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SOME THEOREMS ON PRIME IDEALS IN ALGEBRAIC NUMBER FIELDS

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Let K be an arbitrary algebraic number field. We denote by n the degree of K, by $\boldsymbol{f}$ an arbitrary ideal of K, by $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ prime ideals of K, by $\mu(\boldsymbol{a})$ the Moebius function of the ideal $\boldsymbol{a}$ of K, by $N\boldsymbol{a}$ the norm of $\boldsymbol{a}$, by $(\boldsymbol{a}, \boldsymbol{f})$ the greatest common divisor of $\boldsymbol{a}$ and $\boldsymbol{f}$, and by $h(\boldsymbol{f})$ the number of ideal classes H mod $\boldsymbol{f}$. It is known that

$$(1)\qquad \begin{aligned} &A(x, \boldsymbol{f}) := \sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} 1 = \gamma(\boldsymbol{f})x + R(x, \boldsymbol{f}), R(x, \boldsymbol{f}) = O(x^{1-1/n}) , \\ &\gamma(\boldsymbol{f}) = \alpha \prod_{\boldsymbol{p} | \boldsymbol{f}} \left(1 - \frac{1}{N\boldsymbol{p}}\right) \qquad (\alpha = \alpha(K) > 0) . \end{aligned}$$

According to [1], the proof of the generalized Selberg formula for ideal classes H mod $\boldsymbol{f}$ in K:

$$(2)\qquad \sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} + \sum_{\substack{N\boldsymbol{p}\boldsymbol{q} \leqq x \\ \boldsymbol{p}\boldsymbol{q} \in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x)$$

can be reduced to

$$(3)\qquad \sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} \frac{\mu(\boldsymbol{a})}{N\boldsymbol{a}} \log^2 \frac{x}{N\boldsymbol{a}} = \frac{2}{\gamma(\boldsymbol{f})} \log x + O(1) ,$$

and (3) is established directly in [1]. First, we generalize (3):

THEOREM 1. *Let $r > 1$ be a rational integer; then*

$$\sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} \frac{\mu(\boldsymbol{a})}{N\boldsymbol{a}} \log^r \frac{x}{N\boldsymbol{a}} = \frac{r}{\gamma(\boldsymbol{f})} \log^{r-1} x + \sum_{t=1}^{r-2} c_t(r, \boldsymbol{f}) \log^t x + O(1) ;$$

the constants $c_t(r, \boldsymbol{f})$ resp. the constant in $O(1)$ depends on $K, r, t, \boldsymbol{f}$ resp. $K, r, \boldsymbol{f}$ only.

The formula

$$\sum_{\boldsymbol{a} | \boldsymbol{f}} \mu(\boldsymbol{a}) = \begin{cases} 1 & \text{for } \boldsymbol{f} = 1 , \\ 0 & \text{for } \boldsymbol{f} \neq 1 \end{cases}$$

yields

LEMMA 1. *Let $f(x)$ be a complex valued function $(x \geqq 1)$; then*

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$$g(x) := \sum_{\substack{Na \leqq x \\ (a,f)=1}} f\left(\frac{x}{Na}\right) \quad \textit{implies} \quad f(x) = \sum_{\substack{Na \leqq x \\ (a,f)=1}} \mu(a)\, g\left(\frac{x}{Na}\right).$$

Using the Euler summation formula, we find

$$\sum_{m \leqq x} \frac{1}{m} \log^{r-1} m = \frac{1}{r} \log^r x + a_r + O\left(\frac{1}{x} \log^{r-1} x\right) \quad (r \text{ integer, } >1) . \tag{4}$$

Because of

$$\sum_{\substack{Na \leqq x \\ (a,f)=1}} \frac{1}{Na} \log^{r-1} Na = \sum_{m \leqq x} (A(m, f) - A(m-1, f)) \frac{1}{m} \log^{r-1} m ,$$

(1) and (4) imply

$$\sum_{\substack{Na \leqq x \\ (a,f)=1}} \frac{1}{Na} \log^{r-1} Na = \frac{\gamma(f)}{r} \log^r x + b_r(f) + O(x^{-1/n} \log^{r-1} x) \qquad (r > 1) ; \tag{5}$$

the constants $b_r(f)$ depend on K, r, f only. Because of

$$\sum_{\substack{Na \leqq x \\ (a,f)=1}} \left(\frac{x}{Na}\right)^{1-1/n} \log^{r-1} \frac{x}{Na}$$
$$= \sum_{m \leqq x} (A(m, f) - A(m-1, f)) \left(\frac{x}{m}\right)^{1-1/n} \log^{r-1} \frac{x}{m} ,$$

(1) implies

$$\sum_{\substack{Na \leqq x \\ (a,f)=1}} \left(\frac{x}{Na}\right)^{1-1/n} \log^{r-1} \frac{x}{Na} = O(x) . \tag{6}$$

By the binomial theorem and

$$\sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s} \frac{1}{s+1} = \frac{1}{r} ,$$

(5) yields

$$\begin{aligned} &\sum_{\substack{Na \leqq x \\ (a,f)=1}} \frac{1}{Na} \log^{r-1} \frac{x}{Na} \\ &= \frac{\gamma(f)}{r} \log^r x + \sum_{s=0}^{r-1} d_s(r, f) \log^s x + O(x^{-1/n} \log^{r-1} x) ; \end{aligned} \tag{7}$$

the constants $d_s(r, f)$ depend on K, s, r, f only.

As shown in [1],

$$\sum_{\substack{Na \leqq x \\ (a,f)=1}} \frac{\mu(a)}{Na} = O(1) , \qquad \sum_{\substack{Na \leqq x \\ (a,f)=1}} \frac{\mu(a)}{Na} \log \frac{x}{Na} = O(1) . \tag{8}$$

Proof of Theorem 1. By (3), Theorem 1 is correct for $r=2$. Suppose $r>2$ and

$$\sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^s \frac{x}{N\mathfrak{a}} = \sum_{t=1}^{s-1} c_t(s,\mathfrak{f}) \log^t x + O(1) \qquad (1<s<r)\,. \tag{9}$$

In Lemma 1, let $f(x):=x\log^{r-1}x$; then

$$\begin{aligned} g(x) &= x \sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} \frac{x}{N\mathfrak{a}} \\ &= \frac{\gamma(\mathfrak{f})}{r} x \log^r x + x \sum_{s=1}^{r-1} d_s(r,\mathfrak{f}) \log^s x + O(x^{1-1/n}\log^{r-1}x)\,, \end{aligned} \tag{10}$$

by (7). Lemma 1, (10), (9), (6), and (8) imply

$$\begin{aligned} x\log^{r-1}x &= \sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \mu(\mathfrak{a}) \Big(\frac{\gamma(\mathfrak{f})x}{rN\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} + \frac{x}{N\mathfrak{a}} \sum_{s=1}^{r-1} d_s(r,\mathfrak{f}) \log^s \frac{x}{N\mathfrak{a}} \\ &\qquad + O\Big(\Big(\frac{x}{N\mathfrak{a}}\Big)^{1-1/n} \log^{r-1}\frac{x}{N\mathfrak{a}}\Big)\Big) \\ &= \frac{\gamma(\mathfrak{f})x}{r} \sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} \log^r \frac{x}{N\mathfrak{a}} + \sum_{s=2}^{r-1} d_s(r,\mathfrak{f}) \sum_{t=1}^{s-1} c_t(s,\mathfrak{f}) \log^t x + O(x)\,; \end{aligned}$$

let

$$c_t(r,\mathfrak{f}) := -\frac{r}{\gamma(\mathfrak{f})} \sum_{s=t+1}^{r-1} d_s(r,\mathfrak{f})\, c_t(s,\mathfrak{f}) \qquad (t=1,2,\cdots,r-2)\,.$$

This proves Theorem 1.

The fact that

$$c_{s-1}(s,\mathfrak{f}) = \frac{s}{\gamma(\mathfrak{f})}$$

was not used in the preceding proof.

Now we derive two consequences of (2). The well-known relation (Landau (1903))

$$T(x) := \sum_{N\mathfrak{p}\leq x} \log N\mathfrak{p} = O(x)$$

implies

$$T(x, H \bmod \mathfrak{f}) := \sum_{\substack{N\mathfrak{p}\leq x\\ \mathfrak{p}\in H \bmod \mathfrak{f}}} \log N\mathfrak{p} = O(x) \tag{11}$$

By

$$\sum_{\substack{N\mathfrak{p}\leq x\\ \mathfrak{p}\in H \bmod \mathfrak{f}}} \log^2 N\mathfrak{p} = \sum_{m\leq x} (T(m, H\bmod \mathfrak{f}) - T(m-1, H\bmod\mathfrak{f}))\log m\,,$$

(11) gives

$$\sum_{\substack{Np \leq x \\ p \in H \bmod f}} \log^2 Np = T(x, H \bmod f) \log x + O(x) . \tag{12}$$

According to Landau (1903), we have

$$s(x) := \sum_{Np \leq x} \frac{\log Np}{Np} = \log x + O(1) . \tag{13}$$

Using

$$\sum_{N \quad \leq x} \frac{\log^2 Np}{Np} = \sum_{m \leq x} (S(m) - S(m-1)) \log m ,$$

(13) implies

$$\sum_{Np \leq x} \frac{\log^2 Np}{Np} = \frac{1}{2} \log^2 x + O(\log x) . \tag{14}$$

LEMMA 2. *We have*

$$\sum_{\substack{Npq \leq x \\ pq \in H \bmod f}} \log^2 Np \log Nq = \frac{\log x}{2} \sum_{\substack{Npq \leq x \\ pq \in H \bmod f}} \log Np \log Nq + O(x \log x) .$$

Proof. Denote by $H(q) \bmod f$ the class of all ideals a of K with $aq \in H \bmod f$; then (12), (13) and the definition of $T(x, H \bmod f)$ in (11) give

$$\begin{aligned} \sum_{\substack{Npq \leq x \\ pq \in H \bmod f}} \log^2 Np \log Nq &= \sum_{\substack{Nq \leq x \\ (q, f) = 1}} \log Nq \left(T\left(\frac{x}{Nq}, H(q) \bmod f\right) \log \frac{x}{Nq} \right. \\ &\qquad \left. + O\left(\frac{x}{Nq}\right)\right) \\ &= \sum_{\substack{Npq \leq x \\ pq \in H \bmod f}} \log Nq \log Np (\log x - \log Nq) \\ &\qquad + O(x \log x) . \end{aligned}$$

This proves Lemma 2.

THEOREM 2. *We have*

$$\log x \sum_{\substack{Npq \leq x \\ pq \in H \bmod f}} \log Np \log Nq + 2 \sum_{\substack{Npqr \leq x \\ pq \in r H \bmod f}} \log Np \log Nq \log Nr$$
$$= \frac{2x}{h(f)} \log^2 x + O(x \log x)$$

where the constant in the remainder term depends on K and f only.

Proof. We write (2) for x/Nr and $H(r) \bmod f$ instead of x and

$H \bmod \boldsymbol{f}$, multiply by $\log N\boldsymbol{r}$, and take summation over all prime ideals $\boldsymbol{r}$ with $(\boldsymbol{r}, \boldsymbol{f}) = 1$ and $N\boldsymbol{r} \leqq x$. By (13) and (14), we find

$$\sum_{\substack{N\boldsymbol{pr}\leqq x \\ \boldsymbol{pr}\in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \log N\boldsymbol{r} + \sum_{\substack{N\boldsymbol{pqr}\leqq x \\ \boldsymbol{pqr}\in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r}$$
$$= \frac{x}{h(\boldsymbol{f})} \log^2 x + O(x \log x) .$$

The application of Lemma 2 completes the proof.

THEOREM 3. *If*

$$\sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for the principal class $H_0 \bmod \boldsymbol{f}$, *then*

$$\frac{1}{x} \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for all $h(\boldsymbol{f})$ *classes* $H \bmod \boldsymbol{f}$.

Proof. Suppose

$$\sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_1 \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} = O(x)$$

for a certain ideal class $H_1 \bmod \boldsymbol{f}$. Then (2) implies

$$\sum_{\substack{N\boldsymbol{pq}\leqq x \\ \boldsymbol{pq}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x) , \tag{15}$$

and Theorem 2 gives

$$\sum_{\substack{N\boldsymbol{pqr}\leqq x \\ \boldsymbol{pqr}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} = O(x \log x) . \tag{16}$$

By (15) and (13), we get

$$\begin{aligned} \sum_{\substack{N\boldsymbol{pqr}\leqq x \\ \boldsymbol{pqr}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} &\geqq \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \sum_{\substack{N\boldsymbol{qr}\leqq x/N\boldsymbol{p} \\ \boldsymbol{qr}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{q} \log N\boldsymbol{r} \\ &= \sum_{\substack{N\boldsymbol{p}\leq x \\ \boldsymbol{p}\in H \bmod_0 \boldsymbol{f}}} \log N\boldsymbol{p} \left(\frac{2}{h(\boldsymbol{f})} \frac{x}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O\left(\frac{x}{N\boldsymbol{p}}\right)\right) \\ &= \frac{2x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O(x \log x) \\ &= \frac{x \log x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p}\leqq \sqrt{x} \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} + O(x \log x) . \end{aligned} \tag{17}$$

(17) and (16) imply the contradiction

$$\sum_{\substack{Np \leq \sqrt{x} \\ p \in H_0 \bmod f}} \frac{\log Np}{Np} = O(1) ,$$

and Theorem 3 is proved.

The special case of Theorem 3 for the rational number field was treated in [2].

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