

# Pacific Journal of Mathematics

**AN EFFECTLESS CUTTING OF A VIBRATING MEMBRANE**

HANS F. WEINBERGER

# AN EFFECTLESS CUTTING OF A VIBRATING MEMBRANE

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Let  $G$  be a multiply connected domain bounded by an outer boundary  $\Gamma_0$ , inner boundaries  $\Gamma_1, \Gamma_2, \dots$ , and possibly some other inner boundaries  $\gamma_1, \gamma_2, \dots$ . Let  $u$  be the eigenfunction corresponding to the lowest eigenvalue  $\lambda_1$  of the membrane problem

$$(1) \quad \Delta u + \lambda_1 u = 0 \quad \text{in } G$$

with

$$(2) \quad \begin{aligned} u &= 0 \quad \text{on } \Gamma_0, \Gamma_1, \dots \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \gamma_1, \gamma_2, \dots \end{aligned}$$

We shall show that there exists a cut  $\tilde{\gamma}$  consisting of a finite set of analytic arcs along which  $(\partial u / \partial n) = 0$  which separates any given one of the fixed holes, say  $\Gamma_1$ , from the outer boundary  $\Gamma_0$  and the other holes  $\Gamma_2, \Gamma_3, \dots$ . This means that the membrane  $G$  may be cut in two along  $\tilde{\gamma}$  without lowering its lowest eigenvalue. This fact is used in the preceding paper of J. Hersch to establish an upper bound for  $\lambda_1$ .

We assume that  $\Gamma_0, \Gamma_1, \dots$  have continuous normals and that  $\gamma_1, \gamma_2, \dots$  are analytic. Then it is well-known that  $u$  has the following properties:

- $$(3) \quad \begin{aligned} (a) \quad &u > 0 \text{ in } G, \text{ and } \frac{\partial u}{\partial n} < 0 \text{ on } \Gamma_0, \Gamma_1, \dots \\ (b) \quad &u \text{ is analytic in } G + \gamma_1 + \gamma_2 + \dots \\ (c) \quad &u_{xx} \text{ and } u_{yy} \text{ do not vanish simultaneously.} \end{aligned}$$

(The last property follows from (3a) and (1)).

We define  $G_1$  to be the set of points of  $G$  from which the fall lines, i.e. the trajectories of

$$(4) \quad \begin{aligned} \frac{dx}{dt} &= -u_x \\ \frac{dy}{dt} &= -u_y \end{aligned}$$

reach  $\Gamma_1$ . By property (3a)  $G_1$  contains a neighborhood in  $G$  of  $\Gamma_1$ , and its exterior contains neighborhoods in  $G$  of  $\Gamma_0, \Gamma_2, \dots$ . Since  $u_x$

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and  $u_y$  are continuous,  $G_1$  is open.

Let  $\tilde{\gamma}$  be the part of the boundary of  $G_1$  that lies in  $G$ . Let  $P$  be a point of  $\tilde{\gamma}$  where the gradient of  $u$  does not vanish. Then there is a trajectory  $\gamma$  satisfying (4) through  $P$ . Let  $Q$  be any other point on  $\gamma$ . Since  $P$  is not in  $G_1$ , it follows from the definition that  $Q$  is not in  $G_1$ . On the other hand, if a whole neighborhood of  $Q$  were not in  $G_1$ , it would follow from the continuity of the trajectories with respect to their initial points that a whole neighborhood of  $P$  would be outside  $G_1$ . This would contradict the fact that  $P$  is a boundary point of  $G_1$ .

Thus we have shown that the whole trajectory  $\gamma$  lies in  $\tilde{\gamma}$ . It cannot go to  $\Gamma_1$ . Since the set of points from which trajectories go to  $\Gamma_0, \Gamma_2, \dots$  is also open,  $\gamma$  cannot go to these boundary components.

We note that  $u$  is monotone on  $\gamma$ , and

$$(5) \quad \left| \frac{du}{ds} \right| = |\text{grad } u|.$$

Thus  $\gamma$  is either of finite length, or it must contain a sequence of points  $Q_1, Q_2, \dots$  on which  $\text{grad } u$  approaches zero. These will have a limit point  $Q$  at which  $\text{grad } u = 0$ . (It may be that  $Q$  lies on one of the  $\gamma_i$ . In this case we think of  $u$  extended across  $\gamma_i$  as an analytic function by reflection).

There is a neighborhood of  $Q$  in which the trajectories can be determined by examining the first few terms of the power series for  $u$ . Using property (3c), we find that  $\gamma$  is of finite length. This is, of course, true in both the  $t$  and  $-t$  directions.

The free boundary curves  $\gamma_i$  are composed of trajectories of (4) and critical points, i.e., points where  $\text{grad } u = 0$ . Hence it follows from the uniqueness of the initial value problem for (4) that if  $\gamma$  ends on  $\gamma_i$ , the end point must again be a critical point. Thus, each trajectory  $\gamma$  in  $\tilde{\gamma}$  connects two critical points.

It follows from properties (3b) and (3c) and the implicit function theorem that a critical point  $Q$  is either an isolated critical point or lies on an analytic arc of critical points. These arcs are again isolated.

Thus we have shown that  $\tilde{\gamma}$  is composed of a finite number of analytic arcs of finite length along which  $(\partial u / \partial n) = 0$ , and a finite number of critical points. We delete any isolated points of  $\tilde{\gamma}$ .

The fact that  $\tilde{\gamma}$  separates  $\Gamma_1$  from  $\Gamma_0, \Gamma_2, \dots$  is clear from the definition of  $G_1$ .

The above considerations apply to any function with properties (3).

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