Pacific Journal of Mathematics

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Shaul Foguel

Vol. 13, No. 2 April 1963

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SHAUL R. FOGUEL

Introduction. Let H be a Hilbert space and P an operator with $\|P\| = 1$. Our main problem is to find the weak limits of $P^n x$ as $n \to \infty$. This is applied to Markov Processes and to Measure Preserving Transformations.

Markov Processes. Let (Ω, Σ, μ) be a measure space. Let x_n be a sequence of real valued measurable functions on Ω and:

1. $\mu(x_{n+\alpha} \in A \cap x_{m+\alpha} \in B) = \mu(x_n \in A \cap x_m \in B)$.
2. *Conditional probability that* $x_k \in A$ *given* x_i *and* x_j, $i < j < k$, *is equal to conditional probability that* $x_k \in A$ *given* x_j.

Let $I(\sigma)$ denote the characteristic function of σ. Define $P(n)$ by linear extension of:

$P(n)\, I(x_0 \in A) =$ *Conditional probability that* $x_n \in A$ *given* x_0.

Then:

1′. $\|P(1)\| = 1$

2′. $P(n) = P(1)^n$.

For details see [1] and [2].

We will study limits of

$$(P(1)^n\, I(x_0 \in A),\, I(x_0 \in B)) = \mu(x_n \in A \cap x_0 \in B)\ .$$

Many of the results here appear in particular cases in [1,] [2] and [3].

1. Reduction to unitary operators. For every $x \in H$

a. $\|P^{*k}P^kP^nx - P^nx\|^2 \leqq 2\|P^nx\|^2 - 2\,Re(P^{*k}P^kP^nxP^nx)$
$$= 2(\|P^nx\|^2 - \|P^{n+k}x\|^2) \underset{n\to\infty}{\longrightarrow} 0$$

b. $\|P^kP^{*k}P^nx - P^nx\|^2 \leqq \|P^{*k}P^kP^{n-k}x - P^{n-k}x\|^2 \underset{n\to\infty}{\longrightarrow} 0.$

Therefore:

If weak lim $P^{n_i}x = y$ then $P^{*k}P^ky = P^kP^{*k}y = y$ (here and elsewhere n_i or m_i will denote a subsequence of the integers). This means $\|y\| = \|P^ky\| = \|P^{*k}y\|$. Notice that if $P^*Px = x$ then $\|Px\|^2 = (P^*Px, x) = \|x\|^2$. On the other hand

$$\|Px\|^2 = (P^*Px, x) \leqq \|P^*Px\|\,\|x\| \leqq \|x\|^2 \text{ since } \|P\| = 1.$$

Hence if $\|Px\| = \|x\|$ then $(P^*Px, x) = \|P^*Px\|\,\|x\|$ and thus $P^*Px = x$.

THEOREM 1.1. *Let* $K = \{x \,|\, \|P^kx\| = \|P^{*k}x\| = \|x\|\ k = 1, 2, \cdots\}$

Received November 26, 1962. The research reported in this document has been sponsored in part by Air Force Office of Scientific Research, O.A.R. through the European Office, Aerospace Research, United States Air Force.

then K is a subspace of H, invariant under P and P^*. On K the operator P is unitary. If $x \perp K$ then

$$\text{weak} \lim_{n\to\infty} P^n x = \text{weak} \lim_{n\to\infty} P^{*n} x = 0 .$$

Proof. It is only necessary to prove the last part. If $x \perp K$ and $y = \text{weak} \lim P^{n_i} x$ then by the preceding remark $y \in K$ hence $y = 0$. Now from the weakly sequentially compactness follows: weak lim $P^n x = 0$.

This theorem is a consequence of Theorem 2 of [9] and was reproduced here only because of the elementary proof.

If F is the selfadjoint projection on K and H is finite dimensional, then F is the spectral measure of the circumference of the unit circle in the sence of Dunford's spectral theory, with respect to P. This is no longer true when H is infinite dimensional and P a spectral operator (even a scalar type operator) in the sense of Dunford. These remarks are proved in [4].

LEMMA 2.1. *Let $y = \text{weak} \lim P^{n_i} x$. Then $\|y\|^2 \leqq \limsup |(P^n x, x)|$.*

Proof. Let $x = u + v$ where $u \in K$ and $v \perp K$. Then $y = \text{weak} \lim P^{n_i} u$, $\limsup |(P^n x, x)| = \limsup |(P^n u, u)|$. Now

$$|(y, P^k u)| = \lim_{i\to\infty} |(P^{n_i} u, P^k u)| = \lim |(P^{n_i - k} u, u)|$$

since $u \in K$. Thus

$$\|y\|^2 = \lim |(y, P^{n_i} u)| \leqq \limsup |(P^n u, u)| .$$

This could also be written in the form

$$\limsup |(P^n x, z)| \leqq \|z\| \limsup |(P^n x, x)|^{1/2} .$$

DEFINITION A. Let $H_0 = \{x \mid \lim (P^n x, x) = 0\}$.

THEOREM 3.1. *$x \in H_0$ if and only if weak $\lim P^n x = 0$, if and only if weak $\lim P^{*n} x = 0$. The set H_0 is a closed subspace of H containing $K^\perp$. If T commutes with P or with P^* and $x \in H_0$ then $Tx \in H_0$.*

Proof. The first parts of the theorem follow from Lemma 2.1 and Theorem 1.1. Now if $TP = PT$ and $P^n x \xrightarrow{w} 0$ then $P^n Tx = TP^n x \xrightarrow{w} 0$.

Applications.

1. Markov processes.

a. If $\lim_{n\to\infty} \mu(x_n \in A \cap x_0 \in A) = 0$ then $\lim_{n\to\infty} \mu(x_n \in A \cap x_0 \in B) = 0$ and $\lim_{n\to\infty} \mu(x_0 \in A \cap x_n \in B) = 0$ for every set B.

b. Let $\lim \mu(x_n \in A \cap x_0 \in A) = \mu(x_0 \in A)^2$. Put $x = I(x_0 \in A) - \mu(x_0 \in A)$. (Provided that $\mu(\Omega) < \infty$ so that $1 \in L_2$).
Then

$$\begin{aligned}(P(1)^n x, x) &= (I(x_n \in A) - \mu(x_0 \in A), I(x_0 \in A) - \mu(x_0 \in A)) \\ &= \mu(x_n \in A \cap x_0 \in A) - \mu(x_0 \in A)^2 \to 0 .\end{aligned}$$

Thus for every Borel set B:

$$\lim (I(x_n \in A) - \mu(x_0 \in A), I(x_0 B)) = 0$$

or

$$\mu(x_n \in A \cap x_0 \in B) \to \mu(x_0 \in A)\, \mu(x_0 \in B) .$$

Similarly

$$\mu(x_0 \in A \cap x_n \in B) \to \mu(x_0 \in A)\, \mu(x_0 \in B) .$$

2. Measure preserving transformations. Let φ be a M.P.T. on (Ω, Σ, μ). If $\mu(\varphi^{-n}(A) \cap A) \to 0$ then

$$\lim_{n\to\infty} \mu(\varphi^{-n}(A) \cap B) = \lim_{n\to\infty} \mu(A \cap \varphi^{-n}(B)) = 0 .$$

if $\lim \mu(\varphi^{-n}(A) \cap A) = \mu(A)^2$ and $\mu(\Omega) < \infty$ then

$$\begin{aligned}&\mu(\varphi^{-n}(A) \cap B) \to \mu(A)\mu(B) \\ &\mu(A \cap \varphi^{-n}(B)) \to \mu(A)\mu(B) .\end{aligned}$$

3. Measure theory. Let μ be a positive finite measure on Borel subsets of $(0, 2\pi)$. Define the operator P by $Pf(\vartheta) = e^{i\vartheta}f(\vartheta)$. Then H_0 is the set of all functions f such that

$$\int_0^{2\pi} e^{in\vartheta} |f(\theta)|^2 \mu(d\vartheta) \to 0 .$$

Let $f \in H_0$ and $A_\varepsilon = \{\vartheta \,|\, |f(\vartheta)| \geqq \varepsilon\}$. Define $g_\varepsilon = 1/f$ on A_ε and zero elsewhere. Finally let

$$T_\varepsilon h(\vartheta) = g_\varepsilon(\vartheta)h(\vartheta) .$$

Then T_ε commutes with P and by Theorem 3.1

$$\int_A e^{in\vartheta} \mu(d\vartheta) \to 0$$

where $A = \cup A_\varepsilon$.

By taking unions of such sets one can prove:
There exists a set B such that for every h whose support is contained in B a.e.

$$\int e^{in\vartheta} |h(\vartheta)|^2 \mu(d\vartheta) \to 0$$

and this holds only for such functions.

2. **Positive contractions.** In this section we assume that H is the real Hilbert space $L_2(\Omega, \Sigma, \mu)$ where $\mu \geqq 0$ and $\mu(\Omega) = 1$. An operator S will be called positive if:

a. If $f \geqq 0$ a.e. than $Sf \geqq 0$ a.e.
b. $S1 = 1$.
c. $||S|| = 1$.

We will assume that P is positive. It is easily seen that so are P^*, $P^n P^{*n}$ and $P^{*n} P^n$.

LEMMA 1.2. *Let S be a positive operator on* $L_2(\Omega, \Sigma, \mu)$. *The space*

$$L = \{f \mid Sf = f\}$$

is generate by characteristic functions of a σ *subfield,* Σ', *of* Σ:
$f \in L$ *if and only if* f *is* Σ' *measurable.*

Proof. Let Σ' contain all $\sigma \in \Sigma$ such that $SI(\sigma) = I(\sigma)$. If $Sf = f$ then

$$||f||^2 \geqq (S|f|, |f|) \geqq |(Sf, f)| = ||f||^2$$

hence $S|f| = |f|$ therefore if $f, g \in L$ so do $\max(f, g)$ and $\min(f, g)$. This shows in particular that Σ' is a field and since L is closed it is a σ field.

Now if $f \in L$ so does $f - c$ for any constant, thus it is enough to show that

$$\{\omega \mid f(\omega) > 0\} \in \Sigma' :$$

Let f_+ be the positive part of f, $2f_+ = |f| + f \in L$. Thus $\varepsilon^{-1} \min(\varepsilon, f^+) \in L$ but as $\varepsilon \to 0$ this converges to $I\{\omega \mid f(\omega) > 0\}$.

This Lemma was proved in [8].

THEOREM 2.2. *The space K is generated by characteristic functions of a* σ *subfield* Σ_1 *of* Σ. *If* $\sigma \in \Sigma_1$ *then* $PI(\sigma) = I(\tau)$ *where* $\tau \in \Sigma_1$, *similarly for* P^*.

Proof. The space K is the intersection of the space

$$\{f \mid \|P^n f\| = \|f\|\}, \quad \{f \mid \|P^{*n} f\| = \|f\|\} \quad n = 1, 2, \cdots$$

By Lemma 1 each of this is generated by a σ subfield of Σ. Thus K is generated by the intersection of these subfields.

Now if $\sigma \in \Sigma_1$ then $\sigma' = \Omega - \sigma \in \Sigma_1$ too. The functions $P(I(\sigma))$ and $P(I(\sigma'))$ are positive, bounded by 1 and $(P(I(\sigma)), P(I(\sigma'))) = (P^*P(I(\sigma)), I(\sigma')) = (I(\sigma), I(\sigma')) = 0$. Moreover $P(I(\sigma)) + P(I(\sigma')) = 1$, therefore, both functions are characteristic functions. As K is invariant under P these are characteristic functions of sets in Σ_1.

Let $I(A)$ and $I(B)$ belong to K. Then

$$P(I(A) \cdot I(B)) \leqq \min \{P(I(A)), P(I(B))\} = P(I(A)) \cdot P(I(B)) .$$

On the other hand

$$P^*[(P(I(A)) \cdot P(I(B))] \leqq I(A) \cdot I(B)$$

or

$$P(I(A)) \cdot P(I(B)) \leqq P(I(A) \cdot I(B)) .$$

Therefore

$$P(I(A) \cdot I(B)) = P(I(A)) \cdot P(I(B)) .$$

It could be shown that if $f, g \in K$ and $f \cdot g \in L_2$ then $P(fg) = Pf \cdot Pg$.

Thus if $Pf = \alpha f$ and $Pg = \beta g$ where $|\alpha| = |\beta| = 1$ then $f, g \in K$ and if $f \cdot g \, L_2$ then $P(fg) = \alpha\beta fg$.

If $Pf = \alpha f$ where $|\alpha| = 1$ let $f = |f| h$ then:

$$\|f\|^2 \geqq (P|f|, |f|) \geqq |(Pf, f)| = \|f\|^2 .$$

Therefore, $P|f| = |f|$ necessarily $Ph = \alpha h$. It follows that

$$P(|f| h^2) = \alpha^2 |f| h^2 .$$

This is a Theorem of [8].

Following [1] let us define:

Doeblin's Condition. There exists a positive finite measure ν define on Σ, and a positive ε such that: If $\nu(\sigma) < \varepsilon$ then for some n either

$$\|P^n(I(\sigma))\| < \mu(\sigma)^{1/2}$$

or

$$\|P^{*n}(I(\sigma))\| < \mu(\sigma)^{1/2} .$$

Using the same arguments as in Theorem 3.11 and its corollaries of [1] we conclude.

THEOREM 3.2. *If Doeblin's condition holds then* $\Sigma_1 = \{\sigma_1, \cdots, \sigma_n\}$ *where* σ_i *are disjoint sets such that*

1. $\bigcup_{i=1}^n \sigma_i = \Omega$
2. $P^n(I(\sigma_i)) = I(\sigma_i) = P^{*n}(I(\sigma_i))$.
3. *The operator* $P(P^*)$ *acts as a permutation on the* σ_i *sets.*
4. *For each* $f, g, \in L_2$

$$\lim_{k\to\infty} (P^{nk+d}f, g) = \sum_{i=1}^n \mu(\sigma_i)^{-1} \int_{\sigma_i} f(\omega)\mu(d\omega) \int_{P^d\sigma_i} g(\omega)\mu(d\omega)$$

where $P^d\sigma_i$ *denotes the set whose characteristic function is* $P^d(I(\sigma_i))$.

Thus if x_n is a Markov process and $\mu(\Omega) = 1$ then

$$\lim \mu(x_{kn+d} \in A \cap x_0 \in B) = \sum_{i=1}^n \mu(\sigma_i)^{-1}\mu(x_0 \in A \cap \sigma_i)\mu(x_0 \in B \cap P^d\sigma_i) .$$

For detailed proves of these results and treatment of the case $\mu(\Omega) = \infty$ in the case of Markov processes see [1] and [3].

Measure Preserving Transformations. Let φ be a measure preserving transformation on (Ω, Σ, μ). The operator P is defined on $L_2(\Omega, \Sigma, \mu)$ by $Pf = g$ where $g(\omega) = f(\varphi(\omega))$. It is a positive contraction. Thus the space K is generated by all characteristic functions f that satisfy $\|P^{*n}f\| = \|f\|$, for P is an isometry. Let the restriction of P to K be denoted by U and let Σ_1 be the Boolean algebra that generates K. On Σ_1 φ acts like a measure preserving invertable transformation. (It maps Σ_1 onto itself).

We will use here the terminology of [5]

THEOREM 4.2. *The transformation* φ *on* Σ *is ergodic, weakly mixing or strongly mixing, if and only if,* φ *on* Σ_1 *is ergodic, weakly mixing or strongly mixing, respectively.*

Proof. It is clear that if P satisfies any of the requirements so does U. Conversely:

a. Let U be ergodic. If P was not then for some nonconstant function f, $Pf = f$. But then $P^nf = P^{*n}f = f$ and $f \in K$, so U is not ergodic.

b. Let U be weakly mixing. Given $f = f_1 + f_2$ where $f_1 \in K f_2 \perp K$ then for every g

$$\frac{1}{n}\sum_{j=0}^{n-1} |(P^jf, g) - (f, 1)(1, g)| \leqq \frac{1}{n}\sum_{j=0}^{n-1} |(P^jf_1, g) - (f_1, 1)(1, g)| + \frac{1}{n}\sum_{j=0}^{n-1} |(P^jf_2, g) - (f_2, 1)(1, g)| .$$

The first term tends to zero because U is weakly mixing and g can be replaced by the projection of g on K. The second term is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1}|(P^j f_2, g)|$$

for $(f_2, 1) = 0$. Thus it tends to zero with $(P^n f_2, g)$.

c. Let U be strongly mixing. Put again $f = f_1 + f_2$ $P^n f_1$ tends weakly to $(f_1, 1)\,1 = (f, 1)\,1$ and $P^n f_2$ tends weakly to zero.

COROLLARY. *The transformation φ is weakly mixing, if and only if, P has on the unit circle no eigenvalue except for* 1 *which is a simple eigenvalue.*

This generalizes the 'Mixing Theorem' in [5] page 39.

Proof. The operator U satisfies the same condition and by the 'Mixing Theorem' is weakly mixing. By the previous theorem so is P.

3. The space H_c.

DEFINITION. $H_c = \{x \mid x \in K$ *and the set* $P^n x$ $n = 1, 2, \cdots$ *is conditionally compact*$\}$.

The set H_c is a subspace of H, invariant under P and P^*. $P^{n_i}x$ converges for $x \in K$ iff $(P^{n_i}x, P^{n_j}x) \to_{n_i, n_j \to \infty} \|x\|^2$. This is equivalent to $(P^{*n_i}x, P^{*n_j}x) \to \|x\|^2$ because P is unitary. Thus P could be replaced by P^* in the definition.

THEOREM 1.3. *The following conditions are equivalent*:
a. $x \in K$ *and* $P^n x$ *contains a convergent subsequence.*
b. *There exists a subsequence* m_i *such that* $x = \lim P^{m_i}x$.
c. $\limsup |(P^n x, x)| = \|x\|^2$.

Proof.
$a \Rightarrow b$: Let $P^{n_i}x \to y$ then

$$\|x\|^2 = \|y\|^2 = \lim (P^{n_i}x, P^{n_{i-1}}x) = \lim (P^{n_i - n_{i-1}}x, x)$$

because $x \in K$.
Hence $\|x - P^{n_i - n_{i-1}}x\| \to 0$.
$b \Rightarrow c$: obvious.
$c \Rightarrow a$: Let $\lim |(P^{n_i}x, x)| = \|x\|^2$ and weak $\lim P^{n_i}x = y$. Then $|(y, x)| = \|x\|^2$ while $\|y\| \leqq \|x\|$ hence $y = \alpha x$ where $|\alpha| = 1$.
From [7] page 79 $P^{n_i}x$ converges strongly to αx. Finall if $Z \in H_0$ then:

$$(Z, x) = \lim \alpha^{-1}(Z, P^{n_i}x) = \lim \alpha^{-1}(P^{*n_i}Z, x) = 0 .$$

It is clear that if $x \in H_c$ then condition (a) is satisfied hence the other conditions. In particular $H_c \perp H_0$.

THEOREM 2.3. *If $x \in H_c$ and $y = \lim_{i\to\infty} P^{n_i}x$ then there exists a subsequence k_i so that*

$$x = \lim P^{k_i}y .$$

Proof Let k_i be chosen so that

$$x = \lim P^{n_i + k_i}x .$$

Then

$$\lim \|x - P^{k_i}y\| = \lim \|P^{n_i}x - y\| = 0 .$$

4. Finitely many limits. Let x be such that the sequence $(P^n x, x)$ has finitely many limits. Let these be $c_1, c_2, \cdots, c_r$ where $|c_i| \leqq |c_{i+1}|$.

DEFINITION C. $L = \{z \mid P^n z = z \text{ for some } n\}$. If $z \in L$ then $az \in L$. If $z \in L$ and $y \in L$ then:

$$P^n z = z , \quad P^m y = y \Rightarrow P^{nm}(z + y) = z + y .$$

Thus L is a linear manifold, also $\bar{L} \subset H_c$.

If $z \in H$ let $\{z\}^0$ be the set consisting of z alone and $\{z\}^n$ be the set of all weak limits of $P^m y$ where $y \in \{z\}^{n-1}$.

Let $x = x_0 + x_1$ where $x_0 \in H_0$ $x_1 \perp H_0$. Then

$$(P^n x, x) = (P^n x_0, x_0) + (P^n x_1, x_1), \lim (P^n x_0, x_0) = 0 .$$

Thus we will assume that $x \perp H_0$.

LEMMA 1.4. *For some k $\{x\}^k \cap L \neq 0$.*

Proof. Let $0 \neq y \in \{x\}^1$ then for every n $(y, P^n x)$ is equal to one of the values c_i and:

a. For every $n \geqq 0$ $(P^n y, y)$ can assume only the values c_i $1 \leqq i \leqq r$.

Let $(y, y) = |c_i|$. If for some k $|(P^k y, y)| = (y, y)$ then $P^k y = \lambda y$ with $|\lambda| = 1$. Thus λ must be a root of one for $(P^{nk}y, y) = \lambda^n(y, y)$ assumes finitely many values. Therefore in this case $y \in L$.

If $|(P^n y, y)| < (y, y)$ for every n then

$$\limsup_{n\to\infty} |(P^n y, y)| < (y, y) .$$

Also $\limsup (P^n y, y,) \neq 0$ for $y \perp H_0$. Thus we may choose a subsequence n_i so that $P^{n_i} y$ will converge weakly to $z \neq 0$. Now z satisfies a and $||z|| < ||y||$ by Lemma 2.1.

This procedure cannot be continued more than r times thus at some stage we must get an element of L.

LEMMA 2.4. *If u is the projection of x on $\bar{L}$ then $u \in L$.*

Proof. Let $0 \neq y \in \{x\}^k \cap L$. Then $y \in \{u\}^k + \{x - u\}^k$. Now $y \in L$ and $x - u \perp L$. Also L is invariant under P and P^* hence $\{x - u\}^k \perp L$ and $y \in \{u\}^k$. By Theorem 2.3 $u \in \{\overline{P^n y}\}$ which is a finite set in L.

THEOREM 3.4. *If the sequence $(P^n x, x)$ has finitely many limits then $x = x_0 + x_1$ where $x_0 \in H_0$ and $x_1 \in L$.*

Proof. Let $x_1 = u + v$ where $u \in L$ (by Lemma 2.4.) and $v \perp L$. Now $(P^n v, v) = (P^n x_1, x_1) - (P^n u, u)$ has finitely many limits and by Lemma 1.4 cannot be orthogonal to L unless it is zero.

If limit $(P^n x, x)$ exists then $P x_1 = x_1$.

If L is one dimensional (for instance ergodic transformations) then the conditions of Theorem 3.4 imply that $P x_1 = x_1$.

THEOREM 4.4. *Let $A = \{x$ the sequence $(P^n x, x)$ has finitely many limits$\}$. If linear combinations of elements of A are dense in H, then the eigenvalues of P on the circumference of the unit circle, are roots of 1.*

Proof. Let $Px = \lambda x$ where $|\lambda| = 1$. Let $x_i \in A$ and $y = \Sigma a_i x_i$ where $||x - y|| < 1/2\, ||x||$.
Since $x \perp H_0$ we may assume that for some integers k_i $P^{k_i} x_i = x_i$. Hence for $k = k_1 k_2 \cdots k_n$ we have $P^k y = y$. Thus

$$\lambda^{km} x = P^{km} x = y + P^{km}(x - y) .$$

Therefore

$$|\lambda^{km} - 1|\, ||x|| \leqq ||\lambda^{km} x - y|| + ||y - x|| < ||x|| .$$

This equation cannot be satisfied for all values of m unless λ^k is a root of 1.

5. Semi groups of contractions. Let $P(t)$ be a strongly continuous semi group of contractions $0 \leqq t$. For every $\delta > 0$ $P(\delta)$ defines the subspace $K(\delta)$ as in Theorem 1.1.

LEMMA 1.5. *$x \in K(\delta)$ if and only if*

$$||P(t)x|| = ||P(t)^*x|| = ||x|| \quad 0 \leqq t < \infty .$$

Proof. Trivially the condition is sufficient. If $x \in K(\delta)$ and $t \leqq n\delta$ then

$$||x|| = ||P(n\delta)x|| = ||P(n\delta - t)\, P(t)x|| \leqq ||P(t)x|| \leqq ||x|| .$$

Thus $||P(t)x|| = ||x||$ and similarly $||P(t)^*x|| = ||x||$.

Thus all the spaces $K(\delta)$ are the same and will be denoted by K.

THEOREM 2.5. *The space K is invariant under $P(t)$ and $P(t)^*$ for all t. On K $P(t)$ is unitary. If $x \perp K$ then*

$$\text{weak} \lim_{t\to\infty} P(t)x = 0$$

and by symmetry

$$\text{weak} \lim_{t\to\infty} P(t)^*x = 0 .$$

Proof It was shown that $K = K(t)$ hence by Theorem 1.1 K is invariant under $P(t)$ and $P(t)^*$ and $P(t)$ is unitary on K.

Let $x \perp K$ and let $y \in H$ and $\varepsilon > 0$ be given. Choose η so that

$$||P(s)x - x|| < \varepsilon. \quad \text{if } s \leqq \eta .$$

Choose n_0 so that

$$|(P(n\eta)x, y)| < \varepsilon \quad \text{if } n \geqq n_0 .$$

This is possible by Theorem 1.1. If

$$(n + 1)\eta \geqq t \geqq n\eta > n_0\eta$$

then

$$|(P(t)x, y)| \leqq |(P(n\eta)x, y)| + |(P(t)x - P(n\eta)x, y)| .$$

The first term is less than ε because $n > n_0$. The second term is bounded by

$$\begin{aligned} ||y||\, ||P(t)x - P(n\eta)x|| &= ||y||\, ||P(n\eta)\,(P(t - n\eta)x - x)|| \\ &\leqq ||y||\, ||P(t - n\eta)x - x|| \leqq ||y||\varepsilon \end{aligned}$$

for $0 \leqq t - n\eta \leqq \eta$.

This is proved also in [9] Theorem 4.

Let us assume in this section:

(*) *For some $t_0 > 0$ the operator $P(t_0)\, P(t_0)^*$ is the sum of a compact operator and an operator of norm less then one.*

This is equivalent to:

(**) *For some* $0 < t_0$ *the point* 1 *is isolated in the spectrum of* $P(t_0)\,P(t_0)^*$ *and the space of eigenvectors corresponding to it is finite.*

It is clear that (**) implies (*). Now if 1 is not an isolated point of the spectrum, with finite eigenvectors space, there is a sequence of orthonormal vectors x_n such that

$$||P(t_0)\,P(t_0)^*x_n - x_n|| \to 0\ .$$

(We use here the fact that $P(t_0)\,P(t_0)^*$ is self adjoint). Let

$$P(t_0)\,P(t_0)^* = A + B$$

where B is compact and $||A|| < 1$. Then

$$||Ax_n + Bx_n - x_n|| \to 0\ .$$

But B is compact hence $Bx_n \to 0$ hence

$$||Ax_n - x_n|| \to 0$$

and 1 is the spectrum of A contrary to assumption.

It is easily seen that $P(t)\,P(t)^*$ satisfy, also, the condition if $t > t_0$: $P(t)\,P(t)^* = P(t-t_0)P(t_0)P(t_0)^*P(t-t_0)^*$. Let

$$K(t) = \{x \,|\, ||P(t)^*x|| = ||x||\} = \{x \,|\, P(t)P(t)^*x = x\}\ .$$

Then $K(t_1) \subset K(t_2)$ if $t_1 > t_2$ and $K(t)$ is finite dimensional when $t \geqq t_0$.

For some $s > 0$ dim $K(s)$ is minimal hence $K(s) = K(s+h)$ for all $h \geqq 0$. Let us denote $K(s)$ by K.

LEMMA 3.5. *The space* K *is invariant under* $P(h)^*$ *and* $P(h)$ *for all* $h > 0$.

Proof. If $x \in K$ then $x \in K(s+h)$ hence

$$||P(s+h)^*x|| = ||x||$$

hence

$$||x|| = ||P(s)^*P(h)^*x|| \leqq ||P(h)^*x|| \leqq ||x||$$

or $P(h)^*x \in K$.

Now on the finite dimensional space K, the operator $P(h)^*$ is norm preserving and therefore onto.

If $x \in K$ then for some $y \in K$ $P(h)^*y = x$ and $||x|| = ||y||$. Thus $P(h)x = y \in K$.

We may assume that $s \geqq t_0$.

The subspace $K^\perp$ is also invariant under $P(t)$ and $P(t)^*$. Now

$P(s)\,P^*(s)$ is quasi compact on K and

$$(P(s)\,P^*(s)x, x) < 1 \quad x \in K^{\perp} .$$

Hence on $K^{\perp}$ $||P(s)|| = c < 1$:
The operator $P(s)$ *is quasi compact on* H (in the sense of (*).

Let A be the infinitesimal generator of $P(t)$ then:

1. On K the operator $(1/i)A$ is self adjoint.
2. On $K^{\perp}$

$$\sigma(A) \subset \{\lambda \,|\, Re\,\lambda \leqq \omega_0\}$$

where

$$\omega_0 = \lim_{t \to \infty} t^{-1} \log ||P(t)|| .$$

See [6] corollary to Theorem 11.5.1
Now

$$\omega_0 = \lim_{n \to \infty} (ns)^{-1} \log ||P(ns)|| \leqq \lim_{n \to \infty} (ns)^{-1} \log ||P(s)||^n \leqq s^{-1} \log c < 0 .$$

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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Pacific Journal of Mathematics
Vol. 13, No. 2 April, 1963