Pacific Journal of Mathematics

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JAMES MICHAEL GARDNER FELL

Vol. 13, No. 2 April 1963

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J. M. G. FELL

Introduction. Throughout this paper G is a fixed locally compact group. Let us recall some concepts bearing on the representation theory of G. The family of all unitary equivalence classes of unitary representations of G will be called $\mathscr{T}(G)$. A function φ of positive type on G is *associated* with a subset $\mathscr{S}$ of $\mathscr{T}(G)$ if there is an S in $\mathscr{S}$, and a vector ξ in the space $H(S)$ of S, such that $\varphi(x) = (S_x\xi, \xi)$ for all x in G. An element T of $\mathscr{T}(G)$ is *weakly contained* in a subset $\mathscr{S}$ of $\mathscr{T}(G)$ if every function of positive type on G associated with T can be approximated uniformly on compact sets by sums of functions of positive type associated with $\mathscr{S}$. The notion of weak containment leads to that of the *inner hull-kernel topology* of $\mathscr{T}(G)$: A net $\{T^i\}$ of elements of $\mathscr{T}(G)$ converges to T in this topology if and only if every subnet of $\{T^i\}$ weakly contains T. Relativized to the subset $\hat{G}$ of $\mathscr{T}(G)$ consisting of the irreducible representations of G, this topology becomes the ordinary *hull-kernel topology* of $\hat{G}$. (For these notions and facts see [1] and [2]).

If H is a Hilbert space, the *adjoint space* $\bar{H}$ of H can be defined as the Hilbert space whose underlying set is the same as that of H, and which is conjugate-isomorphic with H under the identity map. If T is a unitary representation of G, the *adjoint representation* $\bar{T}$ is defined by the requirements: $H(\bar{T}) = H(T)^-$, $\bar{T}_x = T_x (x \in G)$. The *Kronecker product* $S \otimes T$ of two unitary representations S and T of G is that representation whose space is $H(S) \otimes H(T)$, and for which $(S \otimes T)_x(\xi \otimes \eta) = (S_x\xi) \otimes (T_x\eta)$. We can also describe the Kronecker product $S \otimes \bar{T}$ as follows: $H(S \otimes \bar{T})$ is the Hilbert space of all Hilbert-Schmidt operators on $H(T)$ to $H(S)$, and $(S \otimes \bar{T})_x(A) = S_xAT_x^{-1}$.

If $\mathscr{S} \subset \mathscr{T}(G)$ and $\mathscr{T} \subset \mathscr{T}(G)$, let $\mathscr{S} \otimes \mathscr{T}$ denote $\{S \otimes T \mid S \in \mathscr{S}, T \in \mathscr{T}\}$.

Throughout this paper I will be the one-dimensional identity representation of G. It is well known and easily verified that if S and T are finite-dimensional unitary representations of G and T is irreducible, $S \otimes \bar{T}$ contains I if and only if S contains T. Can this be generalized to the case where S and T are infinite-dimensional and 'containment' is replaced by 'weak containment'? The main object of this note is to answer this question affirmatively for the case that S is infinite-dimensional but T is still finite-dimensional (Theorem 4). In preparation for this we shall show (Theorem 2) that the Kronecker product oper-

Received January 21, 1963.

ation is continuous with respect to the inner hull-kernel topology of $\mathscr{T}(G)$.

Another by-product of the main result is the following strenthening (Theorem 3) of a remark of Godement ([4], p. 77): If the regular representation R of G weakly contains some finite-dimensional irreducible unitary representation of G, then R weakly contains all unitary representations of G.

1. The continuity of the Kronecker product.

LEMMA 1. *Suppose that $\mathscr{S} \subset \mathscr{T}(G)$ and $T \in \mathscr{T}(G)$; and let K be the set of all those ξ in $H(T)$ such that the function φ defined by $\varphi(x) = (T_x\xi, \xi)(x \in G)$ can be approximated, uniformly on compact sets, by sums of functions of positive type associated with $\mathscr{S}$. Then K is a closed T-invariant linear subspace of $H(T)$.*

Proof. Obviously K is closed in the norm and under scalar multiplication. By the easy argument of [1], p. 368, (ii′), $\sum_{i=1}^{n} a_i T_{x_i}\xi$ is in K whenever $\xi \in K$, the x_i are in G, and the a_i are complex; in particular K is T-invariant. It remains only to show K closed under addition.

Let ξ and η be elements of K; let L and M be the closed invariant subspaces of $H(T)$ generated by ξ and η respectively; and let Q be the closure of $L + M$. By the preceding paragraph

$$L \subset K \quad \text{and} \quad M \subset K . \tag{1}$$

If A is projection onto $L^\perp$, $A(M)$ is a dense subspace of $Q \cap L^\perp$. So by Mackey's form of Schur's Lemma ([7], Theorem 1.2), the restriction of T to the invariant subspace $Q \cap L^\perp$ is equivalent to a subrepresentation of the restriction of T to M. This and (1) show that

$$Q \cap L^\perp \subset K . \tag{2}$$

Putting $\zeta = \xi + \eta$, we have $\zeta = \xi' + \eta'$, where $\xi' \in L$ and $\eta' \in Q \cap L^\perp$. Since L and $Q \cap L^\perp$ are orthogonal and T-invariant,

$$(T_x\zeta, \zeta) = (T_x\xi', \xi') + (T_x\eta', \eta') \tag{3}$$

$(x \in G)$. By (1) and (2) ξ' and η' are in K; so by (3) $\zeta \in K$, and K is closed under addition.

REMARK 1. If A is a C^*-algebra, $\mathscr{T}(A)$ is defined as the set of all equivalence classes of $*$-representations of A. Exactly the same proof shows that Lemma 1 is valid for C^*-algebras, provided that we replace functions of positive type by positive functionals, and uniform approximation on compact sets by weak* approximation.

REMARK 2. According to Lemma 1, T will be weakly contained in $\mathscr{S}$ provided $H(T)$ is generated (under T) by those ξ in $H(T)$ whose associated functions of positive type are approximated by sums of functions of positive type associated with $\mathscr{S}$. For example, we have immediately:

THEOREM 1. *Suppose that* $\mathscr{S}_k \subset \mathscr{T}(G)$ *and* $\mathscr{S}_k$ *weakly contains* $T_k\,(k = 1, 2)$. *Then* $\mathscr{S}_1 \otimes \mathscr{S}_2$ *weakly contains* $T_1 \otimes T_2$.

THEOREM 2. *The map* $\langle S, T\rangle \to S \otimes T$ *(of* $\mathscr{T}(G) \times \mathscr{T}(G)$ *into* $\mathscr{T}(G)$*) is continuous with respect to the inner hull-kernel topology of* $\mathscr{T}(G)$.

Proof. Let $S^i \to S$ and $T^i \to T$ in $\mathscr{T}(G)$. By the definition of the topology of $\mathscr{T}(G)$, we have only to show that the net $\{S^i \otimes T^i\}$ (and hence by the same argument every subnet of it) weakly contains $S \otimes T$. But Theorem 2.2 of [2] clearly shows that the function of positive type associated with each product vector $\xi \otimes \eta$ in $H(S) \otimes H(T)$ can be approximated by functions of positive type associated with the $S^i \otimes T^i$. Hence by Lemma 1 $S \otimes T$ is weakly contained in $\{S^i \otimes T^i\}$.

It should be mentioned that the "easy verification" of the proposition used in the proof of [2], p. 260, Corollary 1, actually requires the above Theorem 1.

2. When does $S \otimes \bar{T}$ weakly contain I? In this section G is assumed to satisfy the second axiom of countability; and we shall consider only unitary representations acting in a separable space.

Suppose that $T \in \hat{G}$ and $S \in \mathscr{T}(G)$. Is it true that $S \otimes \bar{T}$ weakly contains I if and only if S weakly contains T? In general, as we next show, the implication is false in both directions, even if S is assumed irreducible.

Let R be the regular representation of G, and T some irreducible representation weakly contained in R. Clearly $R \cong \bar{R}$. By [6], Theorem 12.2, $R \otimes R$ is a multiple of R. So $R \otimes \bar{R}$ weakly contains I if and only if R does. Choose G so that R does not weakly contain I; for example G might be the free group on two generators, or a non-compact connected semisimple Lie group (see [8]). Then $R \otimes \bar{R}$ does not weakly contain I, and hence, by Theorem 1, nor does $T \otimes \bar{T}$.

For an easy counter-example in the other direction take G to be the "$ax + b$" group, and T to be one of the two infinite-dimensional irreducible representations of G. Then $\bar{T} = I \otimes \bar{T}$ weakly contains I (see [2], Theorem 5.1), but I does not weakly contain T. A "better" example, in which $S \otimes \bar{T}$ weakly contains I but neither S nor T weakly contains the other, will be given in § 3.

However, if T is finite-dimensional, the answer to the question posed above is affirmative (Theorem 4).

LEMMA 2. *If $\mathscr{S} \subset \mathscr{T}(G)$ and $\mathscr{S}$ weakly contains a finite-dimensional irreducible unitary representation T of G, then $\mathscr{S} \otimes \bar{T}$ weakly contains I.*

Proof. $\mathscr{S} \otimes \bar{T}$ weakly contains $T \otimes \bar{T}$ by Theorem 1. Since T is finite-dimensional, $T \otimes \bar{T}$ contains I.

Here is an interesting consequence of Lemma 2:

THEOREM 3. *If the regular representation R of G weakly contains some finite-dimensional irreducible representation T of G, it weakly contains all unitary representations of G.*

Proof. By Lemma 2 $R \otimes \bar{T}$ weakly contains I. But by [2], Lemma 4.2, $R \otimes \bar{T}$ is a multiple of R. Hence R weakly contains I, and the conclusion follows from Godement's remark ([4], p. 77, or [2], p. 260).

LEMMA 3. *Let T be an irreducible finite-dimensional unitary representation of G. To each $\delta > 0$, there is a finite subset F of G and an $\varepsilon > 0$ such that, whenever A is a positive linear operator on $H(T)$ satisfying* (i) $\|A\| = 1$ *and* (ii) $\|AT_x - T_xA\| < \varepsilon$ *for all x in F, then $\|A - E\| < \delta$ (E being the identity operator on $H(T)$).*

Proof. Assume the lemma false. Then there is a $\delta > 0$ and a net $\{A_i\}$ of positive operators in Q such that $A_iT_x - T_xA_i \xrightarrow[i]{} 0$ for all x in G; here Q is the compact set of those positive operators A on $H(T)$ for which $\|A\| = 1$ and $\|A - E\| \geqq \delta$. Replacing $\{A_i\}$ by a subnet, we may assume that $A_i \to A$ in Q. Passing to the limit, we deduce that $AT_x = T_xA$ for all x, whence $A = \lambda E$. Since A is positive and of norm 1, we must have $\lambda = 1$; but this contradicts $\|A - E\| \geqq \delta$.

LEMMA 4. *Suppose that $\mathscr{S} \subset \mathscr{T}(G)$, and T is a finite-dimensional irreducible unitary representation of G such that $\mathscr{S} \otimes \bar{T}$ weakly contains I. Then $\mathscr{S}$ weakly contains T.*

Proof. The family of all finite direct sums of elements of $\mathscr{S}$ weakly contains T if and only if $\mathscr{S}$ does; hence we may assume without loss of generality that $\mathscr{S}$ is closed under finite direct sums. But then I belongs to the quotient closure of $\mathscr{S} \otimes \bar{T}$ ([2], Theorem 1.1).

Let C be a compact subset of G. For fixed $\delta > 0$, choose F and ε as in Lemma 3. Let r be the dimension of $H(T)$; and put $C' = (C \cup F) \cup (C \cup F)^{-1}$.

By [2], Lemma 1.1, there is an S in $\mathscr{S}$ and a unit vector ζ in $H(S \otimes \bar{T})$ such that

$$(4) \qquad \| (S \otimes \bar{T})_x \zeta - \zeta \| < \frac{\varepsilon}{2r^4}$$

for all x in C'. Fixing an orthonormal basis $\xi_1, \cdots, \xi_r$ of $H(T)$, let us write $\zeta = \sum_{i=1}^r \eta_i \otimes \xi_i (\eta_i \in H(S))$, where

$$(5) \qquad 1 = \| \zeta \|^2 = \sum_{i=1}^r \| \eta_i \|^2 .$$

If the matrix of T_x in the basis $\{\xi_i\}$ is $\{\tau_{ij}(x)\}$, we have $\bar{T}_x \xi_i = \sum_{j=1}^r \overline{\tau_{ji}(x)} \xi_j$. So $(S \otimes \bar{T})_x \zeta = \sum_j (\sum_i \overline{\tau_{ji}(x)} S_x \eta_i) \otimes \xi_j$, whence

$$(6) \qquad \| (S \otimes \bar{T})_x \zeta - \zeta \|^2 = \sum_j \left\| \left(\sum_i \overline{\tau_{ji}(x)}\, S_x \eta_i \right) - \eta_j \right\|^2 .$$

By (4) and (6),

$$(7) \qquad \left\| \left(\sum_i \overline{\tau_{ji}(x)}\, S_x \eta_i \right) - \eta_j \right\| < \frac{\varepsilon}{2r^4}$$

$(x \in C', j = 1, \cdots, r)$. From (7) and the unitariness of $\tau(x)$,

$$(8) \qquad \begin{aligned} &\left\| S_x \eta_k - \sum_j \tau_{jk}(x) \eta_j \right\| \\ &\leqq \sum_j | \tau_{jk}(x) | \left\| \left(\sum_i \overline{\tau_{ji}(x)}\, S_x \eta_i \right) - \eta_j \right\| \\ &< \frac{\varepsilon}{2r^3} . \end{aligned}$$

Let A be the linear map of $H(T)$ into $H(S)$ sending ξ_i into $\eta_i (i = 1, \cdots, r)$. Then (8) gives

$$(9) \qquad \| S_x A - A T_x \| < \frac{\varepsilon}{2r^2} \qquad (x \in C') .$$

From this and the symmetry of C',

$$(10) \qquad \| A^* S_x - T_x A^* \| < \frac{\varepsilon}{2r^2} \qquad (x \in C') .$$

By (5), $\| A \| = \| A^* \| \leqq r$ and also

$$(11) \qquad \| A^* A \| \geqq \frac{1}{r} .$$

Hence, denoting $A^*A / \| A^*A \|$ by B, we obtain from (9) and (10) $\| BT_x - T_x B \| < \varepsilon$ $(x \in C')$. Since B is positive, $\| B \| = 1$, and $F \subset C'$, Lemma 3 asserts that $\|B - E\| < \delta$. From this, setting $\eta'_i = \eta_i/\|A\|$, we get

$$(12) \qquad |(\eta'_i, \eta'_j) - \delta_{ij}| < \delta$$

for all i, j. Let $\varphi(x) = (S_x\eta'_1, \eta'_1)(x \in G)$. By (8) and (11) $\|S_x\eta'_1 - \sum_j \tau_{j1}(x)\eta'_j\| < \varepsilon/2r^2$. Combining this with (12) we have for x in C

$$\begin{aligned} |\varphi(x) - \tau_{11}(x)| &\leqq \left|\left(\left(S_x\eta'_1 - \sum_j \tau_{j1}(x)\eta'_j\right), \eta'_1\right)\right| \\ &\quad + \left|\left(\sum_j \tau_{j1}(x)\eta'_j, \eta'_1\right) - \tau_{11}(x)\right| \\ &\leqq \sum_j |\tau_{j1}(x)|\,|(\eta'_j, \eta'_1) - \delta_{j1}| + \frac{\varepsilon}{2r^2}\|\eta'_1\| \\ &\leqq r\delta + \frac{\varepsilon}{2r}, \end{aligned}$$

which is as small as we wish. Thus we have an S in $\mathscr{S}$ and a function φ of positive type associated with S which differs from τ_{11} on C by an arbitrarily small quantity. So $\mathscr{S}$ weakly contains T.

Combining Lemmas 2 and 4 we get:

THEOREM 4. *Let $\mathscr{S}$ be a family of unitary representations of G and T a finite-dimensional irreducible unitary representation of G. Then $\mathscr{S}$ weakly contains T if and only if $\mathscr{S} \otimes \bar{T}$ weakly contains I.*

As a corollary we mention the following weak "Frobenius-like" proposition. As usual, U^S denotes the representation of G induced from the representation S of a subgroup.

COROLLARY. *Let K be a closed subgroup of G, and J and I the identity representations of K and G respectively. We assume that U^J weakly contains I. If $\mathscr{S} \subset \mathscr{T}(K)$, T is a finite-dimensional irreducible unitary representation of G, and $\mathscr{S}$ weakly contains some irreducible component of $T|K$, then $\{U^S \mid S \in \mathscr{S}\}$ weakly contains T.*

Proof. By Theorem 4 $\mathscr{S} \otimes \bar{T}|K$ weakly contains J. Hence by [2], Theorem 4.2, $\{U^{S \otimes \bar{T}|K} \mid S \in \mathscr{S}\}$ weakly contains U^J. By hypothesis the latter weakly contains I; so $\{U^{S \otimes \bar{T}\,K} \mid S \in \mathscr{S}\}$ weakly contains I. But by [2], Lemma 4.2, $U^{S \otimes \bar{T}.K} \cong U^S \otimes \bar{T}$. Hence another application of Theorem 4 gives the required conclusion.

3. A counter-example. Let G be the proper Euclidean group in three-dimensional real space R^3. We observe that the hull-kernel topology of $\hat{G}$ is T_1 (i.e. points are closed). Indeed, the results of [5] show that T_f is completely continuous whenever $T \in \hat{G}$ and $f \in L_1(G)$. So, by [1], Lemma 1.11, $\hat{G}$ is T_1. Thus, if S and T are inequivalent elements of $\hat{G}$, neither weakly contains the other. We shall now construct two inequivalent elements S and T of $\hat{G}$ such that $S \otimes \bar{T}$ weakly contains I (see the beginning of § 2).

Let N and K be the translation and rotation subgroups of G respectively; τ_u will denote translation by u: $\tau_u(v) = u + v (u, v \in R^3)$. Let χ be the fixed character of N defined by $\chi(\tau_u) = e^{iu_1}$. The "stationary subgroup" for χ (consisting of those σ in G such that $\chi(\sigma\tau_u\sigma^{-1}) = \chi(\tau_u)$ for all u) is $Z = HN$, where $H = \{\rho \in K \mid \rho(1, 0, 0) = (1, 0, 0)\}$. Thus, by [6], Theorem 14.1, to each character φ of the Abelian group H we get an irreducible representation T^φ of G, namely, that induced from the character ψ of Z, where

$$\psi(\rho\tau_u) = \varphi(\rho)\chi(\tau_u) \qquad (\rho \in H, u \in R^3) . \tag{13}$$

Further, if φ and φ' are distinct characters of H, T^φ and $T^{\varphi'}$ are inequivalent.

Now let φ and φ' be distinct characters of H. Let $0 < \theta < \pi/2$ and let ρ be the element of K consisting of rotation through an angle θ about the third axis. We verify easily that $Z \cap \rho Z \rho^{-1} = N$. Hence by [3], Theorem 5.4 (the 'weak containment' version of Mackey's Kronecker Product Theorem), $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains the representation of G induced from the character χ_θ of N given by $\chi_\theta(\tau_u) = \chi(\tau_{\rho(u)})\chi(\tau_u)$. (Here $(T^{\varphi'})^-$ is the adjoint of $T^{\varphi'}$). Since this is true whenever $0 < \theta < \pi/2$, we can use [2], Theorem 4.2, to pass to the limit as $\theta \to 0$; we then conclude that $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains U^{χ_0}, where χ_0 is the identity character of N. But U^{χ_0} is obtained by lifting to G the regular representation of the compact group K; hence it contains I as a direct summand. Thus we conclude that $T^\varphi \otimes (T^{\varphi'})^-$ weakly contains I. This is the desired example, since we have already observed that T^φ and $T^{\varphi'}$ are inequivalent irreducible representations of G.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 13, No. 2 April, 1963