerate problems considered by Tersenov [24] and Hu Hsien Sun [14] is also contemplated (this involves an equation of the form $K(t)u_{tt} - u_{xx} + bu_x + eu_t + du - h = 0$ with data given for $t = 0$). As indicated above our results generalize in certain respects those of Krasnov, however the methods employed here are quite different; for example Krasnov relies heavily on a Galerkin type method for existence whereas we employ an energy method based on work of Lions [17]. Further generalizations in our framework are clearly possible (see [16]).

2. Following Lions (see [18] for an extensive bibliography and treatment of operational differential equations) we reformulate (1.2) as follows. Let V and H, $V \subset H$, be Hilbert spaces, V dense in H, with the topology of V being finer than that induced by H.* The norms in V and H are denoted by $\|\ \|$ and $|\ |$ respectively. Let $(u, v) \to a(t, u, v)$ be a continuous sesquilinear form on $V \times V$ for t fixed, $0 \leqq t \leqq b < \infty$, with $a(t, u, v) = \overline{a(t, v, u)}$. Assume that $t \to a(t, u, v) \in C^1[0, b]$ for (u, v) fixed. We recall (see [18]) that the form $a(t, u, v)$ defines an unbounded operator $A(t)$: $D(A(t)) \to H$ by defining $D(A(t))$ to be the set of $u \in V$ such that $v \to a(t,u,v)$ is continuous on V in the topology of H. Then we can write for $u \in D(A(t))$, $(A(t)u, v) = a(t, u, v)$ for $v \in V$. Now let $\{B(t)\}$ be a family of bounded Hermitian operators in H with $t \to B(t) \in \mathscr{E}^1(\mathscr{L}_s(H, H))$ (here $\mathscr{E}^m(G)$ is the space of m-times continuously differentiable functions of t with values in G and $\mathscr{L}_s(H, H)$ is the space of continuous linear maps $H \to H$ with the topology of simple convergence—see [5]).

Let now $\psi > 0$ be a numerical function with $\psi \uparrow$ as $t \to 0$, $\psi \in C^0(0, b]$. Here ψ does not necessarily approach ∞. We assume q is another numerical function such that $q > 0$ on $(0, b]$ with $q \to 0$ as $t \to 0$ (in what follows all limits such as $q \to 0$ will refer to $t \to 0$). Let f be given such that $\psi f \in L^2(H)$ (for the spaces $L^p(H)$ and the integration of vector valued functions see [6; 7]). We assume $q \in C^1(0, b]$. Let $\mathscr{F}_s$ be the Hilbert space of functions u on $[0, s]$ such that $u(0) = 0$, $\psi u' \in L^2(H)$, and $\omega u \in L^2(V)$ with

$$\|u\|^2_{\mathscr{F}_s} = \int_0^s \{\|\omega u\|^2_V + |\psi u'|^2_H\}\, dt \tag{2.1}$$

(ω is a numerical function to be determined, $\omega > 0$, $\omega \to \infty$). Here all derivatives are taken in the sense of vector valued distributions in $\mathscr{D}'(H)$(see [23]) and $\mathscr{F}_s$ may be proved complete by standard arguments. Let now $\mathscr{H}_s$ be the space of functions h which satisfy $h(s) = 0$, $h/\psi \in L^2(H)$, $h'/\psi \in L^2(H)$, and $qh/\omega \in L^2(V)$. Set

* H is also assumed to be separable for simplicity in a later argument; this condition is not necessary however.

$$\tilde{E}_s(u,h) = \int_0^s \{qa(t, u, h) + (B(t)u', h) - (u', h')\}dt \tag{2.2}$$

and define

$$\tilde{L}_s(h) = \int_0^s (f, h)dt . \tag{2.3}$$

We note that (2.2) and (2.3) are well defined for $u \in \mathscr{F}_s, h \in \mathscr{H}_s$, and f as described. Thus assume ω as indicated has been given; then we pose

Problem 1. Find s and $u \in \mathscr{F}_s$ such that for all $h \in \mathscr{H}_s$

$$\tilde{E}_s(u, h) = \tilde{L}_s(h) . \tag{2.4}$$

Naturally we wish to find the best ω in some sense when posing problem 1. Here best will be left vague for the present in remarking only that ω furnishes a measure of how rapidly the solution u tends to 0 as $t \to 0$. We define now $\mathscr{K}_s$ to be the space of functions k such that $k = \int_0^t \varphi h d\xi$ for $h \in \mathscr{H}_s$ where φ is a numerical function to be determined (in general $\varphi \in C'[0, s], \varphi > 0$ on $(0, s]$, and $\varphi \to 0$ as $t \to 0$). Clearly $k' = \varphi h$ and thus $k'/\varphi\psi = h/\psi \in L^2(H)$. For suitable choice of the numerical function $\delta > 0, \delta \to \infty$, we define $\mathscr{K}_s$ as a prehilbert space with norm

$$||k||^2_{\mathscr{K}_s} = \int_0^s \left\{ ||\delta k||^2_V + \left| \frac{k'}{\varphi\psi} \right|^2_H \right\} dt \tag{2.5}$$

LEMMA 1. *Define $v = \varphi/q$ and assume*
(i) $\varphi\psi^2 \in L^\infty$
(ii) $\omega \leqq \delta$
(iii) $\omega^2 v^2 \in L^1$
(iv) $\delta^2 \int_0^t \omega^2 v^2 d\xi \in L^1$ *with* $\varphi, q, \omega, \psi, \delta \in C^0(0, s]$ *all positive on* $(0, s]$.
Then $\mathscr{K}_s \subset \mathscr{F}_s$ algebraically and topologically.

Proof. The following estimates are straightforward

$$|\psi k'| = \left| \frac{\varphi\psi^2 k'}{\varphi\psi} \right| \leqq c \left| \frac{k'}{\varphi\psi} \right| \tag{2.6}$$

$$||\delta k||^2 = \left|\left| \delta \int_0^t \frac{q}{\omega}\, \omega v h d\xi \right|\right|^2 \leqq \delta^2 \int_0^t \omega^2 v^2 d\xi \int_0^t \left|\left| \frac{qh}{\omega} \right|\right|^2 d\xi . \tag{2.7}$$

Thus by (2.7) for $k \in \mathscr{K}_s$ and δ satisfying the hypotheses we have $\int_0^s ||\delta k||^2 d\xi < \infty$; also by (2.6) and the fact $\omega \leqq \delta$ it follows that $||k||_{\mathscr{F}_s} \leqq \tilde{c}||k||_{\mathscr{K}_s}$. From (2.7) we obtain also the result that $||k||^2 \to 0$ as $t \to 0$ which proves that in fact $\mathscr{K}_s \subset \mathscr{F}_s$.

LEMMA 2. *Assume* (i)–(iv) *and*

(v) $1/v\int_0^t \omega^2 v^2 d\xi \in L^\infty$

(vi) $\varphi'\psi^2 \in L^\infty$

(vii) $1/v\delta^2 \in L^\infty$

(viii) $-(1/v)'\, 1/\delta^2 \in L^\infty$, $v' \geqq 0$. *Assume also that* $a(t, u, u) \geqq \alpha \|u\|^2$, *then*

$$2ReE_s(k, k) \geqq \int_0^s \|\delta k\|^2 \left\{-\alpha\left(\frac{1}{v}\right)' \frac{1}{\delta^2} - \frac{c_1}{v\delta^2}\right\} dt \tag{2.8}$$
$$+ \int_0^s \left|\frac{k'}{\varphi\psi}\right|^2 \{\varphi'\psi^2 - 2\beta\varphi\psi^2\}\, dt$$

where, for $k = \int_0^t \varphi h d\xi$, $E_s(u, k) = \tilde{E}_s(u, h)$.

Proof. Formally we have

$$2ReE_s(k, k) = \frac{q}{\varphi} a(t, k, k)\Big|_0^s - \int_0^s \left\{\left(\frac{q}{\varphi}\right)' a(t, k, k) - \left(\frac{q}{\varphi}\right) a'(t, k, k)\right\} dt\Big\} \tag{2.9}$$
$$+ 2Re\int_0^s \frac{1}{\varphi}(Bk', k')dt - \varphi|h|^2\Big|_0^s + \int_0^s \varphi'|h|^2\, dt .$$

Noting that $\lim \varphi|h|^2 = \lim 1/\varphi|k'|^2 = \theta^2 \geqq 0$ will exist if all the other terms make sense we have

$$\frac{q}{\varphi} a(t, k, k) \leqq \frac{c}{v}\|k\|^2 \leqq \frac{c}{v}\int_0^t \omega^2 v^2 d\xi \int_0^t \left\|\frac{qh}{\omega}\right\|^2 d\xi \tag{2.10}$$

which vanishes as $t \to 0$. Note by the Banach Steinhaus theorem it follows that (see [18])

$$|a(t, u, h)| \leqq c\|u\|\,\|h\| \tag{2.11}$$

$$|a'(t, u, h)| \leqq c_1\|u\|\,\|h\| \tag{2.12}$$

$$\left|\int_0^s \frac{1}{\varphi}(Bk', k')dt\right| \leqq \beta\int_0^s \left|\frac{k'}{\varphi\psi}\right|^2 \varphi\psi^2 dt < \infty . \tag{2.13}$$

Moreover under the hypotheses above

$$\int_0^s \frac{\varphi'}{\varphi^2}|k'|^2 dt = \int_0^s \varphi'\psi^2\left|\frac{k'}{\varphi\psi}\right|^2 dt < \infty \tag{2.14}$$

$$\left|\int_0^s \frac{q}{\varphi} a'(t, k, k)dt\right| \leqq c_1\int_0^s \frac{1}{v\delta^2}\|\delta k\|^2 dt < \infty \tag{2.15}$$

$$-\int_0^s \left(\frac{q}{\varphi}\right)' a(t, k, k)dt \leqq c\int_0^s -\left(\frac{1}{v}\right)' \frac{1}{\delta^2}\|\delta k\|^2 dt < \infty \tag{2.16}$$

Thus (2.9) is valid and (2.8) follows.

The formula (2.8) indicates the properties desired of δ and φ in order to obtain an estimate $ReE_s(k, k) \geqq \Omega \|k\|^2_{\mathscr{H}s}$ thus enabling us to apply the Lions projection theorem (see [18]). We will give here a natural choice for δ, φ etc. without seeking the best possible result. To this end set

$$\varphi = \hat{c}\int_0^t \frac{d\xi}{\psi^2}\,. \tag{2.17}$$

Then $\varphi \in C^1[0, b]$, $\varphi \to 0$, and since ψ is monotone $\varphi/\varphi' = \psi^2\int_0^t d\xi/\psi^2 \leqq Nt$. Hence $\varphi\psi^2 = \hat{c}\varphi/\varphi' \to 0$ also and thus $1/\varphi\psi \to \infty$. Next let $R \neq 0$ be a constant and

$$-\left(\frac{1}{v}\right)'\frac{1}{\delta^2} = R;\ v = \frac{1}{\left[\delta_1 + \int_t^s R\delta^2 d\xi\right]} \tag{2.18}$$

where $\delta_1 > 0$ is determined by $v(s)$. Thus $v \to 0$ corresponds to $\delta \notin L^2$ and in any case, noting $v' = Rv^2\delta^2$,

$$\frac{1}{v}\int_0^t \omega^2 v^2 d\xi \leqq \frac{1}{v}\int_0^t \delta^2 v^2 d\xi = \frac{1}{R}\left[1 - \frac{v(0)}{v(t)}\right] = \frac{1}{R}\left\{1 - \frac{\delta_1 + \int_t^s R\delta^2 d\xi}{\delta_1 + \int_0^s R\delta^2 d\xi}\right\}. \tag{2.19}$$

(This shows that $\int_0^t \omega^2 v^2 d\xi < \infty$ and that $1/v\int_0^t \omega^2 v^2 d\xi \leqq M$. The last term in (2.19) is taken to be zero if $\delta \notin L^2$ or $v(0) = 0$, and $v(0)/v(t)$ is seen to be bounded by one in all other cases.) Thus (i), (ii) (by assumption), (iii), (v), (vi), and (viii) hold. Also the $\varphi'\psi^2$ term dominates in the second integral of (2.8) for s small. Now for (vii) we note that $1/v\delta^2 = (v/v')R$ and $v' = (\varphi/q)'$; thus

$$\frac{v'}{v} = \frac{\varphi'}{\varphi} - \frac{q'}{q} = \frac{\varphi'}{\varphi}\left[1 - \frac{q'\psi^2}{q}\int_0^t \frac{d\xi}{\psi^2}\right]. \tag{2.20}$$

If we assume for example that $(q'\psi^2/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$ for t small then $v'/v \geqq \varepsilon_1\varphi'/\varphi \to \infty$ since $\varphi, \varphi' > 0$ on $(0, b]$ and $\varphi/\varphi' \to 0$. In any case if $v'/v \to \infty$ then $v/v' \to 0$ and $1/v\delta^2 \to 0$ which means not only that (vii) holds but that the $-\alpha(1/v)'1/\delta^2$ term dominates in the first integral of (2.8) for s small. Note here that φ and hence v are defined on $[0, b]$ independently of s by say (2.17) whereas (2.18) determines δ^2 on any interval $(0, s]$ for v given. Finally with regard to (iv) there are various hypotheses on ω and v which would work but we assume simply that

$$\omega^2 = \frac{v'}{v^{2-\varepsilon}},\ 0 < \varepsilon < 1 \tag{2.21}$$

Then if say $v \in C^0[0, b]$

$$\begin{aligned}\int_0^s \delta^2\left(\int_0^t \omega^2 v^2 d\xi\right) dt &= \int_0^s \frac{v'}{Rv^2}\left(\int_0^t v' v^\varepsilon d\xi\right) dt \\ &= \frac{1}{R(1+\varepsilon)} \int_0^s \frac{v'}{v^{1-\varepsilon}} dt = \frac{1}{R\varepsilon(1+\varepsilon)} v^\varepsilon(t)\Big|_0^s .\end{aligned} \tag{2.22}$$

It should be noted that $v \in C^0[0, b]$ now implies that $\omega \leqq c\delta$ since $\omega^2/\delta^2 = Rv^\varepsilon$ and this would be a condition equivalent to (ii). We remark that $v \to 0$ implies $\omega \notin L^2$ since $\int_t^s \omega^2 d\xi = \int_t^s v'/v^{2-\varepsilon} d\xi = 0(1/v^{1-\varepsilon})$. This proves

LEMMA 3. *Assume* $a(t, u, u) \geqq \alpha||u||^2$, $v'/v \to \infty$, $v \in C^0[0, b]$, $\omega^2 = v'/v^{2-\varepsilon}$, $\varphi = \hat{c}\int_0^t d\xi/\psi^2$, *and* $v = 1/\delta_1 + \int_t^s R\delta^2 d\xi$. *Then* $\omega \leqq c\delta$ *and* (i), (iii)–(viii) *hold with* $ReE_s(k, k) \geqq \Omega||k||^2_{\mathscr{K}_s}$ *for s sufficiently small.*

Using the above lemmas and the Lions projection theorem (see [18]) there results

THEOREM 1. *Under the hypotheses of Lemma* 3 *and the conditions on* $a(t, u, v)$, $B(t)$ *stipulated above there exist functions* ω ($\omega \notin L^2$ *if* $v \to 0$) *such that for s small problem* 1 *has a solution.*

Proof. We need only check that the map $u \to E_s(u, k)$: $\mathscr{F}_s \to C$ is continuous for $k \in \mathscr{K}_s$ fixed and that the map $k \to L_s(k) = \tilde{L}_s(h)$: $\mathscr{K}_s \to C$ is continuous. This verification is immediate.

Now since $q > 0$ on $(0, b]$ we can treat $qa(t, u, v)$ as a nondegenerate form on say $[s/2, b]$ and apply Lions' results for such problems (see [17; 18]). We want to solve

Problem 2. Find $u \in \mathscr{F}_b$ such that $\tilde{E}_b(u, h) = \tilde{L}_b(h)$ for all $h \in \mathscr{H}_b$.

Thus suppose the problem has been solved for $[0, s]$, that is suppose problem 1 has been solved with solution u_1. Then following [17] let $p \in C^1$ with $p = 1$ on $[0, 2/3\, s]$ and $p = 0$ in a neighborhood of s. Set $u_2 = u - pu_1$; then $u_2 = 0$ on $[0, 2/3\, s]$ and $u_2 = u$ for $t \geqq s$. The problem 2 for u becomes

$$\begin{aligned}\tilde{E}_b(u_2, h) &= \int_0^b (f, h) dt - \int_0^b p'[(Bu_1, h) + (u_1', h)] dt \\ &\quad - \int_0^b \{qa(t, u_1, ph) + (Bu_1', ph) - (u_1', (ph)')\} dt .\end{aligned} \tag{2.23}$$

Now if $h \in \mathscr{H}_b$ we see that $ph \in \mathscr{H}_s$; hence

$$\tilde{E}_b(u_2, h) = \int_0^b (f, h - ph) dt - \int_0^b p'[(Bu_1, h) + (u_1', h)] dt . \tag{2.24}$$

In particular we see that everything vanishes on say $[0, s/2]$; hence we pose the Cauchy problem with initial data given at $s/2$ as follows. Let $\mathscr{F}_{s/2\ s_1}$ be the space of u such that $\omega u \in L^2(v)$ and $\psi u' \in L^2(H)$ on $[s/2, s/2 + s_1]$ with $u(s/2) = 0$. The space $\mathscr{H}_{s/2\ s_1}$ corresponding to $\mathscr{H}_s$ is defined similarly on $[s/2, s/2 + s_1]$. We extend ω and δ to be constant on $[s, b]$; then since ψ, ω, δ etc. are positive and continuous we may define say $\mathscr{F}_{s/2,s_1}$ in terms of $u \in L_2(V)$ and $u' \in L^2(H)$. Let $\tilde{E}_{s/2\ s_1}$ denote the terms in $\tilde{E}_b$ integrated over $[s/2, s/2 + s_1]$, and denote the right side of (2.24) integrated from $s/2$ to $s/2 + s_1$ by $\tilde{\tilde{L}}_{s/2\ s_1}(h)$. Then consider

Problem 3. Find $u_2 \in \mathscr{F}_{s/2\ s_1}$ such that $\tilde{E}_{s/2,s_1}(u_2, h) = \tilde{\tilde{L}}_{s/2\ s_1}(h)$ for all $h \in \mathscr{H}_{s/2,s_1}$.

Problem 3 has a (unique) solution for s_1 sufficiently small by [17] and the above extension procedure may be repeated in steps of length $s_1/2$. Thus u will eventually be determined on $[0, b]$ satisfying problem 2. Hence

THEOREM 2. *Under the hypotheses of Theorem* 1 *there exists a solution of problem* 2.

3. Suppose now that $\tilde{E}_s(u, h) = 0$ for all $h \in \mathscr{H}_s$. Let $h = -\int_t^s Jud\xi$, $h' = Ju$, $J \to \infty$. Then

LEMMA 4. *Assume*
(a) $J^2/\omega^2\int_0^t d\xi/\psi^2 \in L^1$
(b) $J/\omega\psi \in L^\infty$
(c) $J^2/\omega^2\int_0^t (q^2/\omega^2)\, d\xi \in L^1$. *Then* $h \in \mathscr{H}_s$ *if* $u \in \mathscr{F}_s$ *and* $h = -\int_t^s Jud\xi$.

Proof. Clearly $h'/\psi = (J/\omega\psi)\omega u \in L^2(V)$ (hence certainly $h'/\psi \in L^2(H)$) and $h(s) = 0$; also

$$\left|\frac{h}{\psi}\right|^2 \leqq c\left|\left|\frac{h}{\psi}\right|\right|^2 \leqq \left(\frac{1}{\psi}\int_t^s \frac{J}{\omega}\,||\omega u||\, d\xi\right)^2 \leqq \frac{1}{\psi^2}\int_t^s \frac{J^2}{\omega^2}\, d\xi \int_t^s ||\omega u||^2 d\xi \tag{3.1}$$

$$\int_0^s\left|\left|\frac{q}{\omega}\, h\right|\right|^2 d\xi \leqq \int_0^s \frac{q^2}{\omega^2}\left(\int_t^s \frac{J^2}{\omega^2}\, d\xi\right)dt \int_0^s ||\omega u||^2\, d\xi\,. \tag{3.2}$$

Using the Fubini and Tonelli theorems (see e.g. [19]) the lemma follows.

We note now explicitly the fact that if $u \in L^2(H)$ and $u' \in L^2(H)$ (u' taken in $\mathscr{D}'(H)$ on $(0, s)$) then u may be identified with a continuous function and $u(0) = 0$ makes sense. Indeed for u, determined almost everywhere, we see that $u' \in L^1(H)$ on $[0, s]$ and clearly $D\tilde{u} = u'$ in $\mathscr{D}'(H)$ where $\tilde{u} = \int_0^t u'd\xi \in \mathscr{E}^0(H)$ (see [23]). Thus $D(\tilde{u} - u) = 0$ and by [21] for any $h \in H$, $(\tilde{u} - u, h) = c_h$ in $\mathscr{D}'$. Hence $(\tilde{u} - u, h) = c_h$

almost everywhere as a function and thus u may be identified scalarly with the continuous function $\tilde{u}$. Since H is separable we may then identify u with a continuous function and $u(0) = 0$ is meaningful (see [23], [22]). Hence $u = \tilde{u}$ follows. Thus setting $u = \int_0^t u' d\xi$, $h = -\int_t^s h' d\xi$

$$(3.3)\quad |(u, h)| = \left| -\int_0^t \int_t^s (u'(\xi), h'(\eta)) d\eta d\xi \right|$$

$$\leqq \sup \left| \frac{\psi(\eta)}{\psi(\xi)} \right| \int_0^t \int_t^s |\psi u'| \left| \frac{h'}{\psi} \right| d\eta d\xi \leqq \frac{N}{2} \int_0^t \int_t^s \left\{ |\psi u'|^2 + \left| \frac{h'}{\psi} \right|^2 \right\} d\eta d\xi$$

$$\leqq \frac{N}{2} \left\{ \int_0^t (s - t) |\psi u'|^2 d\xi + t \int_t^s \left| \frac{h'}{\psi} \right|^2 d\eta \right\} .$$

Thus $(u, h) = 0$ at $t = 0$ and we note that $\int_0^s (Bu', h) dt = -\int_0^s (B'u, h) dt - \int_0^s (Bu, h') dt$. Hence $\tilde{E}_s(u, h) = 0$ becomes, with h as above

$$(3.4)\quad \int_0^s \left\{ \frac{q}{J} a(t, h', h) - (B'u, h) - J(Bu, u) - J(u', u) \right\} dt = 0 .$$

Set now $\tilde{\theta}^2 = \lim q/J\, a(t, h, h)$ which will exist if everything else makes sense in the following. Then we have

LEMMA 5. *Assume* (a)–(c) *from Lemma* 4 *and*
(d) $J \int_0^t d\xi / \psi^2 \in L^\infty$
(e) $-J'/\omega^2 \in L^\infty$; $J' < 0$
(f) $J \to \infty$; $J/J' \to 0$
(g) $(q/J)'/(q/J) \to \infty$. *Then if* $h = -\int_t^s J u d\xi$, $u \in \mathscr{F}_s$, *and if* $a(t, h, h) \geqq \alpha \|h\|^2$ *it follows that*

$$(3.5)\quad \int_0^s \left\{ \alpha \left(\frac{q}{J} \right)' \frac{\omega^2}{q^2} - c_1 \left(\frac{q}{J} \right) \frac{\omega^2}{q^2} \right\} \left\| \frac{qh}{\omega} \right\|^2 dt$$

$$+ \int_0^s \left\{ -\frac{J'}{\omega^2} - \frac{2\beta J}{\omega^2} - \frac{\hat{\beta}}{\omega^2} \int_t^s J d\xi - \frac{\hat{\beta} t J}{\omega^2} \right\} |\omega u|^2 dt \leqq 0$$

Proof. By (d) we have

$$J|u|^2 \leqq J \left(\int_0^t |\psi u'| \frac{d\xi}{\psi} \right)^2 \leqq J \int_0^t \frac{d\xi}{\psi^2} \int_0^t |\psi u'|^2 d\xi \to 0$$

whereas from (e) there results $-J'|u|^2 = -J'/\omega^2 |\omega u|^2 \in L^1$. Next by (f) and (e) it follows that $\lim Jq/\omega^2 = \lim (J/-J')(-J'q/\omega^2) = 0$; hence $Jq/\omega^2 \in L^\infty$ and

$$(3.6)\quad \int_0^s \left(\frac{q}{J} \right)' \|h\|^2 d\xi \leqq \int_0^s \left(\frac{q}{J} \right)' \left(\int_t^s J \|u\| d\xi \right)^2 dt$$

$$\leqq \int_0^s \left(\frac{q}{J} \right)' \left(\int_t^s \frac{J^2}{\omega^2} d\xi \int_t^s \|\omega u\|^2 d\xi \right) dt \leqq \left(\int_0^s \|\omega u\|^2 d\xi \right) \int_0^s \frac{Jq}{\omega^2} d\xi .$$

Note here $q/J \to 0$ and $q/J = \int_0^t (q/J)'\, d\xi$; also by (g) surely $\int_0^s q/J \|h\|^2\, d\xi < \infty$. Now by (f) it follows that $J|u|^2 = (J/J')\, J'|u|^2 \in L^1$ and finally we remark that

$$\left|2Re\int_0^s (B'u,\, h)\, d\xi\right| \leqq \hat{\beta}\int_0^s\int_t^s J(\xi)\, \{|u(t)|^2 + |u(\xi)|^2\}\, d\xi dt \tag{3.7}$$
$$\leqq \hat{\beta}\left\{\int_0^s |\omega u|^2 \left(\frac{1}{\omega^2}\int_t^s J d\xi\right) dt + \int_0^s \frac{Jt}{\omega^2}\, |\omega u|^2\, dt\right\}.$$

Here the Jt/ω^2 term makes sense since $Jt/\omega^2 = (J/-J')(-J't/\omega^2) \to 0$ by (e) and (f). Then we note that

$$\frac{1}{\omega^2}\int_t^s J d\xi = \left(\frac{-J'}{\omega^2}\right)\left(\frac{J}{-J'}\right)\left(\frac{1}{J}\int_t^s J d\xi\right);$$

but by 1' Hospital's rule $\lim 1/J\int_t^s J d\xi = \lim J/-J^1 = 0$ (here note that $J' \neq 0, J \neq 0$ for $t > 0$). Hence we may write

$$\tilde{\theta}^2 + \int_0^s \left\{\left(\frac{q}{J}\right)' a(t,\, h,\, h) + \left(\frac{q}{J}\right) a'(t,\, h,\, h)\right\} dt \tag{3.8}$$
$$+ 2Re\int_0^s (B'u,\, h)dt + 2Re\int_0^s J(Bu,\, u)dt$$
$$- \int_0^s J'\,|u|^2\, dt + J\,|u(s)|^2 = 0\,.$$

The lemma follows immediately.

Now let $\omega^2 = v'/v^{2-\varepsilon}$ as before and consider the following choice for the function J

$$J = j + \check{c}\int_t^s \omega^2 d\xi;\ -\frac{J'}{\omega^2} = \check{c}\,. \tag{3.9}$$

It follows that (e) holds (we assume ω, v etc. are as before) and since $v = \varphi/q$ (d) is a consequence of the fact that

$$\check{c}\int_t^s \omega^2 d\xi \int_0^t \frac{d\eta}{\psi^2} \leqq \check{c}\varphi\int_t^s \delta^2 d\xi = \check{c}\varphi\int_t^s -\left(\frac{1}{v}\right)'\frac{d\xi}{R} \tag{3.10}$$
$$= \check{c}\,\frac{\varphi}{R}\left[\frac{1}{v(t)} - \frac{1}{v(s)}\right] = \frac{\check{c}}{R}\left[q(t) - \varphi(t)\,\frac{q(s)}{\varphi(s)}\right].$$

Note now that with the above choice of ω we can write J in the form $J = j + \check{c}\int_t^s v'/v^{2-\varepsilon}\, d\xi = j - (\check{c}/1-\varepsilon)\,(1/v(s))^{1-\varepsilon} + (\check{c}/1-\varepsilon)\,(1/v(t))^{1-\varepsilon}$. If j is taken to be $j = (\check{c}/1-\varepsilon)\,(1/v(s))^{1-\varepsilon}$ then

$$J = \frac{\check{c}}{1-\varepsilon}\left(\frac{1}{v}\right)^{1-\varepsilon};\ \frac{J}{J'} = \frac{-1}{1-\varepsilon}\left(\frac{v}{v'}\right). \tag{3.11}$$

Thus if $v/v' \to 0$ then $J/-J' \to 0$. Moreover since $\omega^2 = (v'/v)\,(1/v)^{1-\varepsilon}$ it follows that $\omega \to \infty$ if $v \to 0$ and $v/v' \to \infty$ and also by (3.11) $J \to \infty$ if $v \to 0$. Hence if $v'/v \to \infty$ and $v \to 0$ then (f) holds and $\omega \to \infty$.

Consider now condition (a); using (d) we have $J^2/\omega^2 \int_0^t d\xi/\psi^2 \leqq c\,J/\omega^2 = -\check{c}c\,J/J' \to 0$ which implies (a). For (c) we note

$$(3.12)\qquad \int_0^s \frac{J^2}{\omega^2}\left(\int_0^t \frac{q^2}{\omega^2}d\xi\right)dt$$

$$\leqq \int_0^s \left\{\frac{j^2 + 2j\check{c}\int_t^s \omega^2 d\xi + \left(\check{c}\int_t^s \omega^2 d\xi\right)^2}{\omega^2}\right\}\left(\int_0^t \frac{q^2}{\omega^2}d\xi\right)dt\,.$$

However $1/\omega^2 \int_t^s \omega^2 d\xi = v^{2-\varepsilon}/v' \int_t^s v'/v^{2-\varepsilon}\,d\xi = (1/1-\varepsilon)\,\{v/v' - c/\omega^2\}$ and if $v/v' \to 0$ and $\omega \to \infty$ it follows that the first two integrals in (3.12) exist. The last integral in (3.12) is bounded by

$$c\int_0^s \left[\frac{1}{\omega^2}\int_t^s \omega^2 d\xi\right]\left[\int_t^s \omega^2 d\xi \int_0^t \frac{d\eta}{\omega^2}\right]dt\,.$$

The first term in the integrand vanishes as $t \to 0$ by the above remarks and using 1' Hospital's rule on the second term we note that $\lim \int_t^s \omega^2 d\xi \int_0^t d\eta/\omega^2 = \lim \left(\int_t^s \omega^2 d\xi\right)^2\Big/\omega^4$ which is zero by the above (note here if $\omega \in L^2$ (3.12) is seen immediately to exist and no recourse to the preceding argument is intended). Thus if $v'/v \to \infty$ and $\omega \to \infty$ (c) surely holds.

Now since $J/\omega\psi = (\check{c}/1-\varepsilon)\,1/\omega\psi v^{1-\varepsilon}$ it follows that (b) holds if $\omega^2 v^{2-2\varepsilon} > c/\psi^2$ or $(v'/v)\varepsilon > c/\psi^2$. It is not necessary that $\psi \uparrow \infty$ in general; when $v \to 0$ (b) will hold if $v' > c/\psi^2$. Thus (b) holds if $v \to 0$ and

$$(3.13)\qquad 1 - \left(\frac{\psi^2 q'}{q}\right)\int_0^t \frac{d\xi}{\psi^2} > \tilde{c}q$$

since $v' = \varphi'/q - \varphi q'/q^2$ and $\varphi = \hat{c}\int_0^t d\xi/\psi^2$. In particular (3.13) holds if for example $(\psi^2 q'/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$, since $q \to 0$ (see here also equation (2.20)). This proves

LEMMA 6. *Assume* (h) $(q'\psi^2/q)\int_0^t d\xi/\psi^2 \leqq 1 - \varepsilon_1$ *for t small. Then if* $J = (\check{c}/1-\varepsilon)\,1/v^{1-\varepsilon}$ $(J' = -\check{c}\omega^2)$ *and* $v \to 0$ *it follows that* $v'/v \to \infty$ *and* (a)–(f) *hold.*

We recall that φ and v are defined independently of s (see (2.17)) and our constructions and proofs have shown that for t small enough the $(q/J)'\,\omega^2/q^2$ and $-J'/\omega^2$ terms will dominate in the first and second integrals respectively of (3.5). It remains to check only a few terms in order to see whether by suitable choice of s this domination prevails over $[0, s]$. Now by (3.11) J/J' is independent of s as is J/ω^2 (indeed a priori ω^2 and δ^2 depend only on v). Now since $-J' = \check{c}\omega^2 > 0$ we have J monotone decreasing and clearly

$$\frac{1}{J(t)} \int_t^s J(\xi) d\xi \leqq s - t \leqq b \, .$$

Hence referring to the proof of Lemma 5 we can establish domination over an interval $[0, s]$ in the second integral of (3.5). There remains the $(q/J)'$ term for which we may write

$$\frac{\left(\frac{q}{J}\right)'}{\left(\frac{q}{J}\right)} = \frac{q'}{q} + (1 - \varepsilon)\frac{v'}{v} = \frac{\varphi'}{\varphi}\left\{1 - \varepsilon\left[1 - \frac{q'\varphi}{q\varphi'}\right]\right\}, \tag{3.14}$$

Thus in particular the ratio in (3.14) is a priori independent of s and the desired domination may be obtained on an interval $[0, s]$ by choosing s sufficiently small. Thus we have proved

LEMMA 7. *If the hypotheses of Lemma* 6 *hold and* (g) *is true it follows that for suitably small* s, $\int_0^s |\omega u|^2 \, dt \leqq 0$.

Clearly the condition (h) in Lemma 6 is much stronger than is necessary but it gives a manageable criterion. We note now that if $q' \geqq 0$ then by (h) $\varepsilon_1 \leqq [1 - q'\varphi/q\varphi'] \leqq 1$ and from (3.14) it results that $(q/J)'/(q/J) \geqq (1 - \varepsilon)\,\varphi'/\varphi \to \infty$. Thus if q is monotone, for any ε, $0 < \varepsilon < 1$, (g) is a consequence of (h). Another case of interest would be if $1 - q'\varphi/q\varphi' \leqq \tilde{Q}$; then if $\varepsilon \leqq 1/\tilde{Q}$ (g) holds. A somewhat better result may be obtained as follows. We note that

$$\frac{q'\varphi}{q\varphi'} = \frac{q'\psi^2}{q}\int_0^t \frac{d\xi}{\psi^2} = \frac{(\text{loq}\, q)'}{\left(\text{loq}\int_0^t \frac{d\xi}{\psi^2}\right)'} \, .$$

Then assume that $Q = \lim\, (q'\psi^2/q)\int_0^t d\xi/\psi^2$ exists as $t \to 0$. We note that the conditions needed to apply l'Hospital's rule hold and thus $Q = \lim \text{loq}\, q/\text{loq} \int_0^t d\xi/\psi^2$. Therefore for t small (h) implies that

$$\log q/\log \int_0^t \frac{d\xi}{\psi^2} \leqq 1 - \varepsilon_2\,, \quad 0 < \varepsilon_2 < \varepsilon_1 \, .$$

But for t small the logarithms are negative and thus $\text{loq}\; q \geqq \text{loq}\left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ or $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2} = c\varphi^{1-\varepsilon_2}$. Conversely if $q \geqq c\varphi^{1-\varepsilon_2}$ and if $Q = \lim q'\varphi/q\varphi'$ exists then $Q \leqq 1 - \varepsilon_3$ for some ε_3, $0 < \varepsilon_3 < \varepsilon_2$.

Hence if Q exists as defined and $q \geqq c\varphi^{1-\varepsilon_2}$ then (h) holds and moreover $v = \varphi/q \leqq \varphi/c\varphi^{1-\varepsilon_2} = (1/c)\varphi^{\varepsilon_2} \to 0$. We note that by construction if Q exists then $Q = \lim \log q/\log \int_0^t d\xi/\psi^2 \geqq 0$; hence $\varepsilon[1 - q'\varphi/q\varphi'] < \varepsilon(1 + \varepsilon_4)$ for t small enough and $\varepsilon_4 > 0$ given. Choose now ε_4 such that $\varepsilon(1+\varepsilon_4)<1$ or $\varepsilon_4 < (1-\varepsilon)/\varepsilon$ then from (3.14) $(q/J)'/(q/J) \geqq c\varphi'/\varphi$ for t small. This proves

THEOREM 3. *Assume* $Q = \lim (q'\psi^2/q)\int_0^t d\xi/\psi^2$ *exists and that* $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$, $0 < \varepsilon_2 < 1$. *Then* (h) *holds,* $v \to 0$, *and* $(q/J)'/(q/J) \to \infty$ *for* $J = c/v^{1-\varepsilon}$ *as above. Hence for* s *small enough the solution of problem* 1 *is unique.*

Again using [17] we conclude

THEOREM 4. *Assume* $a(t, u, u) \geqq \alpha \|u\|^2$, $t \to a(t, u, v) \in C^1[0, b]$, $t \to B(t) \in \mathscr{E}^1(\mathscr{L}_s(H, H))$, $a(t, u, v) = \overline{a(t, v, u)}$, $q \in C^1(0, b]$, $q > 0$ *for* $t > 0$, $q \to 0$ *as* $t \to 0$, $\psi \in C^0(0, b]$, $\psi > 0$, $\psi \uparrow$ *as* $t \to 0$, $\psi f \in L^2(H)$, $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ $(0 < \varepsilon_2 < 1)$, *and* $Q = \lim (q'\psi^2/q)\int_0^t d\xi/\psi^2$ *exists. Then there exists a unique solution of problem* 2 *for spaces* $\mathscr{F}_b$, $\mathscr{H}_b$ *based on functions* $\omega \notin L^2$ $(\omega \in C^0(0, b])$.

We note now that if $Q \neq 0$ then $q' < 0$ for t small is not possible. Moreover if $\log q/\log \int_0^t d\xi/\psi^2 \geqq \varepsilon_4 > 0$ then $q \leqq \left(\int_0^t d\xi/\psi^2\right)^{\varepsilon 4}$ and we may assume $\varepsilon_4 < 1$ since if $q \leqq \gamma^{1+\eta}$, $\eta \geqq 0$, $\gamma \to 0$, then $q \leqq \gamma^{\varepsilon 4}$ for any $\varepsilon_4 < 1$ when t is small. In fact $\varepsilon_4 < 1$ is necessary if we are to have $q \geqq c\varphi^{1-\varepsilon_2}$ and thus the case $Q \neq 0$ with $q \geqq \left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2}$ amounts to an estimate of the form $\left(\int_0^t d\xi/\psi^2\right)^{1-\varepsilon_2} \leqq q \leqq \left(\int_0^t d\xi/\psi^2\right)^{\varepsilon 4}$, $0 < \varepsilon_2 < 1$, $\varepsilon_2 + \varepsilon_4 \leqq 1$. Finally we remark that under the hypotheses of Theorem 4 if $\lim q'\psi^2$ exists then by l'Hospital's rule $\lim q'\psi^2 = \lim q\Big/\int_0^t d\xi/\psi^2 = \lim \check{c}\, q/\varphi = \infty$. This implies that $\psi \uparrow \infty$ if q' is bounded but in a case such as $q = t^{1/2}$, $\psi \uparrow \infty$ is not required.

4. Let now $\hat{\mathscr{K}}_s$ be the completion of $\mathscr{K}_s$ for the norm $\|\ \ \|_{\mathscr{K}_s}$. Then we may pose problem 1 for $\hat{\mathscr{K}}_s$ instead of $\mathscr{F}_s$ (call this problem 1′) and repeating the procedures of §§ 2 and 3 there will exist a function $\hat{u} \in \hat{\mathscr{K}}_s$ solving problem 1′ if s is small enough. It may be easily seen that the elements adjoined to $\mathscr{K}_s$ by completion correspond to functions $\hat{k}$ such that $\delta\hat{k} \in L^2(V)$, $\hat{k}'/\varphi\psi \in L^2(H)$, and $\hat{k}(0) = 0$. Moreover the injection i: $\mathscr{K}_s \to \mathscr{F}_s$ may be extended by continuity to a continuous map $\hat{i}$: $\hat{\mathscr{K}}_s \to \mathscr{F}_s$.

LEMMA 8. $\hat{\mathscr{K}}_s \subset \mathscr{F}_s$ *algebraically and topologically.*

Proof. We need only show, after the above remarks, that $\hat{i}$ is an injection. Let $k_n \to \hat{k}$ in $\hat{\mathscr{K}}_s$, $k_n \in \mathscr{K}_s$, and assume that $i(k_n) = k_n \to 0 = \hat{i}(\hat{k})$. We want to show that $\hat{k} = 0$ in $\hat{\mathscr{K}}_s$. First $k_n = i(k_n) \to 0$ in $\mathscr{F}_s$ means in particular that $\omega k_n \to 0$ in $L^2(V)$. Hence (see [6], p. 133) there is a subsequence $\|\omega k_{n_p}\|^2 \to 0$ almost everywhere. Therefore $\|\delta k_{n_p}\|^2 \to 0$ almost everywhere and by the assumption $k_n \to \hat{k}$ in $\hat{\mathscr{K}}_s$ we know $\delta k_{n_p} \to \delta\hat{k}$ in $L^2(V)$. Theorefore we must have (see [6], p. 133 again) $\delta k_{n_p} \to 0$ in $L^2(V)$, and $\delta\hat{k} = 0$ in $L^2(V)$ (similarly $\hat{k}'/\varphi\psi = 0$ in $L^2(H)$); thus in particular $\hat{k} = 0$ which shows that $\hat{i}(\hat{k}) = 0$ implies $\hat{k} = 0$.

Let now $\hat{u} \in \hat{\mathscr{K}}_s$ be the solution of problem 1′ above. Then $\hat{u} \in \mathscr{F}_s$ by Lemma 8 and by the uniqueness Theorem 3 we must have $\hat{u} = u$ for s small where u is the solution of problem 1. Hence

THEOREM 5. *Let the hypotheses of Theorem* 4 *hold. Then there exists a unique solution u of problem* 2 *which belongs to* $\hat{\mathscr{K}}_b$.

Now consider the proof of the Lions projection theorem given say in [17] (see also [18]). We have $ReE_s(k, k) \geqq \Omega\|k\|^2_{\hat{\mathscr{K}}_s}$ for $k \in \mathscr{K}_s$ and wish to solve $E_s(u, k) = L_s(k)$ for $u \in \hat{\mathscr{K}}_s$ (the equation holding for all $k \in \mathscr{K}_s$). Then we write, following Lions, $L_s(k) = ((\chi, k))_{\hat{\mathscr{K}}_s}$, $\chi \in \hat{\mathscr{K}}_s$, and $E_s(u, k) = ((u, Lk))_{\hat{\mathscr{K}}_s}$, $Lk \in \hat{\mathscr{K}}_s$. Here $L: \mathscr{K}_s \to \hat{\mathscr{K}}_s$ is a densely defined linear operator in $\hat{\mathscr{K}}_s$. But $k \in \mathscr{K}_s$

$$\Omega\|k\|^2_{\hat{\mathscr{K}}_s} \leqq |((k, Lk))_{\hat{\mathscr{K}}_s}| \leqq \|k\|_{\hat{\mathscr{K}}_s}\|Lk\|_{\hat{\mathscr{K}}_s} \tag{4.1}$$

which implies L is one-to-one. Moreover if $R_0 = L(\mathscr{K}_s)$ then L^{-1} is a bounded operator on R_0 and may be extended by continuity to $\bar{R}_0$ defining $\hat{L}^{-1}: \bar{R}_0 \to \hat{\mathscr{K}}_s$. Let $P: \hat{\mathscr{K}}_s \to \bar{R}_0$ be the projection and set $R = \hat{L}^{-1}P$ which is thus everywhere defined and continuous on $\hat{\mathscr{K}}_s$. Then we want to find u such that $((u, Lk)) = ((\chi, L^{-1}Lk)) = ((\chi, RLk)) = ((R^*\chi, Lk))$ for all $k \in \mathscr{K}_s$. Thus a solution is $u = R^*\chi$ and by the subsequent uniqueness result $u = R^*\chi$ is the only solution. Using this sketch of the proof of the projection theorem we can bound u. Indeed $\|u\|_{\hat{\mathscr{K}}_s} \leqq \|R^*\chi\|_{\hat{\mathscr{K}}_s} \leqq c\|\chi\|_{\hat{\mathscr{K}}_s}$ since R^* is bounded. Moreover

$$\begin{aligned} |((\chi, k))| &= \left|\int_0^s\left(\psi f, \frac{h}{\psi}\right)dt\right| \leqq \left(\int_0^s|\psi f|^2\,dt\int_0^s\left|\frac{h}{\psi}\right|^2 dt\right)^{1/2} \\ &\leqq \left(\int_0^s|\psi f|^2\,dt\int_0^s|k'/\varphi\psi|^2\,dt\right)^{1/2} \leqq \left(\int_0^s|\psi f|^2\,dt\right)^{1/2}\|k\|_{\hat{\mathscr{K}}_s} = F\|k\|_{\hat{\mathscr{K}}_s}\,. \end{aligned} \tag{4.2}$$

This means (see [5], p. 111) since $\mathscr{K}_s$ is dense in $\hat{\mathscr{K}}_s$ that $\|\chi\| \leqq F = \left(\int_0^s|\psi f|^2\,dt\right)^{1/2}$. Therefore we have proved

THEOREM 6. *Under the hypotheses of Theorem* 4 *and for s suf-*

ficiently small the (unique) solution of problem 1 *satisfies the estimate* $||u||_{\hat{\mathscr{H}}_s} \leqq c\left(\int_0^s |\psi f|^2\, dt\right)^{1/2}$.

The estimate can clearly be extended to $[0, b]$ which given

COROLLARY. *Under the hypotheses of Theorem* 6 *the unique solution of problem* 2 *satisfies the estimate* $||u||_{\hat{\mathscr{H}}_b} \leqq c\left(\int_0^b (\psi f|^2\, dt\right)^{1/2}$.

REFERENCES

1. R. Barancev, *Expansion theorems connected with boundary value problems for the equation* $u_{xx} - K(x)u_{tt} = 0$ *in the strip* $0 \leqq x \leqq 1$ *with a degeneracy or a singularity on the boundary*, Doklady Akad. Nauk, SSSR, T. 121, (1958), 9-12.
2. I. Berezin, *On the Cauchy problem for linear equations of the second order with initial data on the parabolic line*, Mat. Sbornik, **24** (1949), 301-320.
3. L. Bers, *On the continuation of a potential gas flow across the sonic line*, NACA Tech. Note 2058, 1950.
4. ———, *Mathematical aspects of subsonic and transonic gas dynamics*, Wiley, New York, 1958
5. N. Bourbaki, *Espaces vectoriels topologiques*, Chap. 3-5, Paris, 1955.
6. ———, *Int´gration*, Chap. 1-4, Paris, 1952.
7. ———, *Intégration vectorielle*, Chap. 6, Paris, 1959.
8. R. Carroll, *Quelques problèmes de Cauchy dégénérés avec des coefficients opérateurs*, Comptes Rendus, Paris, T. 253, (1961), 1193-1195.
9. R. Conti, *Sul problema di Cauchy per l'equazione* $y^{2\alpha}K^2(x, y)t_{xx} - t_{yy} = f(x, y, t, t_x, t_y)$ *con i dati sulla linea parabolica*, Annali di Matematica, **31** (1950), 303-326.
10. F. Frankl', *On Cauchy's problem for equations of mixed elliptic-hyperbolic type with initial data on the transition line*, Izvestia Akad. Nauk, SSSR, **8** (1944), 195-224.
11. P. Germain and P. Bader, *Solutions élémentaires de certaines équations aux deriveés partielles du type mixte*, *Bull.* de la Soc. Math. de France, T. 81 (1953), 145-174.
12. G. Hellwig, *Anfangs-und Randwertprobleme bei partiellen Differential gleichungen von wechselndem Typus auf dem Rändern*, Math. Zeitschrift, Bd. 58, (1953), 337-357.
13. ———, *Anfangswertprobleme bei partiellen Differential-gleichungen mit Singularitäten*, Journal of Rational Mech. and Analysis, **5** (1956), 395-418.
14. Hu-Hsien Sun, *On the uniqueness of the solution of degenerate equations and the rigidity of surfaces*, Doklady Akad. Nauk, SSSR. T. 122. (1958), 770-773.
15. M. Krasnov, *Mixed boundary value problems and the Cauchy problem for degenerate hyperbolic equations*, Doklady Akad. Nauk, SSSR. T. 107, (1956), 789-792.
16. ———, *Mixed boundary value problems for degenerate linear hyperbolic differential equations of the second order*, Mat. Sbornik, **49** (31), (1959), 29-84.
17. J. L. Lions, *Problemi misti nel senso di Hadamard classici e generalizzati*, Rend. del Sem. Mat. e Fis. di Milano, **28** (1959), 1-47.
18. ———, *Equations différentielles opérationelles et problèmes aux limites*, Grund. d. Math. Wiss., Bd. 111, Berlin, 1961.
19. E. McShane, *Integration*, Princeton, 1949.
20. M. Protter, *The Cauchy problem for a hyperbolic second order equation with data on the parabolic line*, Canadian Journal of Math., **6** (1954), 542-553.
21. L. Schwartz, *Theorie des distributions*, Vols 1, 2, Paris, 1950-51.
22. ———, *Espaces de fonctions différentiables à valeurs vectorielles*, Journal d'analyse Math., **4** (1954-55), 88-148.

23. ———, *Théorie des distributions à valeurs vectorielles*, Annales de l'institut Fourier, pp. 1–141, 1957 et pp. 1–209, 1958.

24. S. Tersenov, *On an equation of hyperbolic type degenerating on the boundary*, Doklady Akad. Nauk. SSSR, T. **129** (1959), 276–279.

25. A. Weinstein, *The singular solutions and the Cauchy problem for generalized Tricomi equations*, Comm. Pure and Appl. Math., **7** (1954), 105–116.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 13, No. 2 April, 1963