

Pacific Journal of Mathematics

**A SUBFUNCTION APPROACH TO A BOUNDARY VALUE
PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS**

JERROLD WILLIAM BEBERNES

A SUBFUNCTION APPROACH TO A BOUNDARY VALUE PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

J. W. BEBERNES

1. Introduction. Consider the ordinary, second-order differential equation

$$(1.1) \quad y'' = f(x, y, y')$$

where $f(x, y, y')$ is a real-valued function defined on the region

$$T = \{(x, y, y') \mid a \leq x \leq b, |y| < \infty, |y'| < \infty\},$$

a and b finite.

The purpose of this paper is to determine sufficient conditions which when placed on $f(x, y, y')$ guarantee the existence of a unique solution of the two-point boundary value problem (BVP):

$$(1.2) \quad y'' = f(x, y, y'), \quad y(a) = \alpha, \quad y(b) = \beta.$$

A solution of the BVP: $y'' = f(x, y, y')$, $y(x_1) = y_1$, $y(x_2) = y_2$, where $a \leq x_1 \leq x_2 \leq b$ will be defined to be a function $y(x)$ which is of class C^2 and satisfies (1.1) on (x_1, x_2) , which is continuous on $[x_1, x_2]$, and which assumes the given boundary values at x_1 and x_2 .

The following assumptions will be placed on $f(x, y, y')$ as needed.

(A₀) $f(x, y, y')$ is continuous on T .

(A₁) $f(x, y, y')$ is a non-decreasing function of y for each fixed x and y' in T .

(A₂) $f(x, y, y')$ satisfies a Lipschitz condition with respect to y' on each fixed compact subset to T .

The primary results of this paper are the following two theorems.

THEOREM 6.2. *If*

(1) $f(x, y, y')$ satisfies A₀, A₁, and A₂,

(2) *there exists a positive continuous function $\phi(u)$ defined for $u \geq 0$ such that*

$$|f(x, y, y') - f(x, y, 0)| \leq K_s \phi(|y'|)$$

where K_s is a constant depending on compact subsets S of

$$\{(x, y) \mid a \leq y \leq b, |y| < \infty\}, \quad (x, y) \in S, \quad |y'| < \infty,$$

Received February 13, 1963. Research was in part supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command. This paper is a portion of a doctoral thesis written under the supervision of Professor Lloyd K. Jackson at the University of Nebraska.

and

$$\int_0^{\infty} \frac{udu}{\phi(u) + 1} = \infty,$$

(3) $|f(x, 0, y') - f(x, 0, 0)| \leq K|y'|$ for $a \leq x \leq b$ and $|y'| < \infty$, then there exists a unique solution of the BVP (1.2) of class $C^2[a, b]$.

THEOREM 6.3. *If given the BVP: $y'' = f(x, y, y')$, $y(a) = 0 = y(b)$, with*

(1) $f(x, y, y')$ satisfying A_0, A_1, A_2 , and

(2) $|f(x, 0, y') - f(x, 0, 0)| \leq K|y'|$ for all $a \leq x \leq b$ and $|y'| < +\infty$, then there exists a unique solution of the BVP of class $C^2[a, b]$.

Lees [4] proved that if $f(x, y, y')$ satisfies A_0, A_1 , and, in place of A_2 , a uniform Lipschitz condition with respect to y' , then there exists a unique solution of the BVP (1.2) of class $C^2[a, b]$. Note that Lees' result is immediate from Theorem 6-2. Whereas Lees used the method of finite differences, we shall attack the BVP (1.2) employing the theory of subfunctions developed by Fountain and Jackson [3]. In [3], Fountain and Jackson utilized the theory of subfunctions to construct a so-called generalized solution of the BVP (1.2) in the sense that if a solution exists it will be this generalized solution by uniqueness. To construct this generalized solution, they assumed $f(x, y, y')$ satisfies A_0, A_1 , and, in place of A_2 , that:

(A₂') $f(x, y, y')$ satisfies a Lipschitz condition with respect to y and y' on each fixed compact subset of T .

In this paper, we shall construct the generalized solution of the BVP (1.2) as in [3] under assumptions A_0, A_1 and A_2 . Most of the proofs carry over with only slight modifications. By knowing properties of this generalized solution, additional conditions can then be imposed on $f(x, y, y')$ to assure a solution of the BVP (1.2).

2. A "local" existence theorem and a maximum principle. The two theorems in this section form the basis for the construction of the generalized solution of the BVP (1.2). The "local" existence theorem is known and was first proven by E. Picard [6, pp. 9-36] in a slightly weaker form.

THEOREM 2.1. *Let $f(x, y, y')$ satisfy A_0 , then given $M, N > 0$ there is a $\delta(M, N) > 0$ such that the BVP:*

$$(2.1) \quad y'' = f(x, y, y'), \quad y(x_1) = y_1, \quad y(x_2) = y_2$$

has a solution of class C^2 on $[x_1, x_2]$ for any points (x_1, y_1) and (x_2, y_2) with $x_1, x_2 \in [a, b]$, $|x_1 - x_2| \leq \delta$, $|y_1| \leq M$, $|y_2| \leq M$, and $|(y_1 - y_2)/(x_1 - x_2)| \leq N$.

The details of the proof of this result will not be given. Note that the BVP (2.1) has a solution if and only if the BVP:

$$(2.2) \quad \begin{aligned} y'' &= f(x, z(x) + px + q, z'(x) + p) \\ z(x_1) &= 0 \quad z(x_2) = 0 \end{aligned}$$

has a solution, and that the BVP (2.2) has a solution if there is a function $z(x) \in C'[x_1, x_2]$ which is a solution of

$$(2.3) \quad z(x) = \int_{x_1}^{x_2} G(x, s) f(s, z(s) + ps + q, z'(s) + p) ds$$

where

$$G(x, s) = \begin{cases} (x_1 - s)(x_2 - x)(x_2 - x_1)^{-1}, & x_1 \leq s \leq x \\ (x_1 - x)(x_2 - s)(x_2 - x_1)^{-1}, & x \leq s \leq x_2. \end{cases}$$

The Schauder-Tychonoff fixed-point theorem [2, p. 456] can be employed to show the existence of a solution of (2.3) of class C' on $[x_1, x_2]$.

The following corollary follows as an immediate consequence of the proof of Theorem 2.1.

COROLLARY 2.1. *Let $M > 0, N > 0$ be fixed and let $\delta(M, N) > 0$ be as in Theorem 2.1. Then given any $\varepsilon > 0$ and any $\delta_1, 0 < \delta_1 \leq \delta(M, N)$, there is an $\eta, 0 < \eta \leq \delta_1$, such that for any points (x_1, y_1) and (x_2, y_2) with $x_1, x_2 \in [a, b], |x_1 - x_2| \leq \eta, |y_1| \leq M, |y_2| \leq M$, and $|(y_1 - y_2)(x_1 - x_2)^{-1}| \leq N$, there is a solution $y(x)$ of the BVP (2.1) on $[x_1, x_2]$ such that $|y(x) - w(x)| \leq \varepsilon$ and $|y'(x) - w'(x)| \leq \varepsilon$ where $w(x)$ is the linear function with $w(x_1) = y_1$ and $w(x_2) = y_2$.*

For any function g defined on $[x_1, x_2]$ and $x_0 \in (x_1, x_2)$, define $\bar{D}g(x_0) = \limsup [g(x_0 + \delta) - g(x_0 - \delta)]/2\delta$ as $\delta \rightarrow 0$ and $\underline{D}g(x_0) = \liminf [g(x_0 + \delta) - g(x_0 - \delta)]/2\delta$ as $\delta \rightarrow 0$.

LEMMA 2.1. *If $f(x, y, y')$ satisfies A_0 and A_1 and if the functions $\phi(x)$ and $\psi(x)$ satisfy:*

- (i) $\phi, \psi \in C[c, d] \cap C'(c, d)$ for $[c, d] \subset [a, b]$,
- (ii) $\underline{D}\phi'(x) \geq f(x, \phi(x), \phi'(x))$ and $\bar{D}\psi'(x) \leq f(x, \psi(x), \psi'(x))$ on (c, d) with at least one of these being strict inequality on (c, d) , and
- (iii) $\phi(c) - \psi(c) \leq M$ and $\phi(d) - \psi(d) \leq M$ where $M \geq 0$, then

$$\phi(x) - \psi(x) < M \text{ on } (c, d).$$

Observe that it suffices to consider only the case $M = 0$. If we assume the conclusion is false, we have an almost immediate contradiction.

LEMMA 2.2. *If $f(x, y, y')$ satisfies $A_0, A_1,$ and $A_2,$ and if there is a function ϕ such that:*

(i) $\phi \in C'[x_1, x_2]$ where $[x_1, x_2] \subset [a, b],$

(ii) $\underline{D}\phi'(x) \geq f(x, \phi(x), \phi'(x))$ on $(x_1, x_2),$

then given $\varepsilon > 0$ there is a function $\phi_1 \in C'[x_1, x_2]$ such that $\phi(x) - \varepsilon \leq \phi_1(x) \leq \phi(x)$ on $[x_1, x_2]$ and

$$\underline{D}\phi_1'(x) > f(x, \phi_1(x), \phi_1'(x)) \quad \text{on } (x_1, x_2).$$

Proof. Let $|\phi(x)| + |\phi'(x)| \leq R$ on $[x_1, x_2]$ and let

$$T^* = \{(x, y, y') \mid x_1 \leq x \leq x_2, |y| \leq R + 1, |y'| \leq R + 1\}.$$

Let $K > 0$ be such that $|f(x, y, y'_1) - f(x, y, y'_2)| \leq K|y'_1 - y'_2|$ for all $(x, y, y'_1), (x, y, y'_2) \in T^*.$ Let $z(x)$ be a solution of $z'' = (K + 1)z'$ on $[x_1, x_2]$ which satisfies the conditions $0 \leq z(x) \leq \varepsilon$ and $-1 \leq z'(x) < 0$ on $[x_1, x_2].$ Then on (x_1, x_2)

$$\begin{aligned} \underline{D}[\phi'(x) - z'(x)] - f(x, \phi(x) - z(x), \phi'(x) - z'(x)) \\ \geq f(x, \phi(x), \phi'(x)) - f(x, \phi(x), \phi'(x) - z'(x)) - z''(x) \\ \geq -K|z'(x)| - (K + 1)z'(x) \\ = -z'(x) > 0. \end{aligned}$$

Hence, $\phi_1(x) \equiv \phi(x) - z(x)$ is the desired function.

A dual statement holds by reversing the inequalities. We shall refer to the dual of a result by using an asterisk (for example, Lemma 2.2*).

THEOREM 2.2 (Maximum Principle). *If $f(x, y, y')$ satisfies $A_0, A_1,$ and $A_2,$ and if there exist functions ϕ and ψ satisfying:*

(i) $\phi, \psi \in C'(c, d) \cap C[c, d]$ for some $[c, d] \subset [a, b],$

(ii) $\underline{D}\phi'(x) \geq f(x, \phi(x), \phi'(x))$ and $\bar{D}\psi'(x) \leq f(x, \psi(x), \psi'(x))$ on $(c, d),$

(iii) $\phi(c) - \psi(c) \leq M$ and $\phi(d) - \psi(d) \leq M$ for some $M \geq 0,$ then $\phi(x) - \psi(x) \leq M$ on $[c, d].$

Proof. It suffices to consider only the case where $M = 0.$ Let $M = 0$ and assume conclusion is false. Then there exist $x \in (c, d)$ such that $\phi(x) - \psi(x) > 0.$ Let $N = \max[\phi(x) - \psi(x)] > 0, x \in [c, d],$ and let $x_0 \in (c, d)$ be such that $\phi(x_0) - \psi(x_0) = N.$ There exist $x_1, x_2 \in (c, d)$ such that $\phi(x_1) - \psi(x_1) \leq N/2, \phi(x_2) - \psi(x_2) \leq N/2$ and $x_1 < x_0 < x_2.$ Choose ε such that $0 < \varepsilon \leq N/2.$ By Lemma 2.2*, there exists a function $\psi_1(x) \in C'[x_1, x_2]$ with

$$\psi(x) \leq \psi_1(x) \leq \psi(x) + \varepsilon \quad \text{and} \quad \bar{D}\psi_1'(x) < f(x, \psi_1(x), \psi_1'(x)) \quad \text{on } (x_1, x_2).$$

Then $\phi(x) - \psi_1(x) < N/2$ on (x_1, x_2) by Lemma 2.1. In particular, $\phi(x_0) - \psi_1(x_0) < N/2$ so

$$\phi(x_0) < \psi_1(x_0) + N/2 \leq \psi(x_0) + \varepsilon + N/2 \leq \psi(x_0) + N$$

which is a contradiction.

Uniqueness of solutions of the BVP is immediate from the Maximum Principle.

COROLLARY 2.2. *Let $f(x, y, y')$ satisfy $A_0, A_1,$ and A_2 . Then the solution of the BVP: $y'' = f(x, y, y'), y(x_1) = y_1, y(x_2) = y_2,$ for $[x_1, x_2] \subset [a, b],$ if it exists, is unique.*

The following example shows that the Maximum Principle does not remain valid if we replaced A_2 by the following weaker assumption:

(A_2^*) $f(x, y, y')$ satisfies a Hölder condition with exponent $\alpha, 0 < \alpha < 1,$ with respect to y' on each fixed compact subset of $T.$ Consider the BVP: $y'' = f(x, y, y') = A^{1-\alpha} |a|^{p(\alpha-1)} p^{1-\alpha} (p-1) |y'|^\alpha$ with $y(-a) = y(a) = A > 0$ where $\alpha = (p-2)(p-1)^{-1}, p > 2.$ $f(x, y, y')$ satisfies $A_0, A_1,$ and $A_2^*,$ but $y_1(x) = A$ and $y_2(x) = A|a|^{-p}|x|^p$ are distinct solutions of the BVP.

3. Subfunctions. The concept of sub- and superfunctions and their properties are fundamental for the remainder of this paper. For this reason some of the known results for subfunctions due to Fountain and Jackson [3] will be stated without proof. In [3] these results were proven assuming $f(x, y, y')$ satisfies $A_0, A_1,$ and $A_2^*.$ If $f(x, y, y')$ satisfies $A_0, A_1,$ and $A_2,$ the proofs can be carried through as in [3] with only slight modifications.

Throughout this section, we shall assume that $f(x, y, y')$ satisfies A_0 although no further explicit mention of this fact will be made. I will denote an interval of $[a, b], cl(I)$ the closure of $I,$ and I° the interior of $I.$

DEFINITION 3.1. A real-valued function s defined on I is said to be subfunction on I in case $s(x) \leq y(x)$ on $[x_1, x_2]$ for any $[x_1, x_2] \subset I$ and any solution y of (1.1) on $[x_1, x_2]$ with $s(x_1) \leq y(x_1)$ and $s(x_2) \leq y(x_2).$

Superfunctions are defined dually by reversing inequalities in the preceding definition, and dual results hold.

Subfunctions satisfy the following properties.

(3.1). If $s(x)$ is a subfunction on $I,$ then the right- and left-hand limits exist in the extended reals with appropriate limits existing at the endpoints of $cl(I), s(x_0) \leq \max [s(x_0 - 0), s(x_0 + 0)]$ for $x_0 \in I^\circ,$ and the number of discontinuities of $s(x)$ on I is at most countable.

(3.2). If $s(x)$ is a bounded subfunction on I with $cl(I)$ having

endpoints x_1 and x_2 , then, if

$$d^+s(x_0) = \limsup [s(x) - s(x_0 + 0)]/(x - x_0)$$

and

$$d_+s(x_0) = \liminf [s(x) - s(x_0 + 0)]/(x - x_0) \quad \text{as } x \rightarrow x_0^+$$

with $d^-s(x_0)$ and $d_-s(x_0)$ analogously defined, $d^-s(x_0) = d_-s(x_0)$ for all $x_1 < x_0 \leq x_2$ and $d^+s(x_0) = d_+s(x_0)$ for all $x_1 \leq x_0 < x_2$ and, hence, $s(x)$ has a finite derivative almost everywhere on I .

(3.3). The supremum of a collection of subfunctions bounded above at each point of I is a subfunction on I .

(3.4). If $s(x)$ is a subfunction of class C' on I , then $\underline{D}s'(x) \geq f(x, s(x), s'(x))$ on I° .

(3.5). If $f(x, y, y')$ satisfies A_1 and A_2 , if $s(x) \in cl(I) \cap C'(I^\circ)$, and if $\underline{D}s'(x) \geq f(x, s(x), s'(x))$ on I° , then $s(x)$ is a subfunction on I .

(3.6). Let s_1 be a subfunction on I and s_2 a subfunction on $[x_1, x_2] \subset cl(I)$. Assume further that $s_2(x_i) \leq s_1(x_i)$, $i = 1, 2$, in case $x_i \in I^\circ$. Then

$$s(x) = \begin{cases} s_1(x), & x \notin [x_1, x_2] \\ \max [s_1(x), s_2(x)], & x \in [x_1, x_2] \end{cases}$$

is a subfunction on I .

(3.7). If $f(x, y, y')$ satisfies A_1 and A_2 , if $s(x)$ and $S(x)$ are continuous sub- and superfunctions on $cl(I)$, if one of $s(x)$ and $S(x)$ is of class C' on I° , and if $s(x_i) \leq S(x_i)$, $i = 1, 2$, where x_i are the endpoints of I , then $s(x) \leq S(x)$ on $cl(I)$.

(3.8). If $f(x, y, y')$ satisfies A_1 and A_2 and if $s(x)$ is a continuous subfunction on I , then $s(x) - M$ is a continuous subfunction on I where M is an arbitrary nonnegative constant.

4. **A generalized solution of the boundary value problem.** A function $H(x)$ will be defined which will be referred to as the "generalized solution" of the BVP in the sense that if a solution of the BVP exists and if $f(x, y, y')$ satisfies certain to-be-determined conditions, then $H(x)$ is that solution.

DEFINITION 4.1. A function $\phi(x)$ is said to be an underfunction

with respect to the BVP (1.2) in case $\phi(x)$ is a subfunction on $[a, b]$ with $\phi(a) \leq \alpha$ and $\phi(b) \leq \beta$.

Overfunctions are defined dually. For the construction of the "generalized solution," the following assumptions will be needed. Assume

(A₃) $f(x, y, y')$ is such that with respect to the given BVP there is an underfunction which is continuous on $[a, b]$ and there is an overfunction which is continuous on $[a, b]$ and of class C' on (a, b) .

DEFINITION 4.2. Let $\{\phi\}$ be the family of all underfunctions with respect to the BVP (1.2) which are continuous on $[a, b]$. Define

$$H(x) = \sup [\phi(x) \mid \phi \in \{\phi\}]$$

for each $x \in [a, b]$. $H(x)$ is called the generalized solution of the BVP.

$H(x)$ satisfies the following properties the proofs of which are given in [3], assuming $f(x, y, y')$ satisfies A₀, A₁, A₁' and A₃. If $f(x, y, y')$ satisfies A₀, A₁, A₂, and A₃, the proofs of these results carry through with some modification. Assuming that $f(x, y, y')$ satisfies A₀, A₁, A₂, and A₃, then:

(4.1). $H(x)$ is both a bounded sub- and superfunction on $[a, b]$ with $H(x) = \min [H(x + 0), H(x - 0)]$, $x \in (a, b)$.

(4.2). $H(x)$ is a solution of $y'' = f(x, y, y')$ on an open subset of $[a, b]$, the complement of which is of measure 0.

(4.3). $DH(x_0 - 0) = DH(x_0 + 0)$ on (a, b) where

$$DH(x_0 \pm 0) = \lim [H(x) - H(x_0 \pm 0)]/(x - x_0) \text{ as } x - x_0 \pm 0.$$

More specifically, let E be the set of interior points of $[a, b]$ at which $H(x)$ does not have a finite derivative, then (i) if $H(x + 0) > H(x - 0)$, $DH(x + 0) = DH(x - 0) = +\infty$, (ii) if $H(x + 0) < H(x - 0)$, $DH(x + 0) = DH(x - 0) = -\infty$; (iii) if $x \in E$ is a point of discontinuity, $DH(x + 0) = DH(x - 0) = \pm\infty$.

(4.4) (i) if $DH(b - 0) \neq -\infty$, $H(b - 0) = H(b)$, (ii) if $H(b - 0) < \beta$, $DH(b - 0) = +\infty$, and (iii) if $DH(b - 0)$ is finite, $H(b - 0) = H(b) = \beta$. Similar results hold at $x = a$.

Consider the BVP: $y'' = f(x, y, y') = -18x(y')^4$ with $y(-1) = -1$ and $y(1) = 1$. $f(x, y, y')$ satisfies A₀, A₁, A₂, and A₃. However, no solution of this BVP exists for if it did the solution would be $y_0(x) = x^{1/3}$. It can be shown that $H(x) = y_0(x)$ on $[-1, 1]$. Although the

generalized solution exists, this example points out the fact that more stringent conditions must be imposed on $f(x, y, y')$ to guarantee that a solution of the BVP exists.

DEFINITION 4.3. A differential equation $y'' = f(x, y, y')$ is said to satisfy property (P) in case given any $A > 0$ and any compact subset $S \subset \{(x, y) \mid a \leq x \leq b, |y| < \infty\}$, there exists $B_{(A, S)} > 0$ such that for any solution $y(x)$ of $y'' = f(x, y, y')$ with the initial condition $y(x_0) = y_0$ and $|y'(x_0)| \leq A$ where $(x_0, y_0) \in S$ the inequality $|y'(x)| < B_{(A, S)}$ holds as long as $(x, y(x)) \in S$.

THEOREM 4.1. Let $f(x, y, y')$ satisfy $A_i, i = 0, \dots, 3$, and let (1.1) satisfy property (P), then $H(x)$ is a solution of the BVP (1.2) of class C^2 on $[a, b]$.

The proof follows from (4.2), (4.3), and (4.4).

5. "Natural" conditions for $f(x, y, y')$. By Theorem 4.1, a solution of the BVP (1.2) exists provided assumptions $A_i, i = 0, \dots, 3$, and property (P) are satisfied. Of these, assumption A_3 and property (P) are unnatural in the sense that they are not imposed directly on $f(x, y, y')$. "Natural" conditions will be given in this section which imply A_3 and P.

Sufficient conditions to assure that $y'' = f(x, y, y')$ satisfies property (P) are given in the following theorem which is similar to a result due to Nagumo [5, pp. 861-863].

THEOREM 5.1. Let

(1) $f(x, y, y')$ satisfy A_0

(2) $\phi(u)$ be a positive continuous function of $u(u \geq 0)$ such that

$$(5.1) \quad |f(x, y, y') - f(x, y, 0)| \leq K_s \phi(|y'|)$$

where K_s is a constant depending on compact subsets

$$S \subset \{(x, y) \mid a \leq x \leq b, |y| < \infty\}, \quad (x, y) \in S, \quad |y'| < \infty,$$

and

$$(5.2) \quad \int_0^\infty \frac{u du}{\phi(u) + 1} = +\infty,$$

then $y'' = f(x, y, y')$ satisfies property (P).

Proof. Let S be any compact subset of $\{(x, y) \mid a \leq x \leq b, |y| < \infty\}$, then for any $(x_1, y_1), (x_2, y_2) \in S$ there is an $L > 0$ such that $|y_1 - y_2| < L$.

By (5.2), given any $A > 0$ there exists $B_{(A,S)} > A$ such that

$$(5.3) \quad \int_A^{B_{(A,S)}} \frac{u du}{\phi(u) + 1} > L \cdot K$$

where $K = \max [K_s, \max_{(x,y) \in S} |f(x,y,0)|]$.

Let $y(x)$ be an arbitrary solution of $y'' = f(x,y,y')$ with initial conditions $y(x_0) = y_0, y'(x_0) = A$ where $(x_0, y_0) \in S$. Let $x_1 \leq x \leq x_2$ be an interval such that $(x, y(x)) \in S$ and $x_1 \leq x_0 \leq x_2$. Claim $y'(x) < B_{(A,S)}$ on $[x_1, x_2]$. If not, there would be two points t_1 and t_2 in $[x_1, x_2]$ such that $y'(t_1) = A, y'(t_2) = B_{(A,S)}$, and $A \leq y'(x) \leq B_{(A,S)}$ for $x \in [t_1, t_2]$ assuming without loss of generality that $t_1 < t_2$.

By (5.1), $f(x,y,y') \leq K(\phi(|y'|) + 1)$ and thus

$$\frac{y'y''}{\phi(|y'|) + 1} \leq Ky'.$$

Therefore,

$$\int_A^{B_{(A,S)}} \frac{u du}{\phi(u) + 1} = \int_{t_1}^{t_2} \frac{y'y'' dx}{\phi(|y'|) + 1} \leq K \int_{t_1}^{t_2} y' dx \leq K \cdot L$$

which contradicts (5.3), and hence $y'(x) < B_{(A,S)}$ on $[x_1, x_2]$. Analogously, one can show that $y'(x) > -B_{(A,S)}$ on $[x_1, x_2]$ whenever $y'(x_0) \geq -A$.

The following theorem imposes sufficient conditions which imply A_3 and, in fact, more.

THEOREM 5.2. *Let $f(x,y,y')$ satisfy $A_i, i = 0, 1, 2$, and*

$$(5.4) \quad |f(x, 0, y') - f(x, 0, 0)| \leq K|y'|$$

for all $a \leq x \leq b$ and $|y'| < +\infty$, then there exists an overfunction $\psi(x)$ of class C^2 on $[a, b]$ and an underfunction $\phi(x)$ of class C^2 on $[a, b]$.

Proof. Let $R > 0$ be such that $|f(x, 0, 0)| \leq R$ on $[a, b]$. Define

$$\psi(x) \equiv \varepsilon_1(N - e^{m(x-a)}) \text{ on } [a, b]$$

where $m = K + 1, \varepsilon_1 = Rm^{-1} + 1$, and

$$N = \max [\varepsilon_1^{-1} \cdot \alpha + 1, \varepsilon_1^{-1} \beta + e^{m(b-a)}, e^{m(b-a)}].$$

Using (3.5), it is easily verified that $\psi(x)$ is an overfunction of class $C^2[a, b]$.

Similarly, $\phi(x) = \varepsilon_2(N - e^{m(x-a)})$ where $\varepsilon_2 = -(Rm^{-1} + 1)$ and $N = \max [\alpha \varepsilon_2^{-1} + 1, \beta \varepsilon_2^{-1} + e^{m(b-a)}, e^{m(b-a)}]$ is an underfunction of class $C^2[a, b]$.

The following slightly stronger result can be proved: If $f(x,y,y')$ satisfies A_0, A_1, A_2 , and if

$$|f(x, 0, y') - f(x, 0, 0)| \leq K[|y'| \cdot \ln |y'| + 1]$$

where $x \in [a, b]$ and $|y'| < \infty$, then there exists an underfunction ϕ and an overfunction ψ of class C^2 on $[a, b]$ with respect to the BVP (1.2).

The proof of this result follows by observing that any solution of $y'' = K_2 y' \ln y'$ where $y(x) \geq 0$ and $y'(x) \geq e$ on $[a, b]$ is a superfunction with respect to (1.1) provided $K_2 \leq -(2K + M)$ where $M = \max |f(x, 0, 0)|$.

In general, one cannot replace (5.4) by the weaker condition:

$$(5.5) \quad |f(x, 0, y') - f(x, 0, 0)| \leq K|y'|^{1+v}$$

where

$$x \in [a, b], \quad |y'| < \infty, \quad \text{and } v > 0;$$

for consider the BVP:

$$\begin{aligned} y'' &= 1 + (y')^3 \\ y(0) &= 0 \quad y(\pi/2) = \beta \neq 0. \end{aligned}$$

There exists no continuous underfunction on $[0, \pi/2]$. However, if the length of the interval is suitably restricted, the following result holds.

THEOREM 5.3. *Let $f(x, y, y')$ satisfy A_i , $i = 0, 1, 2$, and (5.5), then, if $b - a < (vK_1)^{-1}$, there exists an overfunction $\psi(x)$ and an underfunction $\phi(x)$ of class $C^2[a, b]$ with respect to the BVP (1.2).*

Proof. Observe that any solution of the differential equation

$$(5.6) \quad y'' = -K_1(y')^{1+v}$$

where $K_1 = K + \max |f(x, 0, 0)|$, $x \in [a, b]$, satisfying the conditions $y(x) \geq 0$ and $y'(x) \geq 1$ on $[a, b]$ is a superfunction on $[a, b]$ with respect to (1.1).

If $v \neq 1$, the function $\psi(x)$ defined by

$$\psi(x) = [K_1(v - 1)]^{-1} \cdot ([vK_1(x - b) + 1]^{(v-1)/v} - [vK_1(a - b) + 1]^{(v-1)/v}) + \eta$$

where $\eta = \max [\alpha, \beta + [K_1(v - 1)]^{-1} \cdot ([vK_1(a - b) + 1]^{(v-1)/v} - 1), 0]$ is a solution of (5.6) and, if $b - a < (vK_1)^{-1}$, then $\psi(x) \geq 0$ and $\psi'(x) \geq 1$ on $[a, b]$ with $\psi(a) \geq \alpha$ and $\psi(b) \geq \beta$. Thus, $\psi(x)$ is an overfunction of class $C^2[a, b]$ with respect to the BVP (1.2) for $v \neq 1$.

If $v = 1$, the function

$$\psi(x) = K_1^{-1} \ln |K_1(x - b) + 1| + \eta$$

where

$$\eta = \max [0, \alpha - K_1^{-1} \cdot \ln |K(a - b) + 1|, \beta]$$

is an overfunction of class $C^2[a, b]$, provided $b - a < (vK_1)^{-1}$.

An underfunction $\phi(x)$ can similarly be constructed.

6. Some existence theorems. Having found sufficient conditions which when imposed upon $f(x, y, y')$ imply assumptions A_3 and P, we proceed to state and prove some existence theorems for the BVP (1.2).

THEOREM 6.1. *If $f(x, y, y')$ satisfies $A_i, i = 0, 1, 2$, and if*

$$(6.1) \quad |f(x, 0, y') - f(x, 0, 0)| \leq K |y'| \quad \text{where } x \in [a, b] \text{ and } |y'| < \infty,$$

then $H(x)$, the generalized solution of the BVP (1.2), has a finite derivative at each point of (a, b) . In fact, $|H'(x_0)| \leq M$ where the constant M depends on $x_0, a, b, \alpha, \beta, K$, and $R = \max |f(x, 0, 0)|$ on $[a, b]$.

Proof. $H(x)$ exists on $[a, b]$. Let $x_0 \in (a, b)$ and consider the two possible cases: (i) $H(x_0) \leq 0$ and (ii) $H(x_0) > 0$.

(i) Let $H(x_0) \leq 0$. By the known properties of $H(x)$, $H(x_0 - 0)$ and $H(x_0 + 0)$ exist and $H(x_0) = \min [H(x_0 - 0), H(x_0 + 0)]$. Assume without loss of generality that $H(x_0 - 0) = H(x_0)$. Any solution of

$$(6.2) \quad y'' = K_1 y'$$

where $K_1 = K + R$ with $y(x) \leq 0$ and $y'(x) \geq 1$ is a subfunction by (3.5). Let

$$A = \min [\phi(a), H(x_0) - (1 - \exp [K_1(a - x_0)])(K_1 \exp [K_1(a - x_0)])^{-1}],$$

then the function $y_1(x)$ defined by

$$y_1(x) = \frac{H(x_0)}{1 - \exp [K_1(a - x_0)]} [\exp [K_1(x - x_0)] - \exp [K_1(a - x_0)]] \\ + \frac{A}{1 - \exp [K_1(a - x_0)]} [1 - \exp [K_1(x - x_0)]]$$

satisfies differential equation (6.2) with $y_1(a) = A$, $y_1(x_0) = H(x_0)$, $y_1(x) \leq 0$, and $y_1'(x) \geq 1$.

Since $y_1(x)$ is a subfunction of class C^2 on $[a, x_0]$ with $H(a) \geq y_1(a)$ and $H(x_0) = y_1(x_0)$, we assert that $y_1(x) \leq H(x)$ on $[a, x_0]$. Assume not, then there exists an $x_1 \in (a, x_0)$ such that $y_1(x_1) > H(x_1)$. Let $\varepsilon = y_1(x_1) - H(x_1) > 0$, then $y_1(x) - \varepsilon/2$ is a subfunction on $[a, x_0]$ by (3.8). By the definition of $H(x)$, there are continuous underfunctions ϕ_1 and ϕ_2 such that $H(a) - \phi_1(a) \leq \varepsilon/4$ and $H(x_0) - \phi_2(x_0) \leq \varepsilon/4$.

Define $\phi_3(x) = \max [\phi_1(x), \phi_2(x)]$ for $x \in [a, b]$, then $\phi_3(x)$ is a continu-

ous underfunction on $[a, b]$ with $H(a) - \phi_3(a) \leq \varepsilon/4$ and $H(x_0) - \phi_3(x_0) \leq \varepsilon/4$. Define ϕ^* on $[a, b]$ by:

$$\phi^*(x) = \begin{cases} \phi_3(x), & x \notin [a, x_0] \\ \max [\phi_3(x), y_1(x) - \varepsilon/2], & x \in [a, x_0]. \end{cases}$$

$\phi^*(x)$ is a continuous underfunction by (3.6), but $\phi^*(x_1) \geq y_1(x_1) - \varepsilon/2 = H(x_1) + \varepsilon/2$ which contradicts the definition of $H(x)$. Therefore, $y_1(x) \leq H(x)$ for all $x \in [a, x_0]$.

Hence,

$$\frac{H(x_0 - 0) - H(x)}{x_0 - x} \leq \frac{y_1(x_0) - y_1(x)}{x_0 - x} \quad \text{for } a \leq x < x_0$$

and

$$\begin{aligned} DH(x_0 - 0) &\leq y'_1(x_0) \leq K_1[1 - \exp [K_1(a - x_0)]]^{-1} \cdot [H(x_0) - A] \\ &\quad \cdot \exp [K_1 \max [b - x_0, x_0 - a]] \\ &= M_1(a, b, \alpha, \beta, x_0, K, R). \end{aligned}$$

$DH(x_0 - 0) < M_1 < +\infty$ implies, by (4.3), that $H(x_0 - 0) \geq H(x_0 + 0)$. Hence $H(x_0 + 0) = H(x_0 - 0) = H(x_0)$ and $H'(x_0) \leq M_1$.

By a similar argument applied to the right-hand side of x_0 , there exists a constant M_2 depending on $a, b, \alpha, \beta, x_0, K$, and R such that $M_2 \leq H'(x_0)$.

Therefore,

$$M_2 \leq H'(x_0) \leq M_1.$$

(ii) If $H(x_0) > 0$, proof is similar to case (i) and will be omitted.

COROLLARY 6.1. *Under the hypotheses of Theorem 6.1, $H(x)$ is of class C^2 and a solution of (1.1) on (a, b) .*

This follows from (4.1), (4.2), (4.3), and Theorem 6.1.

COROLLARY 6.2. *If*

- (1) $f(x, y, y')$ satisfies A_0, A_1 , and A_2 ,
 - (2) $f(x, y, y') = g(x, y, y') + h(x, y, y')$ on T ,
 - (3) $h(x, 0, y') = 0$ for $x \in [a, b]$ and $|y'| < \infty$,
 - (4) $|g(x, 0, y') - g(x, 0, 0)| \leq K|y'|$ for $x \in [a, b]$ and $|y'| < \infty$,
- then $H(x)$ is of class C^2 and a solution of (1.1) on (a, b) .

Our primary result is the following theorem.

THEOREM 6.2. *If*

- (1) $f(x, y, y')$ satisfies A_0, A_1 , and A_2 ,

(2) *there exists a positive continuous function $\phi(u)$ defined for $u \geq 0$ such that*

$$|f(x, y, y') - f(x, y, 0)| \leq K_s \phi(|y'|)$$

where K_s is a constant depending on compact subsets S of

$$\{(x, y) \mid x \in [a, b], |y| < \infty\}, \quad (x, y) \in S, \quad |y'| < \infty,$$

and

$$\int_0^\infty \frac{udu}{\phi(u) + 1} = +\infty,$$

(3) $|f(x, 0, y') - f(x, 0, 0)| \leq K|y'|$ for $x \in [a, b]$ and $|y'| < \infty$, then $H(x)$ is the solution of the BVP (1.2) of class $C^2[a, b]$.

Proof. By Theorems 4.1, 5.1, and 5.2, $H(x)$ is a solution of the BVP (1.2) in the sense defined. By standard arguments, $H(x)$ is in fact of class $C^2[a, b]$.

COROLLARY 6.3. *If*

(1) $f(x, y, y')$ satisfies A_0 and A_1 ,

(2) $|f(x, y, y'_1) - f(x, y, y'_2)| \leq K_s |y'_1 - y'_2|$ where K_s is a constant depending on compact subsets S of $\{(x, y) \mid x \in [a, b], |y| < \infty\}$, $(x, y) \in S$, and $|y'| < \infty$,

then $H(x)$ is the solution of the BVP (1.2) of class $C^2[a, b]$.

COROLLARY 6.4. *If*

(1) $f(x, y, y')$ satisfies A_0 and A_1 , and

(2) $|f(x, y, y'_1) - f(x, y, y'_2)| \leq K|y'_1 - y'_2|$ for all $(x, y, y'_1), (x, y, y'_2) \in T$, then $H(x)$ is the solution of the BVP (1.2) of class $C^2[a, b]$.

The above corollary was proven by Lees [4] using the method of finite differences.

If the Nagumo condition (Assumption 2 of Theorem 6.2) is dropped, it is still possible to assert the existence of the solution of BVP (1.2) provided some rather severe limitations are placed on the boundary values.

THEOREM 6.3. *If given the BVP: $y'' = f(x, y, y')$, $y(a) = 0 = y(b)$, with*

(1) $f(x, y, y')$ satisfying A_0, A_1, A_2 , and

(2) $|f(x, 0, y') - f(x, 0, 0)| \leq K|y'|$ for all $x \in [a, b]$ and $|y'| < \infty$, then $H(x)$ is the solution of the BVP (1.2).

Proof. By Corollary 6.1, $H(x)$ is of class C^2 and a solution of (1.1) on (a, b) . Thus, it suffices to show, that under the additional stipulation of zero boundary values, $H(x)$ is continuous on $[a, b]$ and assumes the zero boundary values. This is accomplished by constructing an overfunction ψ and an underfunction ϕ both of which are continuous on $[a, b]$ and such that $\phi(a) = \psi(a) = 0$ and $\phi(b) = \psi(b) = 0$.

Let $\phi_1(x)$ be a solution of

$$(6.3) \quad y'' = K_1 y'$$

where $K_1 = K + \max |f(x, 0, 0)|$ for $x \in [a, b]$ satisfying the condition $\phi_1(x) \leq 0$, $\phi_1'(x) \geq 1$, and $\phi_1(b) = 0$, and let $\phi_2(x)$ be a solution of

$$(6.4) \quad y'' = -K_1 y'$$

satisfying the conditions $\phi_2(x) \leq 0$, $\phi_2'(x) \leq -1$, and $\phi_2(a) = 0$, then $\phi_1(x)$ and $\phi_2(x)$ are subfunctions with respect to (1.1) on $[a, b]$. Let $\phi(x) = \max [\phi_1(x), \phi_2(x)]$, then $\phi(x)$ is a continuous underfunction such that $\phi(a) = 0 = \phi(b)$.

Similarly, there exists a continuous overfunction $\psi(x)$ such that $\psi(a) = 0 = \psi(b)$.

Thus, $\phi(x) \leq H(x) \leq \psi(x)$ on $[a, b]$ and result follows.

COROLLARY 6.5. *If*

(1) $g(x, y, y')$ satisfies $A_0, A_1,$ and $A_2,$

(2) $g(x, 0, y') = 0$ for all $a \leq x \leq b$ and $|y'| < \infty,$

(3) $f(x, y, y') = h_1(x) + h_2(x)y' + h_3(y) + g(x, y, y')$ where $h_i, i = 1, 2, 3,$ are continuous on their respective domains and h_3 is non-decreasing,

then there exists a solution of the BVP: (1.1) with $y(a) = 0 = y(b)$.

REFERENCES

1. E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
2. N. Dunford and J. Schwartz, *Linear Operators, Part I: General Theory*, Interscience, 1958.
3. L. Fountain and L. Jackson, *A Generalized Solution for the Boundary Value Problem for $y'' = f(x, y, y')$* , Pacific J. Math., **12** (1962), 1251-1272.
4. M. Lees, *A Boundary Value Problem for Nonlinear Ordinary Differential Equations*, J. Math., Mech., **10** (1961), 423-430.
5. M. Nagumo, *Über die Differentialgleichung $y'' = f(x, y, y')$* , Proc. Physics-Math. Soc. Japan, Series 3, **19** (1937), 861-866.
6. B. Picard, *Leçons sur Quelques Problèmes aux Limites de la Théorie des Equations Différentielles*, Gauthier-Villars, Paris, 1930.

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

RALPH S. PHILLIPS

Stanford University
Stanford, California

M. G. ARSOVE

University of Washington
Seattle 5, Washington

J. DUGUNDJI

University of Southern California
Los Angeles 7, California

LOWELL J. PAIGE

University of California
Los Angeles 24, California

ASSOCIATE EDITORS

E. F. BECKENBACH
T. M. CHERRY

D. DERRY
M. OHTSUKA

H. L. ROYDEN
E. SPANIER

E. G. STRAUS
F. WOLF

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CALIFORNIA RESEARCH CORPORATION
SPACE TECHNOLOGY LABORATORIES
NAVAL ORDNANCE TEST STATION

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced), and the author should keep a complete copy. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is \$18.00; single issues, \$5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$8.00 per volume; single issues \$2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

| | |
|--|------|
| Dallas O. Banks, <i>Bounds for eigenvalues and generalized convexity</i> | 1031 |
| Jerrold William Bebernes, <i>A subfunction approach to a boundary value problem for ordinary differential equations</i> | 1053 |
| Woodrow Wilson Bledsoe and A. P. Morse, <i>A topological measure construction</i> | 1067 |
| George Clements, <i>Entropies of several sets of real valued functions</i> | 1085 |
| Sandra Barkdull Cleveland, <i>Homomorphisms of non-commutative *-algebras</i> | 1097 |
| William John Andrew Culmer and William Ashton Harris, <i>Convergent solutions of ordinary linear homogeneous difference equations</i> | 1111 |
| Ralph DeMarr, <i>Common fixed points for commuting contraction mappings</i> | 1139 |
| James Robert Dorroh, <i>Integral equations in normed abelian groups</i> | 1143 |
| Adriano Mario Garsia, <i>Entropy and singularity of infinite convolutions</i> | 1159 |
| J. J. Gergen, Francis G. Dressel and Wilbur Hallan Purcell, Jr., <i>Convergence of extended Bernstein polynomials in the complex plane</i> | 1171 |
| Irving Leonard Glicksberg, <i>A remark on analyticity of function algebras</i> | 1181 |
| Charles John August Halberg, Jr., <i>Semigroups of matrices defining linked operators with different spectra</i> | 1187 |
| Philip Hartman and Nelson Onuchic, <i>On the asymptotic integration of ordinary differential equations</i> | 1193 |
| Isidore Heller, <i>On a class of equivalent systems of linear inequalities</i> | 1209 |
| Joseph Hersch, <i>The method of interior parallels applied to polygonal or multiply connected membranes</i> | 1229 |
| Hans F. Weinberger, <i>An effectless cutting of a vibrating membrane</i> | 1239 |
| Melvin F. Janowitz, <i>Quantifiers and orthomodular lattices</i> | 1241 |
| Samuel Karlin and Albert Boris J. Novikoff, <i>Generalized convex inequalities</i> | 1251 |
| Tilla Weinstein, <i>Another conformal structure on immersed surfaces of negative curvature</i> | 1281 |
| Gregers Louis Krabbe, <i>Spectral permanence of scalar operators</i> | 1289 |
| Shige Toshi Kuroda, <i>Finite-dimensional perturbation and a representation of scattering operator</i> | 1305 |
| Marvin David Marcus and Afton Herbert Cayford, <i>Equality in certain inequalities</i> | 1319 |
| Joseph Martin, <i>A note on uncountably many disks</i> | 1331 |
| Eugene Kay McLachlan, <i>Extremal elements of the convex cone of semi-norms</i> | 1335 |
| John W. Moon, <i>An extension of Landau's theorem on tournaments</i> | 1343 |
| Louis Joel Mordell, <i>On the integer solutions of $y(y + 1) = x(x + 1)(x + 2)$</i> | 1347 |
| Kenneth Roy Mount, <i>Some remarks on Fitting's invariants</i> | 1353 |
| Miroslav Novotný, <i>Über Abbildungen von Mengen</i> | 1359 |
| Robert Dean Ryan, <i>Conjugate functions in Orlicz spaces</i> | 1371 |
| John Vincent Ryff, <i>On the representation of doubly stochastic operators</i> | 1379 |
| Donald Ray Sherbert, <i>Banach algebras of Lipschitz functions</i> | 1387 |
| James McLean Sloss, <i>Reflection of biharmonic functions across analytic boundary conditions with examples</i> | 1401 |
| L. Bruce Treybig, <i>Concerning homogeneity in totally ordered, connected topological space</i> | 1417 |
| John Wermer, <i>The space of real parts of a function algebra</i> | 1423 |
| James Juei-Chin Yeh, <i>Orthogonal developments of functionals and related theorems in the Wiener space of functions of two variables</i> | 1427 |
| William P. Ziemer, <i>On the compactness of integral classes</i> | 1437 |