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**ON A CLASSICAL THEOREM OF NOETHER IN IDEAL
THEORY**

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ON A CLASSICAL THEOREM OF NOETHER IN IDEAL THEORY

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A classical result in the ideal theory of commutative rings is that an integral domain D with unit is a Dedekind domain if and only if D is noetherian, of dimension less than two, and integrally closed. [8; 275]. The statement of this theorem is due essentially to Noether [6; 53], though the present statement is a refined version of Noether's theorem. (See Cohen [1; 32] for the historical development of the theorem above.) Noether did not, in fact, require that the domain D contain a unit element. By imposing greater restrictions on the prime ideal factorization of each ideal, she showed that D must contain a unit element.

This paper considers an integral domain J with Property C : Every ideal of J may be expressed as a product of prime ideals.

In particular, it is shown that an integral domain J with property C need not contain a unit element. However, factorization of an ideal as a product of prime ideals is unique and J is noetherian, of dimension less than two, and integrally closed.¹ A domain without unit having these three properties need not have property C . If J does not contain a unit element, J is the maximal ideal of a discrete valuation ring V of rank one such that V is generated over J by the unit element e , and conversely. The structure of all such valuation rings V is known. [4; 62].

If J is an integral domain with quotient field k , then J^* will denote the subring of k generated by J and the unit element e of k . We will assume that all domains considered contain more than one element.

If D is an integral domain, not necessarily containing a unit, and if k is the quotient field of D , the definitions of fractionary ideals of D , of sums, products and quotients of fractionary ideals, and of the fractionary ideal (u_1, u_2, \dots, u_i) of D generated by finitely many elements u_1, u_2, \dots, u_i of k , are generalized in the obvious ways. In particular, D^* is a fractionary ideal of D and if \mathcal{S} is the collection of all nonzero fractionary ideals of D , \mathcal{S} is an abelian semigroup under multiplication with unit element D^* . A fractionary ideal F of

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¹ A domain D with quotient field k is integrally closed if D contains every element x of k with the following property: There exist elements d_0, d_1, \dots, d_n of D such that $x^{n+1} + d_n x^n + \dots + d_1 x + d_0 = 0$.

D is said to be invertible if F has an inverse when considered as an element of \mathcal{S} . A nonzero principal fractionary ideal is invertible and $(d)^{-1} = (1/d)$. A product of fractionary ideals is invertible if and only if each of the factors is invertible. [3; 271].

The following two lemmas may be proved by making minor changes in the usual proofs given in the case of a domain with unit. [8; 272-273]. While the proof of Theorem 1 is definitely a modification of the usual proof for a domain with unit, the author feels enough difficulties arise to prove Theorem 1 here.

LEMMA 1. *If A is an invertible fractional ideal of the integral domain D , then $A^{-1} = D^*$: A . Further, A has a finite module basis over D .*

LEMMA 2. *Suppose A is a proper ideal of the domain D such that A may be expressed as a product of invertible prime ideals of D . This representation is unique if $D \subset D^*$, or unique to within factors of D if $D = D^*$.*

Henceforth in this paper, J will denote an integral domain without unit such that J has property C .

THEOREM 1. *Every nonzero proper prime ideal of J is invertible and maximal.*

Suppose first that there exists a nonzero proper invertible prime ideal P of J such that P is not maximal. We chose a such that $P \subset P + (a) \subset J$. We express $P + (a)$ and $P + (a^2)$ as products of prime ideals: $P + (a) = J^k P_1 \cdots P_r$, $P + (a^2) = J^t Q_1 \cdots Q_s$ where each P_i and each Q_j is a proper ideal of J . In $\bar{J} = J/P$ we have: $(\bar{a}) = \bar{J}^k \bar{P}_1 \cdots \bar{P}_r$, $(\bar{a})^2 = \bar{J}^t \bar{Q}_1 \cdots \bar{Q}_s$. By Lemma 2, $s = 2r$ and by proper labeling $P_i = Q_{2i-1} = Q_{2i}$. If \bar{J} does not contain a unit element, then Lemma 2 implies also that $t = 2k$ so that $P + (a^2) = [P + (a)]^2$. If \bar{J} contains a unit, then $(\bar{a}) = \bar{J}^k \bar{P}_1 \cdots \bar{P}_r$ so that r is positive and $(\bar{a}) = \bar{P}_1 \cdots \bar{P}_r$. Similarly, $(\bar{a})^2 = \bar{Q}_1 \cdots \bar{Q}_s$. Therefore $[P + (a)]^2 = P_1^2 \cdots P_r^2 = P + (a^2)$. For either case, therefore, $P + (a^2) = [P + (a)]^2$. The remainder of the proof of the theorem is the same as the proof appearing in [8; 273].

THEOREM 2. *J is a noetherian domain.*

We first show that J is finitely generated. Thus if J contains a proper nonzero prime ideal P , then $P = (p_1, \dots, p_s)$ is maximal and

finitely generated by Theorem 1 and Lemma 1. Therefore if $d \in J$, $d \notin P$, then $J = (p_1, \dots, p_s, d)$. If (0) is the only proper prime ideal of J , then given $d \in J$, $d \neq 0$, $(d) = J^k$ for some integer $k \geq 1$. Then J is invertible, and hence finitely generated.

It follows that every prime ideal of J is finitely generated. Since J has property C , every ideal of J is finitely generated.

THEOREM 3. *Every nonzero ideal of J is a power of J and, in fact, J is a principal ideal domain.*

Since J is noetherian and $J \subset J^*$, $J^2 \subset J$. [5; 172-73]. We choose $x \in J$, $x \notin J^2$. Because J has property C , (x) is prime. We shall show that $(x) = J$. We suppose that $(x) \subset J$. Because (x) is invertible and $J \subset J^*$, $(x) \supset (x)J \supset (x^2)$. If A is any ideal such that $(x) \supset A \supset (x^2)$ and if P is a prime factor of A , then $P \supseteq (x)$ so that $P = (x)$ or $P = J$. Because $(x) \supset A \supset (x^2)$, $A = (x)J^k$ for some $k \geq 1$. But $x \notin J^2$ so that $x^2 \notin (x)J^k$ for $k \geq 2$. Therefore $k = 1$ and $(x)J$ is the unique ideal properly between (x) and (x^2) .

We next show that (x^2) is a primary ideal. Thus if $a, b \in J$, $ab \in (x^2)$, and $a \notin (x)$, then $b \in (x)$. Hence $(x^2) \subseteq (x^2, b) \subseteq (x)$. Now (x) is maximal and prime in J so that $J/(x)$ contains a unit element \bar{u} . Because $a \notin (x)$, $ua \notin (x)$ so that $uax \notin (x^2)$ and therefore $ux \notin (x^2, b)$. This means $(x^2, b) \not\subseteq (x)J$ so that $(x^2, b) = (x^2)$ by the preceding paragraph. Hence $b \in (x^2)$ and (x^2) is primary.

Now $ua - a \in (x)$ so that $(ua - a)^3 \in (x^2)$. If $z \in J$, then $z(ua - a)^3 = a^3(tz - z) \in (x^2)$ where t is a fixed element of J independent of z . Since $a^3 \notin (x)$ and (x^2) is primary, $tz - z \in (x^2)$ for each $z \in J$ —i.e., $J/(x^2)$ contains a unit element. This means, however, that $V = (x)/(x^2)$ is a vector space over the field $J/(x)$. There is a one-to-one correspondence between subspaces of V and ideals of J between (x) and (x^2) . Hence V has exactly one nonzero proper subspace, which is impossible. Therefore $J = (x)$ as asserted.

If P is a proper prime ideal of J , the argument above shows that $P \subseteq J^2 = (x^2)$. This means for some ideal A of J , $P = A(x)$. Since P is prime, $P = A$. Now $(x) = J \subset J^*$ so that P is not invertible and thus $P = (0)$. Hence J is the only nonzero prime ideal of J . Therefore if A is a nonzero ideal of J , $A = J^k = (x^k)$ for some positive integer k .

A ring R with at most two prime ideals is called a *primary ring*. Theorem 3 shows that J is a primary domain. The author has investigated primary rings in [3].

THEOREM 4. *J^* is a discrete valuation ring of rank one. Conversely if D is a discrete valuation ring of rank one with maximal*

ideal M and if $D = M^*$, then M is a domain without unit having property C .

The proof will use the following.

LEMMA 3. Suppose S is a ring with unit e and that R is a subring of S such that S is generated by R and e . A subset of R is an ideal of S if and only if it is an ideal of R . S is noetherian if and only if R is noetherian.

For the proof of the lemma, see [3].

To prove the theorem, we let $\xi \in k$, the quotient field of J^* . For some elements a and b of J , $\xi = a/b$. By Theorem 3 the ideals (a) and (b) of J compare—i.e. $a/b \in J^*$ or $b/a \in J^*$. Therefore, J^* is a valuation ring. Because J^* is noetherian, J^* is discrete and of rank one. [9; 41].

If M is the maximal ideal of J^* then $J = M^r$ for some r . Then $M^{r+1} \subset J$ implies $M^{r+1} = (M^r)^s$ for some integer s so that $r + 1 = rs$ and $r = 1$ —i.e., $J = M$. Hence J^*/J is a field. Because J^* is generated over J by e , $J^*/J = Z/(p)$ for some prime integer p .

The proof of the converse is an immediate consequence of Lemma 3 and of the fact that a discrete valuation ring of rank one is a Dedekind domain. [8; 278].

It is possible to classify all discrete valuation rings V of rank one such that $V = M^*$ where M is the maximal ideal of V , for if V has this property, so does the completion \bar{V} of V . [2; 60]. If now p is a fixed prime, if Π denotes the prime field with p elements, x an indeterminate over π , if $V_1 = Z_{(p)}$ and $V_2 = (\Pi[x])_{(x)}$ then V_1 and V_2 are discrete valuation rings of rank one and with residue field Π . Further V_1 and V_2 are regular and *unramified* in Cohen's sense. [2; 88]. Thus \bar{V}_1 and \bar{V}_2 are so-called *p-adic* rings. [2; 59–60, 89]. Now \bar{V}_1 has characteristic zero (unequal characteristic case for \bar{V}_1 and its residue field) and \bar{V}_2 has characteristic p (equal characteristic case). The within isomorphism, \bar{V}_1 and \bar{V}_2 are the only two *p-adic* rings of dimension one having residue class field Π . [2; 89]. Now \bar{V}_1 is simply the domain of Hensel's *p-adic* integers and \bar{V}_2 is the domain of formal power series in one indeterminate over the field Π . [7; 242–243]. Finally, \bar{V} is an Eisenstein extension of \bar{V}_1 or \bar{V}_2 , and in case \bar{V} has characteristic p , $\bar{V} \cong \bar{V}_2$. In short we have: If V has characteristic p , then to within isomorphism V is a ring between V_2 and \bar{V}_2 . If V is unramified of characteristic 0, then $V_1 \subseteq V \subseteq \bar{V}_1$. If V is ramified of characteristic zero, then V is isomorphic to a valuation ring contained in an Eisenstein extension of \bar{V}_1 . Conversely,

if V is a ring having any of the three properties just described, V is a discrete valuation ring of rank one having residue field H . [2; 59–60].

We add the following remarks:

In the last paragraph of the proof of Theorem 2, it is not necessary to use the fact that J has property C to conclude J is noetherian. That J is noetherian follows from a theorem of Cohen [1; 29] if all prime ideals of J are finitely generated.

In the proof of Theorem 3, it is not true in general that if $D/(x)$ is a field, that the ring $D/(x^2)$ contains a unit element, and hence that $(x)/(x^2)$ is a vector space over $D/(x)$. One can take D to be the ring of even integers and $x = 6$.

Theorem 3 implies that J is noetherian and of dimension less than two. Using Theorem 4, it is easily seen that J is integrally closed. That these three conditions do not imply that a domain D has property C may be seen by taking D to be the domain of even integers. Theorems 3 and 4 imply a bit more than the above. They even imply that J is a noetherian integrally closed primary domain. It can be shown that a noetherian integrally closed primary domain D without unit is the Jacobson radical of D^* , which is a semi-local ring, and that further, $D^*/D \cong Z/(p_1 p_2 \cdots p_k)$ for some distinct primes p_1, \cdots, p_k . [3]. However, D need not have property C as can be seen by choosing D as the Jacobson radical of Z_M where M consists of all integers relatively prime to 6. An analog to the classical Noether theorem cited earlier in the case of a domain without unit, while obtainable, now seems not as desirable to the author as Theorem 4.

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Pacific Journal of Mathematics

Vol. 13, No. 2

April, 1963

Rafael Artzy, <i>Solution of loop equations by adjunction</i>	361
Earl Robert Berkson, <i>A characterization of scalar type operators on reflexive Banach spaces</i>	365
Mario Borelli, <i>Divisorial varieties</i>	375
Raj Chandra Bose, <i>Strongly regular graphs, partial geometries and partially balanced designs</i>	389
R. H. Bruck, <i>Finite nets. II. Uniqueness and imbedding</i>	421
L. Carlitz, <i>The inverse of the error function</i>	459
Robert Wayne Carroll, <i>Some degenerate Cauchy problems with operator coefficients</i>	471
Michael P. Drazin and Emilie Virginia Haynsworth, <i>A theorem on matrices of 0's and 1's</i>	487
Lawrence Carl Eggan and Eugene A. Maier, <i>On complex approximation</i>	497
James Michael Gardner Fell, <i>Weak containment and Kronecker products of group representations</i>	503
Paul Chase Fife, <i>Schauder estimates under incomplete Hölder continuity assumptions</i>	511
Shaul Foguel, <i>Powers of a contraction in Hilbert space</i>	551
Neal Eugene Foland, <i>The structure of the orbits and their limit sets in continuous flows</i>	563
Frank John Forelli, Jr., <i>Analytic measures</i>	571
Robert William Gilmer, Jr., <i>On a classical theorem of Noether in ideal theory</i>	579
P. R. Halmos and Jack E. McLaughlin, <i>Partial isometries</i>	585
Albert Emerson Hurd, <i>Maximum modulus algebras and local approximation in C^n</i>	597
James Patrick Jans, <i>Module classes of finite type</i>	603
Betty Kvarda, <i>On densities of sets of lattice points</i>	611
H. Larcher, <i>A geometric characterization for a class of discontinuous groups of linear fractional transformations</i>	617
John W. Moon and Leo Moser, <i>Simple paths on polyhedra</i>	629
T. S. Motzkin and Ernst Gabor Straus, <i>Representation of a point of a set as sum of transforms of boundary points</i>	633
Rajakularaman Ponnuswami Pakshirajan, <i>An analogue of Kolmogorov's three-series theorem for abstract random variables</i>	639
Robert Ralph Phelps, <i>Čebyšev subspaces of finite codimension in $C(X)$</i>	647
James Dolan Reid, <i>On subgroups of an Abelian group maximal disjoint from a given subgroup</i>	657
William T. Reid, <i>Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems</i>	665
Georg Johann Rieger, <i>Some theorems on prime ideals in algebraic number fields</i>	687
Gene Fuerst Rose and Joseph Silbert Ullian, <i>Approximations of functions on the integers</i>	693
F. J. Sansone, <i>Combinatorial functions and regressive isols</i>	703
Leo Sario, <i>On locally meromorphic functions with single-valued moduli</i>	709
Takayuki Tamura, <i>Semigroups and their subsemigroup lattices</i>	725
Pui-kei Wong, <i>Existence and asymptotic behavior of proper solutions of a class of second-order nonlinear differential equations</i>	737
Fawzi Mohamad Yaqub, <i>Free extensions of Boolean algebras</i>	761