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## **EQUALITY IN CERTAIN INEQUALITIES**

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**1. Introduction.** Let  $\sigma = (\sigma_1, \dots, \sigma_n)$  be a point on the unit  $(n - 1)$ -simplex  $S^{n-1}$ :  $\sum_{i=1}^n \sigma_i = 1, \sigma_i \geq 0$ . Let  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$  and  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n > 0$  be positive numbers and form the function on  $S^{n-1}$

$$(1.1) \quad F(\sigma) = \sum_{i=1}^n \sigma_i \lambda_i \sum_{i=1}^n \sigma_i \mu_i .$$

The main purpose of this paper is to examine the structure of the set of points  $\sigma \in S^{n-1}$  for which  $F(\sigma)$  takes on its maximum value. In case a convex monotone decreasing function  $f$  is fitted to the points  $(\lambda_i, \mu_i)$  (i.e.  $f(\lambda_i) = \mu_i$ ),  $i = 1, \dots, n$ , then it is not difficult to show that the maximum for  $F(\sigma)$  on  $S^{n-1}$  is the upper bound given by M. Newman [4] in a recent interesting paper. In the case of the Kantorovich inequality [1] the function  $f$  is  $f(t) = t^{-1}$ ,  $\mu_i = \lambda_i^{-1}$ ,  $i = 1, \dots, n$ . In this case a maximizing  $\sigma$  is  $\sigma_1 = 1/2, \sigma_n = 1/2, \sigma_i = 0$ ,  $i = 2, \dots, n - 1$ , and if  $\lambda_1 < \lambda_k < \lambda_n$ ,  $k = 2, \dots, n - 1$ , it is a corollary of our main result (Theorem 2) that this is the only choice possible for  $\sigma \in S^{n-1}$  in order to achieve the maximum value.

We shall assume henceforth in this paper that  $\mu_i = f(\lambda_i)$ ,  $i = 1, \dots, n$ , where  $f$  is a monotone decreasing convex function defined on the closed interval  $[\lambda_1, \lambda_n]$ . In 2 we determine the structure of the set of  $\sigma \in S^{n-1}$  for which  $F(\sigma)$  is a maximum in the case in which  $f$  is assumed to be strictly convex. In 3 we investigate the structure of the set of unit vectors  $x$  for which the function

$$(1.2) \quad \varphi(x) = (Ax, x)(f(A)x, x)$$

assumes its maximum value on the unit sphere  $\|x\| = 1$ . Throughout,  $A$  is a positive definite hermitian transformation on an  $n$ -dimensional unitary space  $U$  with inner product  $(x, y)$ . The eigenvalues of  $A$  are  $\lambda_i$ ,  $0 < \lambda_1 \leq \dots \leq \lambda_n$ , with corresponding orthonormal eigenvectors  $u_i$ ,  $Au_i = \lambda_i u_i$ ,  $i = 1, \dots, n$ . Of particular interest in (1.2) is the choice  $f(t) = t^{-p}$ ,  $p > 0$ .

Finally, in 4, we discuss the applications of the previous results to Grassmann compounds and induced power transformations associated with  $A$ . In two recent papers [2, 5] the Kantorovich inequality was applied to the compound to obtain inequalities involving principal subdeterminants of a positive definite hermitian matrix. We shall prove (Theorem 5) that these inequalities are in fact strict except in

trivial cases. Similar inequalities are obtained for the permanent function together with a discussion of the cases of equality. These inequalities are believed to be new.

2. **Maximum values for  $F$ .** In the rest of the paper  $M$  will systematically denote the maximum value of  $F(\sigma)$ ,  $\sigma \in S^{n-1}$ , and  $m$  will denote the largest of  $\lambda_1\mu_1$  and  $\lambda_n\mu_n$ . Also,  $\gamma$  will denote the number  $(\lambda_1\mu_n + \lambda_n\mu_1)/2$ . The main result of this section is Theorem 2 which describes the structure of those  $\sigma$  for which  $F(\sigma) = M$  when  $f$  is strictly convex. We first prove

**THEOREM 1.** *For any  $\sigma \in S^{n-1}$  there exists a  $\beta \in [0, 1]$  such that*

$$(2.1) \quad F(\sigma) \leq (\beta\lambda_1 + (1 - \beta)\lambda_n)(\beta\mu_1 + (1 - \beta)\mu_n).$$

*If  $f$  is strictly convex and for some  $k$ ,  $1 \leq k \leq n$ ,  $\lambda_1 < \lambda_k < \lambda_n$  and  $\sigma_k > 0$  then there exists a  $\beta \in [0, 1]$  for which (2.1) is a strict inequality.*

To prove Theorem 1 we use the following elementary fact.

**LEMMA.** *If  $0 \leq a_1 \leq a_2 \leq a_3$ , and  $b_1 \geq b_2 \geq b_3 \geq 0$  and*

$$(2.2) \quad (a_1 - a_3)(b_2 - b_3) \geq (a_2 - a_3)(b_1 - b_3)$$

*then for any  $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in S^2$  there exists a  $\beta \in [0, 1]$  such that*

$$(2.3) \quad \sum_{i=1}^3 \alpha_i a_i \sum_{i=1}^3 \alpha_i b_i \leq (\beta a_1 + (1 - \beta)a_2)(\beta b_1 + (1 - \beta)b_3).$$

*If the inequality (2.2) is strict and  $\alpha_2 > 0$  then there exists a  $\beta \in [0, 1]$  such that (2.3) is strict.*

*Proof.* Let  $\theta$  and  $\omega$  in  $[0, 1]$  be so chosen that  $a_2 = \theta a_1 + (1 - \theta)a_3$ ,  $b_2 = \omega b_1 + (1 - \omega)b_3$  and set  $b'_2 = \theta b_1 + (1 - \theta)b_3$ . Then

$$(2.4) \quad b'_2 - b_2 = (\theta - \omega)(b_1 - b_3).$$

Assume first that  $a_3 > a_2$  and  $b_2 > b_3$ . Then  $\theta = (a_2 - a_3)/(a_1 - a_3) > 0$  and  $\omega = (b_2 - b_3)/(b_1 - b_3)$ . Moreover  $\theta \geq \omega$  by (2.2) and if (2.2) is strict then  $\theta > \omega$ . From (2.4)  $b'_2 - b_2 \geq 0$  and we compute that

$$(2.5) \quad L \leq ((\alpha_1 + \theta\alpha_2)a_1 + (\alpha_2(1 - \theta) + \alpha_3)a_3) \\ ((\alpha_1 + \theta\alpha_2)b_1 + (\alpha_2(1 - \theta) + \alpha_3)b_3),$$

where  $L$  is the left side of (2.3). This is (2.3) with  $\beta = \alpha_1 + \theta\alpha_2 \in [0, 1]$ . If (2.2) is strict then  $\theta > \omega$ ,  $b'_2 = b_2$ , and  $\alpha_2 > 0$  together imply that (2.5) is strict.

Suppose next that  $a_2 = a_3$ . From (2.2) and  $(a_1 - a_3) \leq 0$  we have

$(a_1 - a_3)(b_2 - b_3) = 0$  and hence  $a_1 = a_3$  or  $b_2 = b_3$ . The first alternative yields  $a_1 = a_2 = a_3$  and thus  $L = a_1 \sum_{i=1}^3 \alpha_i b_i \leq a_1 b_1$  which is (2.3) with  $\beta = 1$ . If  $b_2 = b_3$  then (2.3) holds with  $\beta = \alpha_1$ . This completes the proof of the lemma.

The proof of Theorem 1 is by induction on  $n$ . The first non-trivial case is  $n = 3$ . In general the convexity of  $f$  implies that

$$(2.6) \quad (\lambda_1 - \lambda_3)(\mu_2 - \mu_3) > (\lambda_2 - \lambda_3)(\mu_1 - \mu_3)$$

and (2.6) is strict if  $\lambda_1 < \lambda_2 < \lambda_3$  and  $f$  is strictly convex. The inequality (2.1) follows from the lemma. If  $n > 3$  we distinguish the two possibilities  $\sigma_1 + \sigma_2 = 1$  and  $\sigma_1 + \sigma_2 < 1$ . In the first case

$$(2.7) \quad F(\sigma) = (\sigma_1 \lambda_1 + \sigma_2 \lambda_2)(\sigma_1 \mu_1 + \sigma_2 \mu_2) .$$

If  $\mu_1 = \mu_n$  and hence  $\mu_i = \mu_1 = \mu_n, i = 1, \dots, n$ , then  $F(\sigma) \leq \lambda_n \mu_n$  which is (2.1) with  $\beta = 0$ . If  $\mu_1 > \mu_n$ , and hence  $\lambda_1 < \lambda_n$ , obtain  $\theta$  and  $\omega$  in  $[0, 1]$  so that  $\lambda_2 = \theta \lambda_1 + (1 - \theta) \lambda_n, \mu_2 = \omega \mu_1 + (1 - \omega) \mu_n$  and set  $\mu'_2 = \theta \mu_1 + (1 - \theta) \mu_n$  to obtain

$$(2.8) \quad \mu'_2 - \mu_2 = (\theta - \omega)(\mu_1 - \mu_n) \geq 0 .$$

The convexity of  $f$  again implies that  $\theta \geq \omega$  with strictness in case  $f$  is strictly convex and  $\lambda_2 > \lambda_n$ . Hence

$$\begin{aligned} F(\sigma) &\leq (\sigma_1 \lambda_1 + (\theta \lambda_1 + (1 - \theta) \lambda_n) \sigma_2)(\sigma_1 \mu_1 + \sigma_2 \mu'_2) \\ &= ((\sigma_1 + \theta \sigma_2) \lambda_1 + (1 - \theta) \sigma_2 \lambda_n)((\sigma_1 + \theta \sigma_2) \mu_1 + (1 - \theta) \sigma_2 \mu_n) \end{aligned}$$

which is (2.1) with  $\beta = \sigma_1 + \theta \sigma_2$ . We proceed to the case  $\sigma_1 + \sigma_2 < 1$ . Let  $\lambda'_3 = \sum_{i=3}^n \sigma_i \lambda_i / (1 - \sigma_1 - \sigma_2), \mu'''_3 = \sum_{i=3}^n \sigma_i \mu_i / (1 - \sigma_1 - \sigma_2)$  and observe that  $\lambda_1 \leq \lambda_2 \leq \lambda'_3, \mu_1 \geq \mu_2 \geq \mu'''_3$  and  $F(\sigma) = (\sigma_1 \lambda_1 + \sigma_2 \lambda_2 + (1 - \sigma_1 - \sigma_2) \lambda'_3)(\sigma_1 \mu_1 + \sigma_2 \mu_2 + (1 - \sigma_1 - \sigma_2) \mu'''_3)$ . We next verify that (2.2) holds for the choices  $\lambda'_3 = a_3, \lambda_2 = a_2, \lambda_1 = a_1, \mu_1 = b_1, \mu_2 = b_2, \mu'''_3 = b_3$ :

$$(2.9) \quad \begin{aligned} &(\lambda_1 - \lambda_3)(\mu_2 - \mu'''_3) - (\mu_1 - \mu'''_3)(\lambda_2 - \lambda'_3) \\ &= \mu_2(\lambda_1 - \lambda'_3) - \mu_1(\lambda_2 - \lambda'_3) + \mu'''_3(\lambda_2 - \lambda_1) ; \end{aligned}$$

and

$$\mu'''_3 = \sum_{i=3}^n f(\lambda_i) \sigma_i / (1 - \sigma_1 - \sigma_2) \geq f\left(\sum_{i=3}^n \lambda_i \sigma_i / (1 - \sigma_1 - \sigma_2)\right) = f(\lambda'_3) = \mu'''_3 .$$

Hence the expression in (2.9) is at least

$$(2.10) \quad \mu_2(\lambda_1 - \lambda'_3) - \mu_1(\lambda_2 - \lambda'_3) + \mu'''_3(\lambda_2 - \lambda_1) .$$

If  $\lambda_2 = \lambda'_3$  the expression (2.10) reduces to 0 and the expression in (2.9) is nonnegative. If  $\lambda_2 < \lambda'_3$  then  $\lambda_1 < \lambda'_3$  and (2.10) becomes  $(\lambda_1 - \lambda'_3)(\lambda_2 - \lambda'_3)\{(\mu_2 - \mu'''_3)/(\lambda_2 - \lambda'_3) - (\mu_1 - \mu'''_3)/(\lambda_1 - \lambda'_3)\} \geq 0$ . Apply

the lemma to obtain  $\beta_1 \in [0, 1]$  for which

$$\begin{aligned} & (\sigma_1\lambda_1 + \sigma_2\lambda_2 + (1 - \sigma_1 - \sigma_2)\lambda_3)(\sigma_1\mu_1 + \sigma_2\mu_2 + (1 - \sigma_1 - \sigma_2)\mu_3'') \\ & \leq (\beta_1\lambda_1 + (1 - \beta_1)\lambda_3)(\beta_1\mu_1 + (1 - \beta_1)\mu_3'') \\ & = \left( \beta_1\lambda_1 + \sum_{i=3}^n (1 - \beta_1)\sigma_i\lambda_i / (1 - \sigma_1 - \sigma_2) \right) \\ & \quad \left( \beta_1\mu_1 + \sum_{i=3}^n (1 - \beta_1)\sigma_i\mu_i / (1 - \sigma_1 - \sigma_2) \right). \end{aligned}$$

This last expression is a product of convex combinations of  $\lambda$ 's and  $\mu$ 's involving only  $n - 1$  terms and satisfying the induction hypothesis. Hence there exists  $\beta \in [0, 1]$  such that

$$\begin{aligned} F(\sigma) & \leq \left( \beta_1\lambda_1 + \sum_{i=3}^n (1 - \beta_1)\sigma_i\lambda_i / (1 - \sigma_1 - \sigma_2) \right) \\ & \quad \left( \beta_1\mu_1 + \sum_{i=3}^n (1 - \beta_1)\sigma_i\mu_i / (1 - \sigma_1 - \sigma_2) \right) \leq (\beta\lambda_1 + (1 - \beta)\lambda_n) \\ & \quad (\beta\mu_1 + (1 - \beta)\mu_n). \end{aligned}$$

This establishes (2.10).

The discussion of the strictness in (2.1) requires the use of (2.1) itself. Let  $k$  be the least integer for which both  $\sigma_k > 0$  and  $\lambda_1 < \lambda_k < \lambda_n$ . Then

$$(2.11) \quad \begin{aligned} F(\sigma) & = (\alpha_1\lambda_1 + \alpha_k\lambda_k + \alpha_{k+p}\lambda_{k+p} + \dots + \alpha_n\lambda_n) \\ & \quad (\alpha_1\mu_1 + \alpha_k\mu_k + \alpha_{k+p}\mu_{k+p} + \dots + \alpha_n\mu_n) \end{aligned}$$

in which  $\alpha_1 + \alpha_k + \alpha_{k+p} + \dots + \alpha_n = 1$ ,  $\alpha_j = \sigma_j$ ,  $j = k + p, \dots, n$ , and  $\lambda_k < \lambda_{k+p}$ . Assume

$$\begin{aligned} \alpha_1 + \alpha_k < 1, \text{ set } \lambda'_{k+p} & = \sum_{i=k+p}^n \sigma_i\lambda_i / (1 - \alpha_1 - \alpha_k), \mu''_{k+p} \\ & = \sum_{i=k+p}^n \sigma_i\mu_i / (1 - \alpha_1 - \alpha_k) \end{aligned}$$

and (2.11) becomes

$$(2.12) \quad \begin{aligned} F(\sigma) & = (\alpha_1\lambda_1 + \alpha_k\lambda_k + (1 - \alpha_1 - \alpha_k)\lambda'_{k+p}) \\ & \quad (\alpha_1\mu_1 + \alpha_k\mu_k + (1 - \alpha_1 - \alpha_k)\mu''_{k+p}). \end{aligned}$$

Clearly  $\lambda_1 < \lambda_k < \lambda'_{k+p}$  and we compute that

$$(2.13) \quad \begin{aligned} & (\lambda_1 - \lambda'_{k+p})(\mu_k - \mu'_{k+p}) - (\mu_1 - \mu'_{k+p})(\lambda_k - \lambda'_{k+p}) \\ & = \mu_k(\lambda_1 - \lambda'_{k+p}) - \mu_1(\lambda_k - \lambda'_{k+p}) + \mu''_{k+p}(\lambda_k - \lambda_1); \end{aligned}$$

$$(2.14) \quad \mu''_{k+p} \geq f(\lambda'_{k+p}) = \mu'_{k+p}.$$

It follows that the expression in (2.13) is at least

$$\{(\lambda_1 - \lambda'_{k+p})(\lambda_k - \lambda'_{k+p})\{(\mu_k - \mu'_{k+p})/(\lambda_k - \lambda'_{k+p}) - (\mu_1 - \mu'_{k+p})/(\lambda_1 - \lambda'_{k+p})\}$$

and in case  $f$  is strictly convex this whole expression is positive. The inequality (2.2) holds strictly with  $\lambda_1 = a_1, \lambda_k = a_2, \lambda'_{k+p} = a_3, \mu_1 = b_1, \mu_k = b_2, \mu'_{k+p} = b_3$  and the strict form of the lemma together with (2.12) implies that there exists  $\beta_1 \in [0, 1]$  such that

$$(2.15) \quad F(\sigma) < \left( \beta_1 \lambda_1 + \sum_{i=k+p}^n (1 - \beta_1) \sigma_i \lambda_i / (1 - \alpha_1 - \alpha_k) \right) \\ \left( \beta_1 \mu_1 + \sum_{i=k+p}^n (1 - \beta_1) \sigma_i \mu_i / (1 - \alpha_1 - \alpha_k) \right).$$

Now apply (2.1) to the right side of (2.15) to obtain a  $\beta \in [0, 1]$  for which  $F(\sigma) < (\beta \lambda_1 + (1 - \beta) \lambda_n)(\beta \mu_1 + (1 - \beta) \mu_n)$ .

Assume now that  $\alpha_1 + \alpha_k = 1$  and then  $F(\sigma)$  becomes  $(\alpha_1 \lambda_1 + (1 - \alpha_1) \lambda_k)(\alpha_1 \mu_1 + (1 - \alpha_1) \mu_k)$ . Choose  $\theta$  and  $\omega$  in  $[0, 1]$  so that  $\lambda_k = \theta \lambda_1 + (1 - \theta) \lambda_n, \mu_k = \omega \mu_1 + (1 - \omega) \mu_n$ , set  $\mu'_k = \theta \mu_1 + (1 - \theta) \mu_n$  and note that  $\mu'_k - \mu_k = (\theta - \omega)(\mu_1 - \mu_n)$ . Then since  $f$  is monotone decreasing and strictly convex,  $\theta - \omega$  and  $\mu_1 - \mu_n$  are both positive. It follows that

$$(\alpha_1 \lambda_1 + (1 - \alpha_1) \lambda_k)(\alpha_1 \mu_1 + (1 - \alpha_1) \mu_k) < ((\alpha_1 + \theta(1 - \alpha_1)) \lambda_1 \\ + (1 - \theta)(1 - \alpha_1) \lambda_n)((\alpha_1 + \theta(1 - \alpha_1)) \mu_1 + (1 - \theta)(1 - \alpha_1) \mu_n).$$

If the quadratic polynomial in  $\beta$  on the right in (2.1) is maximized in  $[0, 1]$  we immediately obtain our main result.

**THEOREM 2.** *If*

$$(2.16) \quad \gamma \geq m \text{ and } \lambda_1 < \lambda_n \text{ and } \mu_1 > \mu_n$$

*then*

$$(2.17) \quad M = (\lambda_n \mu_1 - \lambda_1 \mu_n) / 4(\lambda_n - \lambda_1)(\mu_1 - \mu_n).$$

*If*

$$(2.18) \quad \gamma \leq m \text{ or } \lambda_1 = \lambda_n \text{ or } \mu_1 = \mu_n$$

*then*

$$(2.19) \quad M = m.$$

*Let  $f$  be strictly convex and suppose that*

$$\lambda_1 = \dots = \lambda_p < \lambda_{p+1} \leq \dots \leq \lambda_{n-q} < \lambda_{n-q+1} = \dots = \lambda_n.$$

*Then  $F(\sigma) = M, \sigma \in S^{n-1}$ , if and only if  $\sigma$  has the form*

$$\sigma = (\sigma_1, \dots, \sigma_p, 0, \dots, 0, \sigma_{n-q+1}, \dots, \sigma_n),$$

$\sum_{j=1}^p \sigma_j = \beta_0, \sum_{j=n-q+1}^n \sigma_j = 1 - \beta_0, \text{ where}$

$$(2.20) \quad \beta_0 = \begin{cases} (\gamma - \lambda_n \mu_n) / (\lambda_n - \lambda_1)(\mu_1 - \mu_n) & \text{if (2.16) holds,} \\ 0 \text{ or } 1 & \text{if (2.18) holds.} \end{cases}$$

We remark that if  $\gamma = m$  then the expression on the right in (2.17) reduces to  $m$ .

**3. Applications.** As customary  $f(A)$  will designate the linear transformation defined for any  $x \in U$  by

$$(3.1) \quad f(A)x = \sum_{i=1}^n \mu_i(x, u_i)u_i, (\mu_i = f(\lambda_i)).$$

On the unit sphere  $\|x\| = 1$  define the real valued function

$$(3.2) \quad \varphi(x) = (Ax, x)(f(A)x, x).$$

We compute directly from (3.1) that

$$(3.3) \quad \varphi(x) = \sum_{i=1}^n \lambda_i |(x, u_i)|^2 \sum_{i=1}^n \mu_i |(x, u_i)|^2$$

and by setting  $\sigma_i = |(x, u_i)|^2, i = 1, \dots, n,$  we have  $\sigma = (\sigma_1, \dots, \sigma_n) \in S^{n-1}$  and

$$(3.4) \quad \varphi(x) = F(\sigma).$$

Thus by direct application of Theorem 2 we have

**THEOREM 3.** *Then maximum value of  $\varphi(x)$  for  $x$  on the unit sphere  $\|x\| = 1$  is the number  $M$  in the statement of Theorem 2. Moreover  $\varphi(x_0) = M$  can always be achieved with a unit vector  $x_0$  in the subspace spanned by those eigenvectors of  $A$  corresponding to  $\lambda_1$  and  $\lambda_n$ . If  $f$  is strictly convex and  $\varphi(x_0) = M$  then  $x_0$  must lie in the sum of the null spaces of  $A - \lambda_1 I$  and  $A - \lambda_n I$ . In particular, if  $\lambda_1$  and  $\lambda_n$  are simple eigenvalues of  $A, f$  is strictly convex and  $\varphi(x_0) = M$  then  $x_0$  must lie in the two dimensional subspace spanned by  $u_1$  and  $u_n$ .*

In Theorem 3 take  $f(t) = t^{-p}, p > 0$ . Let  $\theta = \lambda_1/\lambda_n$  denote the condition number of  $A$ . Assume that  $\theta < 1$  (otherwise  $\lambda_1 = \lambda_n$  and  $A$  is a multiple of the identity). There are two cases to consider:  $p > 1; p \leq 1$ . In case  $p > 1, m = \lambda_1^{1-p}$  and the condition (2.16),  $\gamma \geq m,$  becomes

$$(3.5) \quad g(\theta) = \theta^{p+1} - 2\theta + 1 \geq 0.$$

We note that  $g$  is convex,  $g(1) = 0, g'(\theta) = 0$  for  $\theta = (2/(p + 1))^{1/p},$  and

hence  $g$  has precisely one root in  $(0, 1)$ , call it  $\theta_p$ . It is easy to see that  $\theta_p > 1/2$  for all  $p > 1$ . In general, if  $0 < \theta \leq \theta_p$  then Theorem 2 yields

$$(3.6) \quad M = \lambda_1^{1-p}(\theta^{p+1} - 1)^2/4\theta(\theta - 1)(\theta^p - 1) ;$$

and if  $1 \geq \theta > \theta_p$  then

$$(3.7) \quad M = \lambda_1^{1-p} .$$

In case  $p \leq 1$ ,  $m = \lambda_n^{1-p}$  and the condition (2.16),  $\gamma \geq m$ , becomes  $g(\eta) \geq 0$  where  $\eta = \theta^{-1}$ . But  $g(\eta) \geq 0$  for  $\eta \geq 1$  and  $\eta = \theta^{-1} \geq 1$  so the upper bound for  $F(\sigma)$  is  $M$  given in (3.6).

Assume now that  $\lambda_1$  and  $\lambda_n$  are both simple eigenvalues of  $A$  and we examine the structure of the vector  $x_0$  that maximizes  $\varphi(x) = (Ax, x)(A^{-p}x, x)$  on the unit sphere  $\|x\| = 1$ . By Theorem 3 the maximum value of  $\varphi(x) = F(\sigma)$  can only occur for  $\sigma_2 = \dots = \sigma_{n-1} = 0$ . Moreover by (2.20)  $F(\sigma) = M$  for the unique values

$$(3.8) \quad \left. \begin{aligned} \sigma_n &= \sigma_n(\theta) = g(\theta)/2(1 - \theta)(1 - \theta^p) \\ \sigma_1 &= \sigma_1(\theta) = \sigma_n(\theta^{-1}) \end{aligned} \right\} \text{if } g(\theta) \geq 0 \text{ or } p = 1 ;$$

and

$$(3.10) \quad \sigma_1 = 1, \sigma_n = 0 \text{ if } g(\theta) < 0 \text{ and } p > 1 .$$

Summing up these results we have

**THEOREM 4.** *Let  $\theta$  designate the condition number of  $A$ ,  $\theta = \lambda_1/\lambda_n$ . If either  $0 < p \leq 1$ , or  $p > 1$  and  $0 \leq \theta \leq \theta_p$ , then for  $\|x\| = 1$*

$$(3.11) \quad (Ax, x)(A^{-p}x, x) \leq \lambda_1^{1-p}(\theta^{p+1} - 1)^2/4\theta(\theta - 1)(\theta^p - 1) .$$

*If  $p > 1$  and  $\theta_p < \theta$  then for  $\|x\| = 1$*

$$(3.12) \quad (Ax, x)(A^{-p}x, x) \leq \lambda_1^{1-p} .$$

*If  $\lambda_1$  and  $\lambda_n$  are simple eigenvalues of  $A$  then the upper bound in (3.11) is only achieved for unit vectors of the form*

$$(3.13) \quad x_0 = \sqrt{\sigma_n(\theta^{-1})} e^{i\omega_1} u_1 + \sqrt{\sigma_n(\theta)} e^{i\omega_2} u_n ,$$

*$\omega_1, \omega_2$  real. The upper bound in (3.12) is achieved only for unit vectors of the form*

$$x_0 = e^{i\omega} u_1 .$$

In case  $p = 1$  we have the Kantorovich inequality. In this case (3.11) becomes (for  $\|x\| = 1$ )



$$(3.14) \quad (Ax, x)(A^{-1}x, x) \leq (\sqrt{\theta} + \sqrt{\theta^{-1}})^2/4.$$

If  $\lambda_1$  and  $\lambda_n$  are simple eigenvalues then the inequality (3.14) is strict unless

$$(3.15) \quad x = x_0 = (e^{i\omega_1}u_1 + e^{i\omega_2}u_n)/\sqrt{2}, \omega_1, \omega_2 \text{ real}.$$

**4. Determinants and permanents.** In this section we specialize by taking  $U$  to be the unitary space of  $n$ -tuples with inner product  $(x, y) = \sum_{i=1}^n x_i \bar{y}_i$  and  $A$  to be an  $n$ -square hermitian positive semi-definite matrix. If  $1 \leq k \leq n$  then  $C_k(A)$  will denote the  $k$ th compound of  $A$  and if  $x_1, \dots, x_k$  are vectors in  $U$  then  $x_1 \wedge \dots \wedge x_k$  is the Grassmann product of these vectors, sometimes called a pure vector of grade  $k$  [6, p. 16]. The eigenvalues of  $C_k(A)$  are all  $\binom{n}{k}$  numbers  $\lambda_{i_1} \dots \lambda_{i_k}$ , with corresponding eigenvectors  $u_{i_1} \wedge \dots \wedge u_{i_k}$ ,  $1 \leq i_1 < \dots < i_k \leq n$ . The smallest and largest of these eigenvalues are  $\prod_{j=1}^k \lambda_j$  and  $\prod_{j=1}^k \lambda_{n-j+1}$  respectively. It has been noted in [2] and [5] that the Kantorovich inequality applied to  $C_k(A)$  yields

$$(4.1) \quad \det A[i_1, \dots, i_k] \det A^{-1}[i_1, \dots, i_k] \leq (\sqrt{\Delta} + \sqrt{\Delta^{-1}})^2/4$$

where  $\Delta = \prod_{j=1}^k \lambda_j \lambda_{n-j+1}$  and  $A[i_1, \dots, i_k]$  is the principal submatrix of  $A$  lying in rows and columns numbered  $i_1, \dots, i_k$ .

We prove

**THEOREM 5.** *If  $1 \leq k < n - 1$  and  $\lambda_1, \dots, \lambda_k$  together with  $\lambda_n, \dots, \lambda_{n-k+1}$  are simple eigenvalues of  $A$  then the inequality (4.1) is always strict.*

*Proof.* The number  $\det A[i_1, \dots, i_k] \det A^{-1}[i_1, \dots, i_k]$  is a value of the product of quadratic forms associated with  $C_k(A)$  and  $C_k(A^{-1})$ ,

$$(4.2) \quad \begin{aligned} & (C_k(A)x_1 \wedge \dots \wedge x_k, x_1 \wedge \dots \wedge x_k) \\ & (C_k(A^{-1})x_1 \wedge \dots \wedge x_k, x_1 \wedge \dots \wedge x_k), \end{aligned}$$

and according to (3.15), (4.1) will be strict unless

$$(4.3) \quad x_1 \wedge \dots \wedge x_k = \frac{1}{\sqrt{2}}(e^{i\omega_1}u_1 \wedge \dots \wedge u_k + e^{i\omega_2}u_n \wedge \dots \wedge u_{n-k+1}).$$

Let  $p = \min\{k, n - k\}$ ,  $q = \max\{k + 1, n - k + 1\}$  and compute successively the Grassmann products of both sides of (4.3) with  $u_1, \dots, u_p$  and  $u_n, \dots, u_q$ . We obtain

$$(4.4) \quad x_1 \wedge \dots \wedge x_k \wedge u_j = \frac{e^{i\omega_2}}{\sqrt{2}}(u_n \wedge \dots \wedge u_{n-k+1} \wedge u_j), j = 1, \dots, p,$$

and

$$(4.5) \quad x_1 \wedge \cdots \wedge x_k \wedge u_j = \frac{e^{i\omega_2}}{\sqrt{2}}(u_1 \wedge \cdots \wedge u_k \wedge u_j), j = q, \dots, n .$$

Since  $u_1, \dots, u_n$  are linearly independent it follows that the right sides of (4.4) and (4.5) are not 0. Thus

$$(4.6) \quad \langle x_1, \dots, x_k, u_j \rangle = \langle u_1, \dots, u_k, u_j \rangle, j = 1, \dots, p ,$$

and

$$(4.7) \quad \langle x_1, \dots, x_k, u_j \rangle = \langle u_1, \dots, u_k, u_j \rangle, j = q, \dots, n ,$$

where  $\langle x_1, \dots, x_k, u_j \rangle$  denotes the subspace spanned by the vectors inside the brackets. Intersect the  $p$  subspaces on the left in (4.6) and observe that  $\langle x_1, \dots, x_k \rangle$  is a subspace of the intersection. Similarly  $\langle x_1, \dots, x_k \rangle$  is a subspace of the intersection of the  $n - q + 1$  spaces on the left in (4.7). On the other hand

$$\bigcap_{j=1}^p \langle u_n, \dots, u_{n-k+1}, u_j \rangle = \langle u_n, \dots, u_{n-k+1} \rangle$$

and

$$\bigcap_{j=q}^n \langle u_1, \dots, u_k, u_j \rangle = \langle u_1, \dots, u_k \rangle .$$

Hence

$$(4.8) \quad \begin{aligned} & \dim \{ \langle u_1, \dots, u_k \rangle \cap \langle u_n, \dots, u_{n-k+1} \rangle \} \\ & = \dim \left\{ \bigcap_{j=1}^p \langle x_1, \dots, x_k, u_j \rangle \cap \bigcap_{j=q}^n \langle x_1, \dots, x_k, u_j \rangle \right\} > k . \end{aligned}$$

The subspace  $\langle u_1, \dots, u_k \rangle \cap \langle u_n, \dots, u_{n-k+1} \rangle$  is nonempty if and only if  $n - k + 1 \leq k$  in which case its dimension is  $2k - n$ . But the inequality  $2k - n \geq k$  implies that  $k \geq n$ , a contradiction. Thus (4.3) cannot hold and (4.1) is strict.

We remark that in case  $k = n - 1$  then  $p = 1, q = n, x_1 \wedge \cdots \wedge x_k \wedge u_1 = u_n \wedge \cdots \wedge u_2 \wedge u_1, x_1 \wedge \cdots \wedge x_k \wedge u_n = u_1 \wedge \cdots \wedge u_{n-1} \wedge u_n$  and the above argument fails. In fact, it is not difficult to construct examples for which (4.1) is equality.

Once again, if  $1 \leq k \leq n$  then  $P_k(A)$  will denote the  $k$ th induced power matrix of  $A$  and if  $x_1, \dots, x_k$  are vectors in  $U$  then  $x_1 \cdots x_k$  will denote the symmetric or dot product of these vectors [3, p. 49]. The eigenvalues of  $P_k(A)$  are all  $\binom{n+k-1}{k}$  homogeneous products  $\lambda_{i_1} \cdots \lambda_{i_k}$  with corresponding eigenvectors  $u_{i_1} \cdots u_{i_k}, 1 \leq i_1 \leq \cdots \leq i_k \leq n$ . Suppose  $x_1, \dots, x_n$  are orthonormal vectors and the multiplicities

of the distinct integers in the sequence  $i_1 \leq \dots \leq i_k$  are respectively  $m_1, \dots, m_p$ . Let  $\mu = \mu(i_1, \dots, i_k) = m_1! \dots m_p!$ . Then the square of the length of the symmetric product  $x_{i_1} \dots x_{i_k}$  is  $\mu(i_1, \dots, i_k)$  [3, p. 50]. Applying the Kantorovich inequality to  $P_k(A)$  yields

$$(4.9) \quad (P_k(A)x_{i_1} \dots x_{i_k}, x_{i_1} \dots x_{i_k})(P_k(A^{-1})x_{i_1} \dots x_{i_k}, x_{i_1} \dots x_{i_k}) \leq \mu^2(\sqrt{\delta} + \sqrt{\delta^{-1}})^2/4, \quad 1 \leq i_1 \leq \dots \leq i_k \leq n,$$

where  $\delta = (\lambda_1 \lambda_n^{-1})^k$ , and  $x_1, \dots, x_n$  is an orthonormal basis of  $U$ . In particular if we let  $x_i = e_i$ , the unit vector with 1 in the  $i$ th position, 0 elsewhere, then (4.9) becomes

$$(4.10) \quad \text{per } A[i_1, \dots, i_k] \text{ per } A^{-1}[i_1, \dots, i_k] \leq \mu^2(\sqrt{\delta} + \sqrt{\delta^{-1}})^2/4,$$

where  $A[i_1, \dots, i_k]$  is the  $k$ -square matrix whose  $(s, t)$  entry is  $a_{i_s i_t}$ ,  $s, t = 1, \dots, k$ .

**THEOREM 6.** *If  $\lambda_1$  and  $\lambda_n$  are simple eigenvalues of  $A$  and there are at least three distinct integers in the sequence  $i_1 \leq \dots \leq i_k$  then the inequality (4.10) is strict.*

*Proof.* According to (3.15), (4.10) will be strict unless

$$(4.11) \quad e_{i_1} \dots e_{i_k} = \frac{e^{i\omega_1}}{\sqrt{2k!}} u_1 \dots u_1 + \frac{e^{i\omega_2}}{\sqrt{2k!}} u_n \dots u_n.$$

Let  $y$  be an arbitrary vector and compute the inner product of both sides of (4.11) with  $y \dots y$  to obtain

$$(4.12) \quad \prod_{j=1}^k (e_{i_j}, y) = \frac{e^{i\omega_1}}{\sqrt{2k!}} (u_1, y)^k + \frac{e^{i\omega_2}}{\sqrt{2k!}} (u_n, y)^k.$$

Set

$$v_1 = \left( \frac{e^{i\omega_1}}{\sqrt{2k!}} \right)^{1/k} u_1, \quad v_2 = \left( \frac{e^{i\omega_2}}{\sqrt{2k!}} \right)^{1/k} u_n,$$

and write  $e_{i_j} = \alpha_j v_1 + w_j$ ,  $w_j \in \langle v_1 \rangle^\perp$ ,  $j = 1, \dots, k$ . Then for  $y$  any vector in  $\langle v_1 \rangle^\perp$ , (4.12) becomes

$$(4.13) \quad \prod_{j=1}^k (e_{i_j}, y) = \prod_{j=1}^k (w_j, y) = (v_2, y)^k,$$

in which  $w_j, v_2, y$  are in  $\langle v_1 \rangle^\perp$ ,  $j = 1, \dots, k$ . But then from [3, Theorem 3] we conclude that  $w_j = \beta_j v_2$ ,  $j = 1, \dots, k$ , for appropriate scalars  $\beta_1, \dots, \beta_k$  and hence  $e_{i_j} \in \langle v_1, v_2 \rangle$ ,  $j = 1, \dots, k$ . Since there are at least three linearly independent  $e_{i_j}$ , (4.11) must fail and hence (4.10) is strict.

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