Pacific Journal of Mathematics

SOME THEOREMS ON PRIME IDEALS IN ALGEBRAIC NUMBER FIELDS

GEORG JOHANN RIEGER

Vol. 13, No. 2 April 1963

SOME THEOREMS ON PRIME IDEALS IN ALGEBRAIC NUMBER FIELDS

G. J. RIEGER

Let K be an arbitrary algebraic number field. We denote by n the degree of K, by $\boldsymbol{f}$ an arbitrary ideal of K, by $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ prime ideals of K, by $\mu(\boldsymbol{a})$ the Moebius function of the ideal $\boldsymbol{a}$ of K, by $N\boldsymbol{a}$ the norm of $\boldsymbol{a}$, by $(\boldsymbol{a}, \boldsymbol{f})$ the greatest common divisor of $\boldsymbol{a}$ and $\boldsymbol{f}$, and by $h(\boldsymbol{f})$ the number of ideal classes H mod $\boldsymbol{f}$. It is known that

$$A(x, \boldsymbol{f}) := \sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} 1 = \gamma(\boldsymbol{f})x + R(x, \boldsymbol{f}),\ R(x, \boldsymbol{f}) = O(x^{1-1/n}),$$
$$\gamma(\boldsymbol{f}) = \alpha \prod_{\boldsymbol{p}|\boldsymbol{f}} \left(1 - \frac{1}{N\boldsymbol{p}}\right) \qquad (\alpha = \alpha(K) > 0)\,. \tag{1}$$

According to [1], the proof of the generalized Selberg formula for ideal classes H mod $\boldsymbol{f}$ in K:

$$\sum_{\substack{N\boldsymbol{p} \leqq x \\ \boldsymbol{p} \in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} + \sum_{\substack{N\boldsymbol{pq} \leqq x \\ \boldsymbol{pq} \in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x) \tag{2}$$

can be reduced to

$$\sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} \frac{\mu(\boldsymbol{a})}{N\boldsymbol{a}} \log^2 \frac{x}{N\boldsymbol{a}} = \frac{2}{\gamma(\boldsymbol{f})} \log x + O(1)\,, \tag{3}$$

and (3) is established directly in [1]. First, we generalize (3):

THEOREM 1. *Let $r > 1$ be a rational integer; then*

$$\sum_{\substack{N\boldsymbol{a} \leqq x \\ (\boldsymbol{a}, \boldsymbol{f})=1}} \frac{\mu(\boldsymbol{a})}{N\boldsymbol{a}} \log^r \frac{x}{N\boldsymbol{a}} = \frac{r}{\gamma(\boldsymbol{f})} \log^{r-1} x + \sum_{t=1}^{r-2} c_t(r, \boldsymbol{f}) \log^t x + O(1)\,;$$

the constants $c_t(r, \boldsymbol{f})$ resp. the constant in $O(1)$ depends on $K, r, t, \boldsymbol{f}$ resp. $K, r, \boldsymbol{f}$ only.

The formula

$$\sum_{\boldsymbol{a}|\boldsymbol{f}} \mu(\boldsymbol{a}) = \begin{cases} 1 & \text{for } \boldsymbol{f} = 1\,, \\ 0 & \text{for } \boldsymbol{f} \neq 1 \end{cases}$$

yields

LEMMA 1. *Let $f(x)$ be a complex valued function $(x \geqq 1)$; then*

Received July. 20, 1962. This work was supported by the National Science Foundation grant G-16305 to Purdue University.

$$g(x) := \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} f\left(\frac{x}{N\mathfrak{a}}\right) \quad implies \quad f(x) = \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \mu(\mathfrak{a})\, g\left(\frac{x}{N\mathfrak{a}}\right).$$

Using the Euler summation formula, we find

$$(4)\quad \sum_{m\leqq x} \frac{1}{m} \log^{r-1} m = \frac{1}{r}\log^r x + a_r + O\left(\frac{1}{x}\log^{r-1} x\right) \quad (r \text{ integer}, >1).$$

Because of

$$\sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} N\mathfrak{a} = \sum_{m\leqq x} (A(m, \mathfrak{f}) - A(m-1, \mathfrak{f}))\frac{1}{m}\log^{r-1} m,$$

(1) and (4) imply

$$(5)\quad \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1} N\mathfrak{a} = \frac{\gamma(\mathfrak{f})}{r}\log^r x + b_r(\mathfrak{f}) + O(x^{-1/n}\log^{r-1} x) \quad (r>1);$$

the constants $b_r(\mathfrak{f})$ depend on K, r, $\mathfrak{f}$ only. Because of

$$\sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n} \log^{r-1}\frac{x}{N\mathfrak{a}} = \sum_{m\leqq x} (A(m, \mathfrak{f}) - A(m-1, \mathfrak{f}))\left(\frac{x}{m}\right)^{1-1/n}\log^{r-1}\frac{x}{m},$$

(1) implies

$$(6)\quad \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \left(\frac{x}{N\mathfrak{a}}\right)^{1-1/n} \log^{r-1}\frac{x}{N\mathfrak{a}} = O(x).$$

By the binomial theorem and

$$\sum_{s=0}^{r-1} (-1)^s \binom{r-1}{s}\frac{1}{s+1} = \frac{1}{r},$$

(5) yields

$$(7)\quad \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{1}{N\mathfrak{a}} \log^{r-1}\frac{x}{N\mathfrak{a}} = \frac{\gamma(\mathfrak{f})}{r}\log^r x + \sum_{s=0}^{r-1} d_s(r, \mathfrak{f})\log^s x + O(x^{-1/n}\log^{r-1} x);$$

the constants $d_s(r, \mathfrak{f})$ depend on K, s, r, $\mathfrak{f}$ only.

As shown in [1],

$$(8)\quad \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}} = O(1), \qquad \sum_{\substack{N\mathfrak{a}\leqq x\\(\mathfrak{a},\mathfrak{f})=1}} \frac{\mu(\mathfrak{a})}{N\mathfrak{a}}\log\frac{x}{N\mathfrak{a}} = O(1).$$

Proof of Theorem 1. By (3), Theorem 1 is correct for $r=2$. Suppose $r>2$ and

$$\sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}}\frac{\mu(\mathfrak{a})}{N\mathfrak{a}}\log^s\frac{x}{N\mathfrak{a}}=\sum_{t=1}^{s-1}c_t(s,\mathfrak{f})\log^t x+O(1)\qquad(1<s<r)\,. \tag{9}$$

In Lemma 1, let $f(x):=x\log^{r-1}x$; then

$$\begin{aligned}g(x)&=x\sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}}\frac{1}{N\mathfrak{a}}\log^{r-1}\frac{x}{N\mathfrak{a}}\\&=\frac{\gamma(\mathfrak{f})}{r}x\log^r x+x\sum_{s=1}^{r-1}d_s(r,\mathfrak{f})\log^s x+O(x^{1-1/n}\log^{r-1}x)\,,\end{aligned}\tag{10}$$

by (7). Lemma 1, (10), (9), (6), and (8) imply

$$\begin{aligned}x\log^{r-1}x&=\sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}}\mu(\mathfrak{a})\Big(\frac{\gamma(\mathfrak{f})x}{rN\mathfrak{a}}\log^r\frac{x}{N\mathfrak{a}}+\frac{x}{N\mathfrak{a}}\sum_{s=1}^{r-1}d_s(r,\mathfrak{f})\log^s\frac{x}{N\mathfrak{a}}\\&\qquad+O\Big(\Big(\frac{x}{N\mathfrak{a}}\Big)^{1-1/n}\log^{r-1}\frac{x}{N\mathfrak{a}}\Big)\Big)\\&=\frac{\gamma(\mathfrak{f})x}{r}\sum_{\substack{N\mathfrak{a}\leq x\\(\mathfrak{a},\mathfrak{f})=1}}\frac{\mu(\mathfrak{a})}{N\mathfrak{a}}\log^r\frac{x}{N\mathfrak{a}}+\sum_{s=2}^{r-1}d_s(r,\mathfrak{f})\sum_{t=1}^{s-1}c_t(s,\mathfrak{f})\log^t x+O(x)\,;\end{aligned}$$

let

$$c_t(r,\mathfrak{f}):=-\frac{r}{\gamma(\mathfrak{f})}\sum_{s=t+1}^{r-1}d_s(r,\mathfrak{f})\,c_t(s,\mathfrak{f})\qquad(t=1,2,\cdots,r-2)\,.$$

This proves Theorem 1.

The fact that

$$c_{s-1}(s,\mathfrak{f})=\frac{s}{\gamma(\mathfrak{f})}$$

was not used in the preceding proof.

Now we derive two consequences of (2). The well-known relation (Landau (1903))

$$T(x):=\sum_{N\mathfrak{p}\leq x}\log N\mathfrak{p}=O(x)$$

implies

$$T(x,H\bmod\mathfrak{f}):=\sum_{\substack{N\mathfrak{p}\leq x\\\mathfrak{p}\in H\bmod\mathfrak{f}}}\log N\mathfrak{p}=O(x) \tag{11}$$

By

$$\sum_{\substack{N\mathfrak{p}\leq x\\\mathfrak{p}\in H\bmod\mathfrak{f}}}\log^2 N\mathfrak{p}=\sum_{m\leq x}(T(m,H\bmod\mathfrak{f})-T(m-1,H\bmod\mathfrak{f}))\log m\,,$$

(11) gives

$$\sum_{\substack{Np\leq x\\ p\in H\bmod f}} \log^2 Np = T(x, H \bmod f)\log x + O(x)\,. \tag{12}$$

According to Landau (1903), we have

$$s(x) := \sum_{Np\leq x} \frac{\log Np}{Np} = \log x + O(1)\,. \tag{13}$$

Using

$$\sum_{N\ \leq x} \frac{\log^2 Np}{Np} = \sum_{m\leq x} (S(m) - S(m-1))\log m\,,$$

(13) implies

$$\sum_{Np\leq x} \frac{\log^2 Np}{Np} = \frac{1}{2}\log^2 x + O(\log x)\,. \tag{14}$$

LEMMA 2. *We have*

$$\sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log^2 Np \log Nq = \frac{\log x}{2} \sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log Np \log Nq + O(x\log x)\,.$$

Proof. Denote by $H(q) \bmod f$ the class of all ideals a of K with $aq \in H \bmod f$; then (12), (13) and the definition of $T(x, H \bmod f)$ in (11) give

$$\begin{aligned}\sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log^2 Np \log Nq &= \sum_{\substack{Nq\leq x\\ (q,f)=1}} \log Nq \left(T\left(\frac{x}{Nq}, H(q) \bmod f\right)\log\frac{x}{Nq} + O\left(\frac{x}{Nq}\right)\right)\\ &= \sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log Nq \log Np\,(\log x - \log Nq) + O(x\log x)\,.\end{aligned}$$

This proves Lemma 2.

THEOREM 2. *We have*

$$\log x \sum_{\substack{Npq\leq x\\ pq\in H\bmod f}} \log Np \log Nq + 2 \sum_{\substack{Npqr\leq x\\ pq\in r H\bmod f}} \log Np \log Nq \log Nr = \frac{2x}{h(f)}\log^2 x + O(x\log x)$$

where the constant in the remainder term depends on K and f only.

Proof. We write (2) for x/Nr and $H(r) \bmod f$ instead of x and

$H \bmod \boldsymbol{f}$, multiply by $\log N\boldsymbol{r}$, and take summation over all prime ideals $\boldsymbol{r}$ with $(\boldsymbol{r}, \boldsymbol{f}) = 1$ and $N\boldsymbol{r} \leqq x$. By (13) and (14), we find

$$\sum_{\substack{N\boldsymbol{pr}\leqq x \\ \boldsymbol{pr}\in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \log N\boldsymbol{r} + \sum_{\substack{N\boldsymbol{pqr}\leqq x \\ \boldsymbol{pqr}\in H \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r}$$
$$= \frac{x}{h(\boldsymbol{f})} \log^2 x + O(x \log x) .$$

The application of Lemma 2 completes the proof.

THEOREM 3. *If*

$$\sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for the principal class $H_0 \bmod \boldsymbol{f}$, then

$$\frac{1}{x} \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} \longrightarrow \infty \qquad (x \longrightarrow \infty)$$

for all $h(\boldsymbol{f})$ classes $H \bmod \boldsymbol{f}$.

Proof. Suppose

$$\sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_1 \bmod \boldsymbol{f}}} \log^2 N\boldsymbol{p} = O(x)$$

for a certain ideal class $H_1 \bmod \boldsymbol{f}$. Then (2) implies

$$\sum_{\substack{N\boldsymbol{pq}\leqq x \\ \boldsymbol{pq}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} = \frac{2}{h(\boldsymbol{f})} x \log x + O(x) , \tag{15}$$

and Theorem 2 gives

$$\sum_{\substack{N\boldsymbol{pqr}\leqq x \\ \boldsymbol{pqr}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} = O(x \log x) . \tag{16}$$

By (15) and (13), we get

$$\sum_{\substack{N\boldsymbol{pqr}\leqq x \\ \boldsymbol{pqr}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \log N\boldsymbol{q} \log N\boldsymbol{r} \geqq \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \log N\boldsymbol{p} \sum_{\substack{N\boldsymbol{qr}\leqq x/N\boldsymbol{p} \\ \boldsymbol{qr}\in H_1 \bmod \boldsymbol{f}}} \log N\boldsymbol{q} \log N\boldsymbol{r}$$

$$\begin{aligned} &= \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H \bmod_0 \boldsymbol{f}}} \log N\boldsymbol{p} \left(\frac{2}{h(\boldsymbol{f})} \frac{x}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O\left(\frac{x}{N\boldsymbol{p}} \right) \right) \\ &= \frac{2x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p}\leqq x \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} \log \frac{x}{N\boldsymbol{p}} + O(x \log x) \\ &\geqq \frac{x \log x}{h(\boldsymbol{f})} \sum_{\substack{N\boldsymbol{p}\leqq \sqrt{x} \\ \boldsymbol{p}\in H_0 \bmod \boldsymbol{f}}} \frac{\log N\boldsymbol{p}}{N\boldsymbol{p}} + O(x \log x) . \end{aligned} \tag{17}$$

(17) and (16) imply the contradiction

$$\sum_{\substack{Np \leqq \sqrt{x} \\ p \in H_0 \bmod f}} \frac{\log Np}{Np} = O(1) ,$$

and Theorem 3 is proved.

The special case of Theorem 3 for the rational number field was treated in [2].

BIBLIOGRAPHY

1. G. J. Rieger, *Verallgemeinerung der Selbergschen Formel auf Idealklassen mod f in algebraischen Zahlkoerpern,* Math. Zeitschrift **69** (1958), 183-194.

2. H. N. Shapiro, *On primes in arithmetic progressions I.* Ann. of Math., **52** (1950), 217-230.

PURDUE UNIVERSITY AND UNIVERSITY MUNICH, GERMANY

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 13, No. 2 April, 1963

Rafael Artzy, *Solution of loop equations by adjunction* 361
Earl Robert Berkson, *A characterization of scalar type operators on reflexive Banach spaces* 365
Mario Borelli, *Divisorial varieties* 375
Raj Chandra Bose, *Strongly regular graphs, partial geometries and partially balanced designs* 389
R. H. Bruck, *Finite nets. II. Uniqueness and imbedding* 421
L. Carlitz, *The inverse of the error function* 459
Robert Wayne Carroll, *Some degenerate Cauchy problems with operator coefficients* 471
Michael P. Drazin and Emilie Virginia Haynsworth, *A theorem on matrices of* 0*'s and* 1*'s* 487
Lawrence Carl Eggan and Eugene A. Maier, *On complex approximation* 497
James Michael Gardner Fell, *Weak containment and Kronecker products of group representations* 503
Paul Chase Fife, *Schauder estimates under incomplete Hölder continuity assumptions* 511
Shaul Foguel, *Powers of a contraction in Hilbert space* 551
Neal Eugene Foland, *The structure of the orbits and their limit sets in continuous flows* 563
Frank John Forelli, Jr., *Analytic measures* 571
Robert William Gilmer, Jr., *On a classical theorem of Noether in ideal theory* 579
P. R. Halmos and Jack E. McLaughlin, *Partial isometries* 585
Albert Emerson Hurd, *Maximum modulus algebras and local approximation in* C^n 597
James Patrick Jans, *Module classes of finite type* 603
Betty Kvarda, *On densities of sets of lattice points* 611
H. Larcher, *A geometric characterization for a class of discontinuous groups of linear fractional transformations* 617
John W. Moon and Leo Moser, *Simple paths on polyhedra* 629
T. S. Motzkin and Ernst Gabor Straus, *Representation of a point of a set as sum of transforms of boundary points* 633
Rajakularaman Ponnuswami Pakshirajan, *An analogue of Kolmogorov's three-series theorem for abstract random variables* 639
Robert Ralph Phelps, *Čebyšev subspaces of finite codimension in* $C(X)$ 647
James Dolan Reid, *On subgroups of an Abelian group maximal disjoint from a given subgroup* 657
William T. Reid, *Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems* 665
Georg Johann Rieger, *Some theorems on prime ideals in algebraic number fields* 687
Gene Fuerst Rose and Joseph Silbert Ullian, *Approximations of functions on the integers* 693
F. J. Sansone, *Combinatorial functions and regressive isols* 703
Leo Sario, *On locally meromorphic functions with single-valued moduli* 709
Takayuki Tamura, *Semigroups and their subsemigroup lattices* 725
Pui-kei Wong, *Existence and asymptotic behavior of proper solutions of a class of second-order nonlinear differential equations* 737
Fawzi Mohamad Yaqub, *Free extensions of Boolean algebras* 761