Pacific Journal of Mathematics

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Vol. 13, No. 4 June 1963

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Kakutani [1] and Markov [2] have shown that if a commutative family of continuous linear transformations of a linear topological space into itself leaves some nonempty compact convex subset invariant, then the family has a common fixed point in this invariant subset. The question naturally arises as to whether this is true if one considers a commutative family of continuous (not necessarily linear) transformations. We shall show that it is true in a rather special, but non-trivial, case, thus giving some hope that further investigation of the general question will yield positive results. The main result of this paper is the following.

THEOREM. *Let B be a Banach space and let X be a nonempty compact convex subset of B. If $\mathscr{F}$ is a nonempty commutative family of contraction mappings of X into itself, then the family $\mathscr{F}$ has a common fixed point in X.*

Note 1. A mapping $f: X \to X$ is said to be a contraction mapping if $||f(x) - f(y)|| \leqq ||x - y||$ for all $x, y \in X$.

Note 2. If the norm for B is strictly convex, then the above theorem is almost trivial since in this case each contraction mapping has a fixed-point set which is nonempty, compact, and convex. In the general case, however, the fixed-point set of a contraction mapping is not convex. An example illustrating this fact is constructed as follows. Let B be the space of all ordered pairs (a, b) of real numbers, where if $x = (a, b)$, then $||x|| = \max\{|a|, |b|\}$. Define $X = \{x: ||x|| \leqq 1\}$ and $f: X \to X$ as follows: if $x = (a, b)$, then $f(x) = (|b|, b)$. It is easily shown that f is a contraction mapping and that $x = (1, 1)$ and $y = (1, -1)$ are fixed points for f. However, $1/2(x + y) = (1, 0)$ is not a fixed point for f.

In the proof of the theorem we shall make use of the following two lemmas.

LEMMA 1. *Let B be a Banach space and let M be a nonempty compact subset of B and let K be the closed convex hull of M. Let ρ be the diameter of M. If $\rho > 0$, then there exists an element $u \in K$ such that*

Received November 14, 1962.

$$\sup \{\|x - u\| : x \in M\} < \rho .$$

Proof. Since M is nonempty and compact, we may find $x_0, x_1 \in M$ such that $\|x_0 - x_1\| = \rho$. Let $M_0 \subset M$ be maximal so that $M_0 \supset \{x_0, x_1\}$ and $\|x - y\| = 0$ or ρ for all $x, y \in M_0$. Since M is compact and we are assuming $\rho > 0$, M_0 must be finite. Let us assume $M_0 = \{x_0, x_1, \cdots, x_n\}$. Now let us define

$$u = \sum_{k=0}^{n} \frac{1}{n+1} x_k \in K .$$

Since M is compact, we can find $y_0 \in M$ such that $\|y_0 - u\| = \sup \{\|x - u\| : x \in M\}$. Now

$$\|y_0 - u\| \leqq \sum_{k=0}^{n} \frac{1}{n+1} \|y_0 - x_k\| \leqq \rho$$

because $\|y_0 - x_k\| \leqq \rho$ for all $k = 0, 1, \cdots, n$. Therefore, if $\|y_0 - u\| = \rho$, then we must have $\|y_0 - x_k\| = \rho > 0$ for all $k = 0, 1, \cdots, n$, which means that $y_0 \in M_0$ by definition of M_0. But then we must have $y_0 = x_k$ for some $k = 0, 1, \cdots, n$, which is a contradiction. Therefore, $\|y_0 - u\| < \rho$.

LEMMA 2. *Let X_0 be a nonempty convex subset of a Banach space and let f be a contraction mapping of X_0 into itself. If there is a compact set $M \subset X_0$ such that $M = \{f(x): x \in M\}$ and M has at least two points, then there exists a nonempty closed convex set K_1 such that $f(x) \in K_1 \cap X_0$ for all $x \in K_1 \cap X_0$ and $M \cap K_1' \neq \phi$. (K_1' is the complement of K_1.)*

Proof. If we take K as the closed convex hull of M, then by Lemma 1 there exists an element $u \in K$ such that

$$\rho_1 = \sup \{\|x - u\| : x \in M\} < \rho ,$$

where ρ is the diameter of M. Since M has at least two points, we have $\rho > 0$, so that our use of Lemma 1 is valid.

For each $x \in M$ let us define $U(x) = \{y : \|y - x\| \leqq \rho_1\}$. Since $u \in U(x)$ for each $x \in M$, we have $K_1 = \bigcap_{x \in M} U(x) \neq \phi$. It is clear that K_1 is closed and convex. For any $x \in K_1 \cap X_0$ and any $z \in M$ we have $x \in U(z)$, i.e., $\|x - z\| \leqq \rho_1$. Since $M = \{f(y): y \in M\}$, there must exist $y \in M$ such that $z = f(y)$. Since f is a contraction mapping, we have

$$\|f(x) - z\| = \|f(x) - f(y)\| \leqq \|x - y\| \leqq \rho_1 ;$$

i.e., $f(x) \in U(z)$. Since this is true for any $z \in M$, we have $f(x) \in K_1 \cap X_0$. We have shown that $f(x) \in K_1 \cap X_0$ for all $x \in K_1 \cap X_0$.

Since M is compact, there exist $x_0, x_1 \in M$ such that $||x_0 - x_1|| = \rho > \rho_1$. Thus, we see that x_1 does not belong to $U(x_0) \supset K_1$, i.e., $x_1 \in M \cap K_1' \neq \phi$.

Proof of the theorem. One may show by using Zorn's lemma that there exists a minimal nonempty compact convex set $X_0 \subset X$ such that X_0 is invariant under each $f \in \mathscr{F}$. If X_0 consists of a single point, then the theorem is proved. We shall now show that if X_0 consists of more than one point, then we obtain a contradiction.

We may use Zorn's lemma again to show that there exists a minimal nonempty compact (but not necessarily convex) set $M \subset X_0$ such that M is invariant under each $f \in \mathscr{F}$. We will now show that $M = \{f(x): x \in M\}$ for each $f \in \mathscr{F}$. Since each $f \in \mathscr{F}$ is continuous and M is compact, $f(M)$ must also be compact. For all $f \in \mathscr{F}$ we have $f(M) \subset M$. Let us assume that for some $g \in \mathscr{F}$ we have $g(M) = N \neq M$. Now for any $x \in N$ there exists $y \in M$ such that $x = g(y)$. Since all functions in $\mathscr{F}$ commute, we have for all $f \in \mathscr{F}$ $f(x) = f(g(y)) = g(f(y)) \in N$ because $f(y) \in M$. Thus, we have $f(N) \subset N \subset M$ for all $f \in \mathscr{F}$. But since N is a nonempty compact subset of X_0 which is invariant under each $f \in \mathscr{F}$ and since $N \subset M$ and $N \neq M$, we have contradicted the minimality of M. Consequently, our assumption that $M \neq N$ is false. We may assume that M has at least two points; otherwise, the theorem is proved.

We may now apply Lemma 2 to each $f \in \mathscr{F}$. Referring to the notation of Lemma 2, we see that the set $K_1 \cap X_0$ is invariant under each $f \in \mathscr{F}$. Since K_1 is closed, we see that $K_1 \cap X_0$ is a nonempty compact convex subset of X_0. Since $X_0 \cap K_1' \supset M \cap K_1' \neq \phi$, we see that $K_1 \cap X_0 \neq X_0$. Thus, we see that if X_0 has more than one point, then we obtain a contradiction to the minimality of X_0.

REFERENCES

1. S. Kakutani, *Two fixed-point theorems concerning bicompact convex sets*, Proc. Imp. Acad Tokyo **14** (1938), 242-245.

2. A. Markov, *Quelques theoremes sur les ensembles Abeliens*, Doklady Akad. Nauk SSSR (N.S.) **10** (1936), 311-314.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 13, No. 4 June, 1963