

Pacific Journal of Mathematics

AN ANALOGUE OF KOLMOGOROV'S THREE-SERIES THEOREM FOR ABSTRACT RANDOM VARIABLES

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1. Introduction. (Ω, \mathcal{M}, P) is a probability space i.e. Ω is an abstract set of points w , \mathcal{M} is a σ -field of subsets of Ω and P is a nonnegative countably additive set function defined on \mathcal{M} such that $P(\Omega) = 1$. G is a locally compact Hausdorff abelian metric topological group. The group operation in G , as well as in the several other groups to be dealt with, will be denoted by $+$. Let e denote the identity element of G . By the Borel sets of G we mean the sets belonging to the σ -ring generated by the class \mathcal{C} of compact subsets of G . Let \mathcal{D} be the class of subsets of G whose intersection with every compact set is a Borel set. Notice that \mathcal{D} is a σ -field containing the open subsets of G . The character group of G will be denoted by \hat{G} . A single valued mapping f of Ω into G will be called a generalised random variable (g.r.v.) if $f^{-1}(A) \in \mathcal{M}$ whenever $A \in \mathcal{D}$. An immediate consequence of this definition is that if f is a g.r.v. then $\eta(f)$ is an ordinary (complex valued) random variable for every $\eta \in \hat{G}$. A finite or a countably infinite collection of g.r.v.'s is said to be independent if and only if for every finite subset $\{X_i, i = 1, 2, \dots, n\}$ of distinct members of the collection and for every choice of sets $A_j \in \mathcal{D}$, $j = 1, 2, \dots, n$ it is true that $P\{w: X_i(w) \in A_i, i = 1, 2, \dots, n\} = \prod_1^n P\{w: X_i(w) \in A_i\}$.

If G is the real line, \hat{G} is the real line too. For $t \in \hat{G}$ and $x \in G$, $t(x) = \exp(itx)$. Given the random variable X and any real number $c > 0$ we define a new random variable $Y = t_0\alpha$ where $t_0 = c/\pi$ and α is the principal amplitude of $\exp(i\pi X/c)$. The two sets $\{w: -c < X(w) \leq c\}$ and $\{w: X(w) \neq Y(w)\}$ are then seen to be equal. Denoting by N the interval $(-c, c]$, the classical three series theorem [2] may be stated thus: If $\{X_n, n = 1, 2, \dots\}$ is a sequence of independent real valued random variables then $\sum_1^\infty X_n$ exists with probability 1 (a.e.) if and only if, for some $c > 0$, the following three series converge.

- (i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$
- (ii) $\sum_1^\infty EY_n$ and
- (iii) $\sum_1^\infty \text{var } Y_n$.

E and var denote respectively the mathematical expectation and

Received August 1962. This research was supported in part by funds provided under contract No. DA-O4-200-ORD-651 with the office of ordnance Research, U.S. Army and was carried out under the guidance of Professor Herman Rubin at the University of Oregon, U.S.A.

variance. α_n is the principal amplitude of $\exp(i\pi X_n/c)$ and $Y_n = t_0\alpha_n$. The convergence of the above three series is easily seen to be equivalent to the convergence of

(i) $\sum_1^\infty P\{w: X_n(w) \notin N\}$

(ii) $\sum_1^\infty E \log t(X_n)$ and

(iii) $\sum_1^\infty \text{var} \log t(X_n)$ for every $t \in \hat{G}$, $\log t(X_n)$ being defined to be equal to $i\theta_n$ where θ_n is the principal amplitude of $\exp(itX_n)$. It is in this form the classical three series theorem lends itself for extension to the case of generalised random variables. In §2 three lemmas are proved leading to the generalisation. In §3 we give a necessary and sufficient condition for the convergence almost everywhere of $\sum_1^\infty X_n$ in terms only of characters and not using characteristic functions.

The following two known results are quoted for the sake of completeness and ready reference.

THEOREM A. (Cor. (2.1) [4]).

If $\{h_n, n = 1, 2, \dots\}$ is a sequence of continuous homomorphisms on a topological group G_1 to a topological group G_2 which converge pointwise to h throughout some Baire set of the second category then h is continuous.

THEOREM B. (§2.21 [3]).

Let G be a locally compact abelian group. Let N be a compact symmetric neighbourhood of e . Let G' be the subgroup of G generated by N . Then G' contains a discrete subgroup D with a finite number of generators such that the quotient group G'/D is compact and $D \cap (N + N + N) = \{e\}$.

2. For a sequence of real or complex numbers $g_n, n = 1, 2, \dots$ we say that $\prod_1^\infty g_n$ exists if $\prod_n^\infty g_k$ is nonzero for sufficiently large n .

LEMMA 1. For $\eta \in \hat{G}$, a necessary and sufficient condition that $\prod_1^\infty \eta(X_n)$ exists a.e. is that $\prod_1^\infty E\eta(X_n)$ exists.

Proof. If $\prod_1^\infty \eta(X_n)$ exists a.e. then, by the bounded convergence theorem, $\prod_1^\infty E\eta(X_n)$ exists.

Conversely let $\prod_1^\infty E\eta(X_n)$ exist. Hence $\prod_1^\infty |E\eta(X_n)|$ exists. Let $\eta(X_n(w)) = \exp(i\theta_n(w))$ where $\theta_n(w)$ is the principal value of the amplitude. Hence $\theta_1, \theta_2, \dots$ is a bounded, independent sequence of real valued random variables. Let θ'_n be the symmetrised version of θ_n and let $\theta'_n(1)$ be θ'_n truncated at 1. One has (p. 196, [2]) $\text{var} \theta'_n(1) \leq 3\{1 - |E\eta(X_n)|^2\}$. Hence $\sum_1^\infty \text{var} \theta'_n(1) < \infty$. By the classical three series theorem it follows that $\sum_1^\infty \theta'_n$ converges a.e. Consequently (p. 250, [2]) there exist constants α_n such that $\sum_1^\infty (\theta_n - \alpha_n)$ exists

a.e. or equivalently $\prod_1^\infty \exp(-i\alpha_n)E\eta(X_n)$ exists. This implies the convergence of $\sum_1^\infty \alpha_n$ since $\prod_1^\infty E\eta(X_n)$ is assumed to converge. We now conclude $\sum_1^\infty \theta_n$ exists a.e. or, what is same, $\prod_1^\infty \eta(X_n)$ exists a.e.

LEMMA 2. For a given $\eta \in \hat{G}$, the following two sets of conditions are equivalent.

$$(2.1) \quad \prod_1^\infty E\eta(X_n) \text{ exists; } \sum_1^\infty \text{var } \eta(X_n) < \infty$$

$$(2.2) \quad \sum_1^\infty E\theta_n \text{ converges; } \sum_1^\infty \text{var } \theta_n < \infty$$

where $\eta(X_n) = \exp(i\theta_n)$, θ_n being the principal amplitude.

Proof. Suppose (2.2) holds. Therefore by the three series theorem on the line, $\sum_1^\infty \theta_n$ exists a.e. This implies that $\prod_1^\infty \eta(X_n)$ exists a.e. Hence $\prod_1^\infty E\eta(X_n)$ exists by the bounded convergence.

Let now $\alpha_n = E\theta_n$; $\beta_n = \text{var } \theta_n$ and $\theta_n = \alpha_n + y_n$. As in the last lemma, $E\eta(X_n) = (1 + d_n\beta_n/2) \exp(i\alpha_n)$ where $|d_n| \leq 1$.

$$\begin{aligned} E|\eta(X_n) - E\eta(X_n)|^2 &= E|\exp(iy_n) - (1 + d_n\beta_n/2)|^2 \\ &\leq c\beta_n \text{ where } c \text{ is an absolute constant} \\ &= c \text{var } \theta_n . \end{aligned}$$

Hence $\sum_1^\infty \text{var } \eta(X_n) < \infty$.

Conversely, suppose (2.1) holds.

$$\begin{aligned} \text{var } \eta(X_n) &= E|\exp(iy_n) - (1 + d_n\beta_n/2)|^2 \\ &= 1 + |1 + d_n\beta_n/2|^2 - 2 \text{ real part of } E\overline{(1 + d_n\beta_n/2)} \exp(iy_n) \\ &= 1 - |1 + d_n\beta_n/2|^2 . \end{aligned}$$

Hence $\sum_1^\infty \{1 - |1 + d_n\beta_n/2|^2\} < \infty$. Now, $|1 + d_n\beta_n/2|$ is the absolute value of the expectation $E\exp(iy_n)$ and hence is less than or equal to 1. It follows therefore that $\sum_1^\infty \{1 - |1 + d_n\beta_n/2|\} < \infty$. As $1 - |1 + d_n\beta_n/2| \geq \beta_n/2$, this implies that

$$\sum_1^\infty \beta_n < \infty \quad \text{i.e.} \quad \sum_1^\infty \text{var } \theta_n < \infty .$$

From the convergence of $\prod_1^\infty E\eta(X_n)$ and $\sum_1^\infty \beta_n$ and the relation $E\eta(X_n) = (1 + d_n\beta_n/2) \exp(i\alpha_n)$, we see that $\sum_1^\infty E\theta_n = \sum_1^\infty \alpha_n$ converges. Thus (2.1) implies (2.2).

LEMMA 3. A necessary and sufficient condition that $\sum_1^\infty X_n$ exist a.e. is that $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$, and for some compact neighbourhood N of e

$$(2.3) \quad \sum_1^\infty P(w: X_n(w) \notin N) < \infty .$$

Proof. Suppose $\sum_1^\infty X_n$ exists a.e. Consequently, for every compact neighbourhood N of e , $P(w: X_n(w) \notin N \text{ i.o.}^1) = 0$ or, equivalently, $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$ by the Borel-Cantelli lemma. That $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$ follows from the continuity of the characters η .

Conversely, let N be any compact neighbourhood of e for which (2.3) is satisfied. Since $N - N \supseteq N$, we have $P\{w: X_n(w) \notin N - N\} \leq P\{w: X_n(w) \notin N\}$. Hence the symmetric neighbourhood $N - N$ of e also satisfies (2.3). Without loss of generality we may therefore assume that N in (2.3) is symmetric.

Denote by G^* the closed subgroup generated by N . Necessarily G^* is σ -compact. Further, by Theorem B, G^* contains a discrete subgroup D with a finite number of generators such that $G_1 = G^*/D$ is compact and $D \cap (N + N - N) = \{e\}$. Hence by the Borel-Cantelli lemma, (2.3) implies that $P\{w: X_n(w) \notin N \text{ i.o.}\} = 0$; that is, if $A_1 = \{w: X_n(w) \in N \text{ for all } n \geq n_0(w)\}$ then $P(A_1) = 1$. Let σ be the natural mapping of G^* onto G_1 and write $Y_n(w) = \sigma X_n(w)$.

As G_1 is a compact, metric group, G_1 (and consequently \hat{G}_1) satisfies the second axiom of countability. Also \hat{G}_1 is discrete, since G_1 is compact. Further \hat{G}_1 consists precisely of those elements of \hat{G} which are identically one on D (cf: Theorem 34 [5]). In view of (2.3), we have $\prod_1^\infty \xi(Y_n)$ exists a.e. for each $\xi \in \hat{G}_1$. As \hat{G}_1 is countable we conclude that, with probability 1, $\prod_1^\infty \xi(Y_n)$ exists for all $\xi \in \hat{G}_1$. Observe that G_1 , being a compact metric space, is a Baire set of the second category. It is now immediate from Theorem A that $\sum_1^\infty Y_n$ exists a.e.

Let A_2 be a set of probability 1 on which $\sum_1^\infty Y_n$ exists. If $A = A_1 \cap A_2$ then $P(A) = 1$. Let $w \in A$ and $n \geq n_0(w)$. Hence

$$(2.4) \quad X_n(w) + X_{n+1}(w) \in N + N .$$

As $\sigma(N)$ is a neighbourhood of the identity in G_1 and since $\sum_1^\infty Y_n(w)$ exists, it is clear that $Y_n(w) + Y_{n+1}(w) \in \sigma(N)$, if n is larger than a certain $n_1(w)$. That is

$$(2.5) \quad X_n(w) + X_{n+1}(w) \in N + D \quad \text{if } n \geq n_1(w) .$$

From (2.4) and (2.5) and the property $D \cap (N + N - N) = \{e\}$, we conclude that $X_n(w) + X_{n+1}(w) \in N$ if $n \geq \max(n_0, n_1)$. Repeating the argument a finite number of times it is seen that all finite tails of the series $\sum_1^\infty X_n(w)$ lie in N . By exactly similar reasoning, all finite tails lie in any preassigned neighbourhood M of e with $M \subseteq N$. As N is compact, we can show (by arguments similar to the ones

¹ infinitely often

on p. 193 [1]) that $\sum_1^\infty X_n(w)$ exists. Thus on A , which is a set of probability 1, $\sum_1^\infty X_n$ exists. Combining these results, we have

THEOREM 1. *If $\{X_n, n = 1, 2, \dots\}$ is an independent sequence of generalised random variables then $\sum_1^\infty X_n$ exists a.e. if and only if the series*

- (i) $\sum_1^\infty P\{w: X_n(w) \in N\}$, N being any preassigned compact neighbourhood of e ,
- (ii) $\sum_1^\infty E \log \eta(X_n)$ and
- (iii) $\sum_1^\infty \text{var} \log \eta(X_n)$ converge for all $\eta \in \hat{G}$. Here $\log(X_n)$ is taken to be $i\theta_n$ where θ_n is the principal amplitude of $\eta(X_n)$.

3. DEFINITION. The measure μ induced in \mathcal{D} by a generalised random variable f will be called the distribution function of f . The distribution μ will be said to be symmetric if $\mu(A) = \mu(-A)$ for every $A \in \mathcal{D}$. It will be called regular if for every $A \in \mathcal{D}$, $\mu(A) = \sup \{\mu(C): C \subseteq A, C \in \mathcal{E}\}$.

THEOREM 2. *If $\{X_n, n = 1, 2, \dots\}$ is an independent sequence of generalised random variables with regular distributions, then $\sum_1^\infty X_n$ exists a.e. if and only if $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$.*

Proof. If $\sum_1^\infty X_n$ exists a.e. then $\prod_1^\infty \eta(X_n)$ exists a.e. for every $\eta \in \hat{G}$ by the continuity property of η .

Conversely, let $\prod_1^\infty \eta(X_n)$ exist a.e. for each $\eta \in \hat{G}$. The assertion is established through the following steps.

(i) Let G be compact. That the assertion is true in this case is seen by the same reasoning as for G_1 in Lemma 3.

(ii) Let G be discrete. The compact subsets of G are therefore only those subsets with a finite number of elements. As the distribution of each X_n is regular we can find a countable subgroup G_1 such that $P\{w: X_n(w) \in G_1, n = 1, 2, \dots\} = 1$. Observe that \hat{G}_1 is the same as \hat{G} restricted to G_1 . Now let the X_n 's have symmetric distributions. Hence, if $\varphi_n(\eta) = E\eta(X_n)$ then the φ_n 's are real and $\varphi_n(-\eta) = \varphi_n(\eta)$. Now by Lemma 1, $\prod_1^\infty \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$, implies that $\prod_1^\infty \varphi_n(\eta)$ exists. Therefore $g(\eta) = \sum_1^\infty \{1 - \varphi_n(\eta)\}$ exists for every $\eta \in \hat{G}$. If $g_n(\eta) = \sum_1^n \{1 - \varphi_k(\eta)\}$ then the g_n 's are continuous and $g_n(\eta)$ converges monotonically up to $g(\eta)$ as $n \rightarrow \infty$ for each η . Hence $\{\eta: g(\eta) \leq a\} = \bigcap_1^\infty \{\eta: g_n(\eta) \leq a\}$ is a closed set. \hat{G} is a compact metric space and so is complete. Hence it is a set of the second category. Further, $\hat{G} = \bigcup_{n=1}^\infty \{\eta: g(\eta) \leq n\}$ i.e. \hat{G} is the union of a countable number of closed sets. Therefore by the Baire category theorem, one of these closed sets in the union, say the set $A = \{\eta: g(\eta) \leq k\}$, has a nonnull interior V . Trivially g is bounded on V . By the positive

definiteness and symmetry of ϕ_k ,

$$1 - \phi_k^2(\xi) - \phi_k^2(\eta) + 2\phi_k(\xi)\phi_k(\eta)\phi_k(\xi + \eta) - \phi_k^2(\xi + \eta) \geq 0 .$$

Let $a_k^2 = 1 - \phi_k(\xi)$, $b_k^2 = 1 - \phi_k(\eta)$ and $c_k^2 = 1 - \phi_k(\xi + \eta)$. Then the above inequality implies that

$$c_k^2 \leq a_k^2 + b_k^2 - a_k^2 b_k^2 + a_k b_k \sqrt{(2 - a_k^2)(2 - b_k^2)} \leq (a_k + b_k)^2 .$$

Consequently,

$$(3.1) \quad g(\xi + \eta) \leq \{[g(\xi)]^{1/2} + [g(\eta)]^{1/2}\}^2 .$$

For any $\xi \in \hat{G}$ consider the open set $\xi - V$. From (3.1) it is immediate that g is bounded on $\xi - V$. The family $\xi - V$, $\xi \in \hat{G}$ is an open covering for the compact \hat{G} . Therefore there exists a finite subcover from this. As g is bounded on each member of this subcover it follows that g is bounded on \hat{G} .

Let m be the Haar measure of \hat{G} with $m(\hat{G})=1$. As $P\{w: X_n(w) \neq e\} = \int_{\hat{G}} \{1 - \varphi_n(\eta)\} dm(\eta)$, we obtain $\sum_1^\infty P\{w: X_n(w) \neq e\} = \int_{\hat{G}} g(\eta) dm(\eta) < \infty$. Since G is discrete this means that for the compact neighbourhood $N = \{e\}$ of e , $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$. That $\sum_1^\infty X_n$ exists a.e. follows from Lemma 3.

(iii) Let G be discrete but the distributions of the X_n 's need not be symmetric.

Let Y_n , $n = 1, 2, \dots$ be another independent sequence of g.r.v.'s and independent of the X_n 's; let Y_n have the same distribution as X_n , $n = 1, 2, \dots$.

Write $Z_n = X_n - Y_n$. The Z_n 's therefore have symmetric distributions. Also the hypothesis yields that $\prod_1^\infty \eta(Z_n)$ exists a.e. for every $\eta \in \hat{G}$. Hence by (ii) above

$$(3.2) \quad \sum_1^\infty P\{w: Z_n(w) \neq e\} < \infty .$$

The distribution of each X_n is assumed to be regular. Hence there exists a countable set A such that $P\{w: Z_n(w) \in A \text{ for all } n\} = 1$. Now, if $p_n(a) = P\{w: X_n(w) = a\}$, we have

$$\begin{aligned} P\{w: Z_n(w) = e\} &= \sum_{a \in A} P\{w: X_n(w) = a\} P\{w: Y_n(w) = a\} . \\ &= \sum_{a \in A} p_n^2(a) \leq \sup_{a \in A} p_n(a) \end{aligned}$$

Since there can only be a finite number of 'values' of X_n for which the associated probability is larger than any preassigned number, the supremum is attained. Let a_n be any one of the values taken by X_n with probability equal to this supremum. Therefore $P\{w: X_n(w) \neq a_n\} \leq$

$P\{w: Z_n(w) \neq e\}$. Consequently, using (3.2), we obtain

$$(3.3) \quad \sum_1^{\infty} P\{w: X_n(w) \neq a_n\} < \infty$$

$$(3.4) \quad \text{or} \quad \sum_1^{\infty} P\{w: X_n(w) - a_n \notin N\} < \infty .$$

Where N is the compact neighbourhood of e consisting only of itself. From (3.3) we conclude that, with probability 1, $X_n = a_n$ except for a finite number of n 's. This fact together with the hypothesis implies that $\prod_1^{\infty} \eta(a_n)$ exists for every $\eta \in \hat{G}$. That $\prod_1^{\infty} \eta(X_n - a_n)$ exists a.e. for every $\eta \in \hat{G}$ is then immediate. Now using (3.4) we see by lemma 3 that $\sum_1^{\infty} (X_n - a_n)$ exists a.e. By Theorem A or by applying Lemma 3 to the random variables a_n we see however that $\sum_1^{\infty} a_n$ exists since $\prod_1^{\infty} \eta(a_n)$ exists, for every $\eta \in \hat{G}$. Hence $\sum_1^{\infty} X_n$ exists a.e., as was to be proved.

(iv) Let G be any metric abelian locally compact group. Let N be a compact symmetric neighbourhood of e and G^* the closed subgroup generated by N . Necessarily G^* is σ -compact and open. Let σ_1 be the natural mapping of G onto $G_1 = G/G^*$. As G^* is open, G_1 is discrete. Further \hat{G}_1 consists precisely of those elements of \hat{G} which are identically one on G^* . Hence $\prod_1^{\infty} \eta(X_n)$ exists a.e. for each $\eta \in \hat{G}$ implies that $\prod_1^{\infty} \xi(Y_n)$ exists a.e. for each $\xi \in \hat{G}_1$, where $Y_n = \sigma_1 X_n$. By part (iii) above, $P\{w: Y_n(w) \neq e_1 \text{ i.o.}\} = 0$ where e_1 is the identity of G_1 . That is

$$(3.5) \quad P\{w: X_n(w) \notin G^*\} = 0 .$$

In other words, there is probability 1 that all except a finite number of the X_n 's lie in G^* .

As G^* is generated by a compact symmetric neighbourhood of e there exists, by Theorem B, a discrete group D with a finite number of generators such that $G_2 = G^*/D$ is compact and $D \cap (N - N) = \{e\}$. Let e_2 be the identity element of G_2 and σ_2 the natural mapping of G^* onto G_2 . Write $Z_n = \sigma_2 X_n$ if $X_n \in G^*$ and $= e_2$ if $X_n \notin G^*$. Hence $Z_n, n = 1, 2, \dots$ is an independent sequence of g.r.v.'s in G_2 . Recall that \hat{G}^* consists of all the elements of \hat{G} restricted to G^* and that \hat{G}_2 consists precisely of those elements of \hat{G}^* which are identically 1 on D . Using the hypothesis and the equation (3.5) we get $\prod_1^{\infty} \xi(Z_n)$ exists a.e. for every $\xi \in \hat{G}_2$. Therefore we have

$$P\{w: Z_n(w) \notin \sigma_2(N) \text{ i.o.}\} = 0 \quad \text{i.e.} \quad P\{w: X_n(w) \notin N + D \text{ i.o.}\} = 0 .$$

Define $s_n = X_n$ if $X_n \in N + D$ and $s_n = e$ if $X_n \notin N + D$. Then for each s_n we have the unique decomposition $s_n = u_n \pm v_n$ where

$u_n \in N$ and $v_n \in D$. The u_n 's form an independent sequence of g.r.v.'s and so do the v_n 's. It is immediate from the hypothesis that $\prod_1^\infty \eta(s_n)$ exists a.e. for each $\eta \in \hat{G}$. Also, since $\prod_1^\infty \xi(Z_n)$ exists a.e. for each $\xi \in \hat{G}_2$, $\prod_1^\infty \eta(u_n)$ exists a.e. for each $\eta \in \hat{G}$. Hence $\prod_1^\infty \xi(v_n)$ exists a.e. for each $\xi \in \hat{D}$. As D is discrete we have, by part (iii), $P\{w: X_n(w) \neq e \text{ i.o.}\} = 0$. This is equivalent to saying $P\{w: s_n(w) \neq u_n(w) \text{ i.o.}\} = 0$. Or $P\{w: X_n(w) \notin N \text{ i.o.}\} = 0$ i.e. $\sum_1^\infty P\{w: X_n(w) \notin N\} < \infty$. That $\sum_1^\infty X_n$ exists a.e. follows now by Lemma 3.

I thank the referee for his suggestions leading to a shorter proof of Lemma 1.

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