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FREE EXTENSIONS OF BOOLEAN ALGEBRAS

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Introduction. This paper is concerned with the problem of imbedding a Boolean algebra B into an α-complete Boolean algebra B^* in such a way that certain homomorphisms of B can be extended to B^*. We investigate two such imbeddings which arose naturally from the consideration of the work of Rieger and Sikorski in [5] and [7]. In [5] Rieger proved the existence of a certain class of free Boolean algebras and investigated their representability by α-fields of sets. Rieger's theorem on the existence of "the free α-complete Boolean algebra on m generators" is equivalent to the following statement: Every free Boolean algebra B can be imbedded in an α-complete Boolean algebra B^* such that every homomorphism of B into an α-complete Boolean algebra C can be extended to an α-homomorphism of B^* into C. The question now arises: Does this result hold for an arbitrary Boolean algebra B which is not necessarily free? If such an imbedding exists, we call B^* the *free α-extension of B*.

In [7], Sikorski gave a characterization of all the σ-regular extensions of a Boolean algebra B. To obtain this characterization, he first proved that B can be imbedded as a σ-regular subalgebra of a σ-complete Boolean algebra B^* such that every σ-homomorphism of B into a σ-complete Boolean algebra C can be extended to a σ-homomorphism of B^* into C. We call B^* the *free σ-regular extension of B*.

In § 2 of this paper we prove that the free α-extension B_α of B exists uniquely for every Boolean algebra B and every infinite cardinal number α. In § 3 we investigate the representability of B_α by an α-field of sets. We first prove that B_α is isomorphic to an α-field of sets if and only if it is α-representable. A corollary to this result is that the free σ-extension B_σ of an arbitrary Boolean algebra B is isomorphic to a σ-field of sets. The problem of characterizing those Boolean algebras B for which B_α is α-representable for $\alpha \geqq 2^{\aleph_0}$ is also discussed. In § 4 we investigate the α-regular extensions of Boolean algebras for an arbitrary cardinal number α. Sikorski's results on the σ-regular extensions depend on the Loomis-Sikorski theorem which does not hold for uncountable cardinal numbers. We use our results on the free α-extension B_α of B to prove the existence of the free α-regular extension and to give a characterization of the α-regular

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extensions of B.

Our result on the existence of the free α-regular extension of B is a special case of a more general result of Kerstan [3], but it is obtained here by a different method. We also learned through a written communication from Professor Sikorski that he also proved the same result and his proof will appear in [10]. Sikorski's proof is similar to ours; however he works with the free α-complete Boolean algebras while we work with the free α-extension of B (see Theorem 4.1 below). The characterization of the α-regular extensions of B given in Theorem 4.2 does not appear in [3] or [10]; the free α-extensions of Boolean algebras have not been considered in either of these papers.

1. Preliminaries. Throughout this paper, the product (=greatest lower bound) of a set $\{x_t: t \in T\}$ of elements of a Boolean algebra B will be denoted, whenever it exists, by $\prod_{t \in T} x_t$. If A is a subalgebra of B and $x_t \in A$ for every $t \in T$, then the set $\{x_t: t \in T\}$ may have two products, one taken in A and the other in B; we denote these products, whenever they exist, by $\prod^A_{t \in T} x_t$ and $\prod^B_{t \in T} x_t$ respectively. The complement of an element x of B will be denoted by $\bar{x}$, and the symbol "0" will designate the zero element of B.

Definitions of the more familiar Boolean concepts which are not given in this section can be found in [9] or [2]. A homomorphism h of a Boolean algebra A into a Boolean algebra B is an *α-homomorphism* if it preserves α-sums (hence α-products) whenever they exist in A. Equivalently ([9], § 22), h is an α-homomorphism of A into B if and only if $\prod_{x \in S} h(x) = 0$ for every subset S of A such that $|S| \leqq \alpha$ and $\prod_{x \in S} x = 0$. h is an *α-isomorphism* if it is a one-to-one α-homomorphism. h is a *complete homomorphism* (*complete isomorphism*) if it is an α-homomorphism (α-isomorphism) for every infinite cardinal number α. A subalgebra A of a Boolean algebra B is *α-regular* if the injection mapping of A into B is an α-isomorphism. Equivalently, A is an α-regular subalgebra of B if and only if $\prod^B_{x \in S} x = 0$ for every subset S of A such that $|S| \leqq \alpha$ and $\prod^A_{x \in S} x = 0$. A is a *regular* subalgebra of B if it is α-regular for every infinite cardinal number α.

A Boolean algebra B is *free on m generators* (m is an arbitrary cardinal number) if it is generated by a subset E with cardinality m and with the property that every mapping of E into a Boolean algebra C can be extended to a homomorphism of B into C. All free Boolean algebras on m generators are isomorphic ([9], § 14) and will be denoted throughout this paper by A_m. An α-complete Boolean algebra B is a *free α-complete Boolean algebra on m generators* if it is α-generated by a subset E with cardinality m and with the property that every mapping of E into an α-complete Boolean algebra C can be extended to an α-homomorphism of B into C. All free α-complete Boolean algebras on m generators are isomorphic [5] and will be denoted here by A_m^α.

For every Boolean algebra B and every infinite cardinal number α, there exists an α-complete Boolean algebra B^* and a complete isomorphism h of B into B^* such that $h(B)$ α-generates B^* ([9], § 36). This Boolean algebra B^*, which is unique up to isomorphisms, is called the *normal α-completion* (also minimal α-extension) *of* B and will be denoted here by B^α. If B^* is complete and h is a complete isomorphism of B into B^* such that B^* is completely generated by $h(B)$, then B^* is called the *normal completion* (also, minimal extension) *of* B and will be denoted by B^∞. When dealing with the normal α-completion (completion) of B, we shall usually identify B with $h(B)$ and thus consider B as a regular subalgebra of both B^α and B^∞.

The *Stone space* (=Boolean space) of a Boolean algebra B is the compact, Hausdorff, totally disconnected space whose open-and-closed subsets, ordered by set inclusion, form a Boolean algebra isomorphic to B. For every Boolean algebra B and every infinite cardinal number α, $S(B)$ will denote the Stone space of B, $F_0(B)$ the Boolean algebra of open-and-closed subsets of $S(B)$, and F_α (B) the smallest α-field of subsets of $S(B)$ containing $F_0(B)$.

A Boolean algebra B is called *α-representable* if it is isomorphic to an α-regular subalgebra of a quotient algebra F/I, where F is an α-field of sets and I is an α-ideal of F. If B is α-complete, then this definition reduces to: B is α-representable if and only if it is an α-homomorph of an α-field of sets. There are α-complete (even complete) Boolean algebras which are not α-representable for $\alpha \geqq 2^{\aleph_0}$ ([9], § 29). However, for the case $\alpha = \aleph_0$, we have the Loomis-Sikorski theorem ([9], § 29): *Every Boolean algebra is σ-representable.*

2. Free α-extensions.

DEFINITION 2.1. An α-complete Boolean algebra B^* is called a *free α-extension* of the Boolean algebra B if B^* is α-generated by a subalgebra B_0 isomorphic to B such that every homomorphism of B_0 into an α-complete Boolean algebra C can be extended to an α-homomorphism of B^* into C.

We shall show in this section that for every Boolean algebra B and every infinite cardinal number α, the free α-extension of B exists and is unique up to isomorphisms. We denote the free α-extension of B by B_α, and we shall consider B as a subalgebra of B_α, thus identifying it with the subalgebra B_0 of Definition 2.1.

The following lemma follows immediately from Definition 2.1.

LEMMA 2.1. *Let A_m be the free Boolean algebra on m generators and A_m^α the free α-complete Boolean algebra on m generators. Then A_m^α is the free α-extension of A_m.*

LEMMA 2.2. *Let I be an ideal of A_m and I^* the smallest α-ideal of A_m^α containing I. Then $I^* \cap A_m = I$.*

Proof. Let h be the canonical homomorphism of A_m onto A_m/I and let i be an isomorphism of A_m/I onto $F_0(A_m/I)$. Then the homomorphism ih can be extended to an α-homomorphism h^* of A_m^α into $F_\alpha(A_m/I)$. And the kernel of h^* is an α-ideal which contains I^* and intersects A_m in I. Hence $I^* \cap A_m = I$.

THEOREM 2.1. *For every Boolean algebra B and every infinite cardinal number α, the free α-extension of B exists and is unique up to isomorphisms.*

Proof. Let $|B| = m$. Then there exists an ideal I of A_m such that A_m/I is isomorphic to B. Let I^* be the smallest α-ideal of A_m^α containing I. We shall show that A_m^α/I^* is a free α-extension of B.

Lemma 2.2 shows that the subalgebra A_m/I^* of A_m^α/I^* is isomorphic to B. And since A_m^α is α-generated by A_m, it follows that A_m/I^* α-generates A_m^α/I^*. Thus it only remains to show that homomorphisms of A_m/I^* can be extended to A_m^α/I^*. Let h be a homomorphism of A_m/I^* into an α-complete Boolean algebra C. Let f be the canonical α-homomorphism of A_m^α onto A_m^α/I^* and denote the restriction of f to A_m by f'. Then the homomorphism $g = hf'$ has an extension g^* which is an α-homomorphism of A_m^α into C. Since both I^* and the kernel of g^* are α-complete ideals containing I, it follows that I^* is contained in the kernel of g^*. We now define the mapping h^* by:

$$h^*(f(x)) = g^*(x),\ x \in A_m^\alpha\ .$$

Then h^* is the desired extension of h; hence A_m^α/I^* is a free α-extension of B.

The uniqueness of the free α-extension of B follows from the standard argument used to show that all free Boolean algebras on the same number of generators are isomorphic. Indeed, suppose that B has two free α-extensions B_1 and B_2. Let i be an isomorphism of the subalgebra B of B_1 onto the subalgebra B of B_2. Then i can be extended to an α-homomorphism of B_1 onto B_2 and the isomorphism i^{-1} can be extended to an α-homomorphism i_2 of B_2 onto B_1. Let $B^* = \{x \in B_1 : i_2(i_1(x)) = x\}$. Then B^* is an α-complete, α-regular subalgebra of B_1 containing B. Hence $B^* = B_1$, and i_1 is an isomorphism of B_1 onto B_2.

LEMMA 2.3. *Let h be a homomorphism of a Boolean algebra B into an α-complete Boolean algebra C. Then the extension of h to an α-homomorphism of B_α into C is unique.*

Proof. Suppose h has two extensions h_1 and h_2. Let $B^* = \{x \in B_\alpha : h_1(x) = h_2(x)\}$. Then B^* is an α-complete, α-regular subalgebra of B_α containing B. Hence $B^* = B_\alpha$, and $h_1 = h_2$.

A slight modification of the proof of Theorem 2.1 yields the following result.

LEMMA 2.4. *If I is an ideal of B, then the free α-extension of B/I is isomorphic to B_α/I^*, where I^*, is the smallest α-ideal of B_α containing I.*

LEMMA 2.5. *If A is a subalgebra of B, then A_α is isomorphic to the α-complete, α-regular subalgebra A^* of B_α α-generated by A.*

Proof. We only need to show that if h is a homomorphism of A into an α-complete Boolean algebra C, then h can be extended to an α-homomorphism of A^* into C. Thus, we imbed C into its normal completion C^∞. Then, by a known result ([9], § 33.1), h can be extended to a homomorphism h_1 of B into C^∞. Furthermore, h_1 can be extended to an α-homomorphism h_2 of B_α into C^∞. Let h^* be the restriction of h_2 to A^*. Then, since A^* is an α-regular subalgebra of B_α, h^* is an α-homomorphism of A^* into C^∞, and the proof will be complete once we show that $h^*(A^*)$ is contained entirely in C. Since both $h^*(A^*)$ and C are α-complete, α-regular subalgebras of C^∞, their intersection $h^*(A^*) \cap C$ is also an α-complete, α-regular subalgebra of C^∞. And since $h^*(A^*)$ is α-generated by $h(A)$, it follows that $h^*(A^*) = h^*(A^*) \cap C$. Hence $h^*(A^*) \subset C$, and the proof is now complete.

3. **Representability by α-field of sets.** In investigating the representability problem of the free α-extensions of Boolean algebras, the following two natural questions arise: When is the free α-extension B_α of a Boolean algebra B isomorphic to an α-field of sets? And, when is B_α α-representable? The following theorem shows that these two questions are equivalent.

THEOREM 3.1. *For every Boolean algebra B and every infinite cardinal number α, there is an α-homomorphism j^* of B_α onto $F_\alpha(B)$ whose restriction to B is the canonical imbedding of B in $F_0(B)$. Moreover, j^* is one-to-one if and only if B_α is α-representable.*

Proof.[1] Let j be the canonical isomorphism of B onto $F_0(B)$ and

[1] This proof, which is considerably shorter than the one intended, is due to the referee.

extend j to an α-homomorphism j^* of B_α into $F_\alpha(B)$. Since $F_\alpha(B)$ is α-generated by $F_0(B)$, j^* is onto. Since B is a subalgebra of B_α, there is a continuous mapping λ of $S(B_\alpha)$ onto $S(B)$ such that for every $x \in B$, $\lambda^{-1}(j(x)) = i(x)$, where i is the canonical isomorphism of B_α onto $F_0(B_\alpha)$. Let k denote the homomorphism $E \to \lambda^{-1}(E)$, mapping the subsets of $S(B)$ to subsets of $S(B_\alpha)$. Then k is an α-isomorphism which maps $F_0(B)$ into $F_0(B_\alpha)$, since $k(j(x)) = i(x)$ for every $x \in B$. Consequently, k maps $F_\alpha(B)$ into $F_\alpha(B_\alpha)$. If B_α is α-representable, then $F_0(B_\alpha)$ is an α-retract of $F_\alpha(B_\alpha)$; that is, there is an α-homomorphism h of $F_\alpha(B_\alpha)$ onto $F_0(B_\alpha)$ whose restriction to B_α is the identity mapping. Then $i^{-1}hkj^*$ is an α-homomorphism of B_α onto itself which is an extension of the identity mapping on B. Thus, it follows from Lemma 2.3 that $i^{-1}hkj^*(x)=x$ for all $x \in B_\alpha$. Thus j^* is an α-isomorphism.

Since every Boolean algebra is σ-representable (the Loomis-Sikorski Theorem), the last theorem yields the following corollary which answers the representability question for the free σ-extensions of Boolean algebras.

COROLLARY 3.1. *For every Boolean algebra B, B_σ is isomorphic to the σ-field of sets $F_\sigma(B)$.*

The next theorem gives a strong necessary condition that a Boolean algebra B must satisfy in order for B_α to be α-representable when $\alpha \geqq 2^{\aleph_0}$.

LEMMA 3.1. *If B_α is α-representable, then so is every subalgebra and every homomorphic image of B.*

Proof. Let h be a homomorphism of B onto a Boolean algebra C. Imbed C into its normal α-completion C^α and extend h to an α-homomorphism of B_α onto C^α. Since B_α is α-representable, so is C^α. And since C is an α-regular subalgebra of C^α, C itself is α-representable. On the other hand, if A is a subalgebra of B, then it follows from Lemma 2.5 that A_α is α-representable. Hence A is α-representable.

DEFINITION 3.1. A Boolean algebra B is called *super-atomic* if every subalgebra and every homomorphic image of B is atomic.

THEOREM 3.2. *Let B be a Boolean algebra and $\alpha \geqq 2^{\aleph_0}$. If B_α is α-representable, then B is super-atomic.*

Proof. We shall first show that if B_α is α-representable, $\alpha \geqq 2^{\aleph_0}$, then B is atomic. Suppose B is not atomic. Then B has an element

x such that the principal ideal (x), when considered as a Boolean algebra, is atomless. Now the Boolean algebra (x) is isomorphic to a subalgebra A of B. For let P be a prime ideal of (x) and let $P^* = \{\bar{x}: x \in P\}$. Then it is not difficult to show that $A = P \cup P^*$ is a subalgebra of B isomorphic to the Boolean algebra (x). Since A is atomless, it has a subalgebra A' isomorphic to the free Boolean algebra on $\aleph_0$ generators ([1], § 1.7). And since B_α is α-representable, Lemmas 2.5 and 3.1 show that A' is α-representable also. This contradicts the fact that the free Boolean algebra on $\aleph_0$ generators is not α-representable if $\alpha \geqq 2^{\aleph_0}$. Thus we conclude that B is atomic.

The proof of the theorem now follows immediately. If C is a subalgebra of B, then, by Lemma 2.5, C_α is α-representable. Hence C is atomic. On the other hand, if C is a homomorphic image of B, then C_α is an α-homomorph of B_α. Thus C_α is α-representable, hence C is atomic.

Super-atomic Boolean algebras were discussed briefly in [4] and more recently in more detail by G. W. Day [1]. In particular, Day proved ([1], Theorem 16) the converse of Theorem 3.2. Day also gave the following characterization of super-atomic Boolean algebras: *A Boolean algebra B is super-atomic if and only if every subalgebra of B is atomic if and only if every homomorph of B is atomic.* A characterization of super-atomic Boolean algebras with ordered basis is given by Theorem 3.3 of [4].

Combining Day's result ([1], Theorem 16) with Theorem 3.2, we obtain:

THEOREM 3.3. *Let B be a Boolean algebra and $\alpha \geqq 2^{\aleph_0}$. Then B_α is α-representable if and only if B is super-atomic.*

If B is not super-atomic and $\alpha \geqq 2^{\aleph_0}$, then $F_\alpha(B)$ is not isomorphic to B_α; however; we shall now show that $F_\alpha(B)$ is the free α-extension of B "over the class of α-representable Boolean algebras." An α-complete, α-representable Boolean algebra B^* is called the *free α-representable extension of B* if B^* is α-generated by a subalgebra B_0 isomorphic to B such that every homomorphism of B_0 into an α-complete, α-representable Boolean algebra C can be extended to an α-homomorphism of B^* into C. We need the following result of Sikorski ([9], 31.1):

LEMMA 3.2. *Let A_m be the free Boolean algebra on m generators and α an infinite cardinal number. Then $F_\alpha(A_m)$ is the free α-representable extension of A_m.*

A slight modification of the proof of Theorem 2.1 shows the following:

THEOREM 3.4. *For every Boolean algebra B and every infinite cardinal number α, the free α-representable extension of B exists and is unique up to isomorphisms.*

The following theorem can be proved by an argument similar to the one used in the proof of Theorem 3.1.

THEOREM 3.5. *For every Boolean algebra B and every infinite cardinal number α, $F_\alpha(B)$ is the free α-representable extension of B.*

4. Free α-regular extensions.

DEFINITION 4.1. An α-complete Boolean algebra B^* is called an *α-regular extension* of the Boolean algebra B if B^* is α-generated by an α-regular subalgebra B_0 isomorphic to B. If, in addition, every α-complete homomorphism of B_0 into an α-complete Boolean algebra C can be extended to an α-homomorphism of B^* into C, then B^* is called a free *α-regular extension* of B.

σ-regular extensions of Boolean algebras were investigated by Sikorski [7]. In this section we investigate the α-regular extensions of Boolean algebras for an arbitrary infinite cardinal number α. We denote the free α-regular extension of B by B_α^* (its existence and uniqueness are proved in Theorem 4.1). Also, for every Boolean algebra B and every infinite cardinal number α, we define the two ideals I_α and J_α as follows: I_α is the smallest α-ideal of B_α containing all elements u such that $u = \prod_{t\in T}^{B_\alpha} x_t$, where $|T| \leqq \alpha$, each $x_t \in B$, and $\prod_{t\in T}^{B} x_t = 0$. The elements u will be called the generators of I_α. J_α is the smallest α-ideal of $F_\alpha(B)$ containing all the nowhere dense α-closed subsets of the Stone space of B. (A subset E of a topological space X is called *α-closed* if E is the intersection of at most α open-and-closed subsets of X.)

LEMMA 4.1. *Let B be a Boolean algebra and I an α-ideal of B_α such that*: (a) $I \supset I_\alpha$, (b) $I \cap B = (0)$. *Then B_α/I is an α-regular extension of B.*

Proof. Let h be the canonical α-homomorphism of B_α onto B_α/I and observe that h is an isomorphism of B onto the subalgebra B/I. Suppose that $|T| \leqq \alpha$ and, for each $t \in T$, $h(x_t) \in B/I$ such that $\prod_{t\in T}^{B/I} h(x_t) = 0$. Then

$$\prod_{t\in T}^{B_\alpha/I} h(x_t) = h\Big(\prod_{t\in T}^{B_\alpha} x_t\Big) = 0 ,$$

where the last equality follows from the fact that $\prod_{t\in T}^{B_\alpha} x_t \in I_\alpha$ and condition (a) of the hypothesis. Thus B/I is an α-regular subalgebra of B_α/I. Furthermore, since B_α is α-generated by B and h is an α-homomorphism, it follows that B/I α-generates B_α/I. Hence B_α/I is an α-regular extension of B.

THEOREM 4.1. *Let B be a Boolean algebra and α an infinite cardinal number. Then the free α-regular extension B_α^* of B exists and is unique up to isomorphisms. Moreover, B_α^* is isomorphic to B_α/I_α.*

Proof. We shall first show that $I_\alpha \cap B = (0)$. Let B^α be the normal α-completion of B; thus B^α is α-generated by a regular subalgebra B_1 isomorphic to B. Let i be an isomorphism of B onto B_1 and observe that i is a complete isomorphism of B into B^α. Extend i to an α-homomorphism i^* of B_α into B^α and let u be a generator of I_α. Then $u = \prod_{t\in T}^{B_\alpha} x_t$, where $|T| \leqq \alpha$, and $\prod_{t\in T}^{B} x_t = 0$. And

$$i^*(u) = \prod_{t\in T}^{B\alpha} i^*(x_t) = \prod_{t\in T}^{B\alpha} i(x_t) = \prod_{t\in T}^{B_1} i(x_t) = i\Big(\prod_{t\in T}^{B} x_t\Big) = 0 \ .$$

It follows from this that I_α is contained in the kernel J of i^*. And since $J \cap B = (0)$, we have $I_\alpha \cap B = (0)$ also.

Now, it follows from Lemma 4.1 that B_α/I_α is an α-regular extension of B. Let h be an α-homomorphism of B/I_α into an α-complete Boolean algebra C. We wish to extend h to B_α/I_α. Let f be the canonical α-homomorphism of B_α onto B_α/I_α and let $g = hf_1$, where f_1 is the restriction of f to B. Then g can be extended to an α-homomorphism g^* of B_α into C. And, if $u = \prod_{t\in T}^{B_\alpha} x_t$ is a generator of I_α, then

$$g^*(u) = \prod_{t\in T} g^*(x_t) = \prod_{t\in T} g(x_t) = \prod_{t\in T} hf(x_t) = hf\Big(\prod_{t\in T}^{B} x_t\Big) = 0 \ .$$

Therefore I_α is contained in the kernel of g^*. We now define the mapping h^* by

$$h^*(f(x)) = g^*(x) \ , \qquad x \in B_\alpha \ .$$

Then h^* is the desired extension of h, and B_α/I_α is a free α-regular extension of B. The uniqueness of B_α^* can be proved easily by an argument similar to the one used in proving that B_α is unique. (See the proof of Theorem 2.1.)

COROLLARY 4.1. (Sikorski). *For every Boolean algebra B, B_σ^* is isomorphic to $F_\sigma(B)/J_\sigma$.*

Proof. By Corollary 3.1, B_σ is isomorphic to $F_\sigma(B)$, and the ideal I_σ, when considered as a σ-ideal of $F_\sigma(B)$, coincides with J_σ. Thus the conclusion follows from Theorem 4.1.

THEOREM 4.2. *An α-complete Boolean algebra B^* is an α-regular extension of B if and only if B^* is isomorphic to B_α/I, where I is an α-ideal of B_α satisfying the following two conditions*: (a) $I \supset I_\alpha$; (b) $I \cap B = (0)$.

Proof. Suppose B^* is an α-regular extension of B. Then B^* is α-generated by an α-regular subalgebra B_0 isomorphic to B. Let i be an isomorphism of B/I_α onto B_0. Then i is an α-isomorphism of B/I_α into B^*, hence it can be extended to an α-homomorphism i^* of B_α/I_α onto B^*. Let $I = \{x \in B_\alpha: i^*([x]_{I_\alpha}) = 0\}$. Then B^* is isomorphic to B_α/I and I satisfies conditions (a) and (b). The converse was proved in Lemma 4.1.

Theorem 4.2 and Corollary 3.1 yield the following result of Sikorski [7].

COROLLARY 4.2. *A σ-complete Boolean algebra B^* is a σ-regular extension of B if and only if B^* is isomorphic to $F_\sigma(B)/I$, where I is a σ-ideal of $F_\sigma(B)$ satisfying the conditions*: (a) $I \supset J_\sigma$; (b) $I \cap F_0(B) = \phi$.

The following result is well known ([9], § 35 and 23.2).

THEOREM 4.3. *The normal completion B^∞ of a Boolean algebra B is isomorphic to B^*_∞. That is, B^∞ has the property that every complete homomorphism of B into a complete Boolean algebra C can be extended to a complete homomorphism of B^∞ into C.*

Using Theorem 3.5 and arguments similar to the ones used in the proofs of Theorems 4.1 and 4.2, we obtain the following two theorems which also can be proved by using Sikorski's methods for the σ-case (see [9], § 36).

THEOREM 4.4. *For every Boolean algebra B and every infinite cardinal number α, B^*_α is isomorphic to $F_\alpha(B)/J_\alpha$ if and only if B^*_α is α-representable.*

THEOREM 4.5. *Let B be a Boolean algebra for which B^*_α is α-representable. Then an α-complete Boolean algelra B^* is an α-regular extension of B if and only if B^* is isomorphic to $F_\alpha(B)/I$, where I is an α-ideal of $F_\alpha(B)$ satisfying the conditions*: (a) $I \supset J_\alpha$; (b) $I \cap F_0(B) = \phi$.

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