Pacific Journal of Mathematics

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Vol. 13, No. 4 June 1963

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Throughout this paper suppose that L denotes a connected, totally ordered topological space in which there is no first or last point, and whose topology is that induced by the order.

A topological space S is said to be homogeneous provided it is true that if $(x, y) \in S \times S$, there is a homeomorphism f from S onto S such that $f(x) = y$. Let H denote the set of all homeomorphisms from L onto L, and let I denote the set of all homeomorphisms which map a closed interval of L onto a closed interval of L. Let $H_0(I_0)$ denote the set of all elements of $H(I)$ which preserve order.

THEOREM 1. *If L is homogeneous, then L satisfies the first axiom of countability.*

Proof. It suffices to show that for some point z of L there exists an increasing sequence $x_1, x_2, \cdots$ and a decreasing sequence $y_1, y_2, \cdots$ such that each of these sequences converges to z. Suppose there is no such point. Let $P_1, P_2, \cdots$ denote an increasing sequence which converges to a point P and $Q_1, Q_2, \cdots$ a decreasing sequence which converges to a point Q. There is an element g in H such that $g(P) = Q$. In view of the preceding supposition, g is order reversing. There is a point R such that $g(R) = R$, and R is the limit of a sequence $R_1, R_2, \cdots$ which is either increasing or decreasing. Suppose the sequence is decreasing. The sequence $g(R_1), g(R_2), \cdots$ is increasing and converges to R. This yields a contradiction. The case where $R_1, R_2, \cdots$ is increasing is similar.

THEOREM 2. *The space L is homogeneous if and only if each pair of closed subintervals of L are topologicaly equivalent.*

Proof. Part 1. Suppose each pair of closed subintervals of L are topologically equivalent and $(x, y) \in L \times L$. There exist elements z and w of L such that $z < x < w$ and $z < y < w$, and an element g of I from $[z, x]$ onto $[z, y]$. If g is order reversing there is an element g' of I_0 from $[z, x]$ onto $[z, y]$ which may be constructed as follows: Let t denote the point of $[z, x]$ such that $g(t) = t$. g' is defined by

Received September 5, 1962, and in revised form June 20, 1963. Abstract 62 T-102, Notices, V. 9, June 1962. The author wishes to express his appreciation to the National Science Foundation for financial support.

$g'(u) = \begin{cases} u, & z \leqq u \leqq t \\ gg(u), & t < u \leqq x \end{cases}$. In any event, let g' and h' denote elements of I_0 which map $[z, x]$ and $[x, w]$, respectively, onto $[z, y]$ and $[y, w]$, respectively. The function f defined by

$$f(u) = \begin{cases} u\,, & u < z \text{ or } u > w \\ g'(u)\,, & z \leqq u \leqq x \\ h'(u)\,, & x < u \leqq w \end{cases}$$

is an element of H_0 such that $f(x) = y$.

Part 2. Suppose L is homogeneous.

LEMMA 1. *If $(x, y) \in L \times L$, there is an element f of H_0 such that $f(x) = y$. Furthermore, if $f \in I$ there is an element g of I_0 having the same domain and range, respectively, as f.*

Proof. Suppose $g \in H$ and $g(x) = y$, but g is not in H_0. There is a point b such that $b = g(b)$ and an element h of H such that $h(x) = b$. The function $f = gh^{-1}g^{-1}h$ is in H_0 and $f(x) = y$. The proof of the second part of Lemma 1 follows easily from the first part and the proof of Part 1 of Theorem 2.

LEMMA 2. *Suppose $[a, b]$ is a closed interval and f and g are elements of I_0 defined on $[a, b]$ such that $f(a) = g(a)$ $(f(b) = g(b))$, but that $f(x) < g(x)$ for $a < x \leqq b$ $(a \leqq x < b)$. If $f(a) < x_0 < f(b)$ $(g(a) < x_0 < g(b))$ and $x_1, x_2, \cdots$ is a sequence such that $x_n = fg^{-1}(x_{n-1})$ $(x_n = gf^{-1}(x_{n-1}))$ for $n \geqq 1$, then $x_0, x_1, x_2, \cdots$ is a decreasing (increasing) sequence which converges to $f(a)$ $(f(b))$.*

Proof of first part. The inequality $a < g^{-1}(x_0) < f^{-1}(x_0) < b$ implies that $f(a) < x_1 = fg^{-1}(x_0) < x_0 < f(b)$. Suppose it has been established that $f(a) < x_n < x_{n-1} < f(b)$. The preceding implies that $a < g^{-1}(x_n) < f^{-1}(x_n) < b$, which implies that $f(a) < x_{n+1} = fg^{-1}(x_n) < x_n < f(b)$. Therefore, $x_0, x_1, x_2, \cdots$ is a decreasing sequence bounded below by $f(a)$, and thus converges to a point $x \geqq f(a)$. Suppose $x > f(a)$. Since $gf^{-1}(x) > x$, there is a positive integer n such that $gf^{-1}(x) > x_n > x$, which implies that $x > fg^{-1}(x_n) = x_{n+1}$. This yields a contradiction, so $x = f(a)$.

LEMMA 3. *If $c \in L$ there exist an interval $[a, b]$ and elements f and g of I_0 with domain $[a, b]$ such that $f(a) = g(a) = c$ and $f(x) < g(x)$, for $a < x \leqq b$; or if $c \in L$ there exists an interval $[a, b]$ and elements f and g of I_0 with domain $[a, b]$ such that $f(b) = g(b) = c$ and $f(x) < g(x)$, for $a \leqq x < b$.*

Proof. Suppose that for each element (x, y) of $L \times L$ there is a unique element f of H_0 such that $f(x) = y$. Let $u_1, u_2, \cdots$ denote an increasing sequence converging to a point u, and for each n, let f_n denote the element of H_0 such that $f_n(u) = u_n$. If x is an element of L and n a positive integer, then $f_n(x) < f_{n+1}(x) < x$; for if this is not the case, the graph of f_n intersects the graph of f_{n+1}, or the graph of f_{n+1} intersects the graph of the identity homeomorphism, and in either event there is a contradiction to the unique homeomorphism hypothesis. If for some x, the sequence $f_1(x), f_2(x), \cdots$ converges to a point $y < x$, the element g of H_0 such that $g(x) = y$ has the property that its graph either intersects the graph of the identity function or the graph of f_n, for some n. Therefore, for any x in L, the sequence $f_1(x), f_2(x), \cdots$ is increasing and converges to x.

For each positive integer j, let $a_{j1}, a_{j2}, \cdots$ and $b_{j1}, b_{j2}, \cdots$ denote sequences such that (1) $a_{j1} = f_j^{-1}(u)$ and $b_{j1} = f_j(u)$, and (2) $a_{jn} = f_j^{-1}(a_{j,n-1})$ and $b_{jn} = f_j(b_{j,n-1})$, for $n > 1$. Suppose $u < x$ and (r, s) is an open interval containing x. Let n denote an integer such that $r < f_n(x)$ and $x < f_n(s)$. Since $u < x < f_n(s)$, it follows that $a_{n1} = f_n^{-1}(u) < s$. If a_{n1} is not in (r, s), let K denote the set of all a_{nj} such that $a_{nj} < x$ and let $z = \text{l.u.b. } K$. If $z \leqq r$, there is an element a_{nj} of K such that $f_n(z) < a_{nj} \leqq z < f_n(x)$, which implies that $z < f_n^{-1}(a_{nj}) = a_{n,j+1} < x$, which is a contradiction. In any event, some a_{nj} is an element of (r, s). The preceding argument clearly indicates that $\sum (a_{ij} + b_{ij})$ is a countable set dense in L, so L is a real line and the unique homeomorphism hypothesis is contradicted.

There exist elements h and k of H_0 and points s and t of L such that $h(s) = k(s)$, but $h(t) < k(t)$. Suppose $s < t$. Let a denote the largest element x of L such that $h(x) = k(x)$ and $x < t$. There is an element p of I_0 with domain $[k(a), k(t)]$ such that $p(k(a)) = c$. The functions $f = p(h)$ and $g = p(k)$ and the interval $[a, t]$ satisfy the first conclusion of the lemma. The case $t < s$ yields the second conclusion.

LEMMA 4. *Suppose $[a, b]$ is a closed interval and c is a point. If $x > c$, there is a point y in (c, x) and an element f of I_0 mapping $[a, b]$ onto $[c, y]$.*

Proof. Let U denote the set of all $x > c$ such that there is a homeomorphism from $[a, b]$ onto $[c, x]$, and let V denote the set of all $x < c$ such that there is a homeomorphism from $[a, b]$ onto $[x, c]$. The sets U and V exist because of the existence of elements h_1 and h_2 of H_0 such that $h_1(a) = c$ and $h_2(b) = c$. Let $u = \text{g.l.b. } U$, $v = \text{l.u.b. } V$ and suppose that $c < u$.

Case 1. Suppose the first conclusion of Lemma 3 holds There exists a point u_1, an interval $[p, q]$, and elements f and g of I_0 having domain $[p, q]$, and such that (1) $c < u_1 < u$, (2) $f(p) = g(p) = u_1$, and (3) $f(x) < g(x)$, for $p < x \leqq q$. There is a point r such that $p < r < q$, $g(r) < u$, and $g(r) < f(q)$, and an element k of I_0 having domain $[p, q]$ such that (1) $k(r) = u$, and (2) $k(x) \geqq g(x)$ for $x \in [p, q]$. The function h defined on $[p, q]$ by $h(x) = kg^{-1}f(x)$ is an element of I_0 such that (1) $h(q) > u$, (2) $h(p) = k(p)$, and (3) $h(x) < k(x)$, for $p < x \leqq q$. There is a point x_0 such that $u \leqq x_0 < h(q)$ and an element f_0 of I_0 mapping $[a, b]$ onto $[c, x_0]$. Let $x_1, x_2, \cdots$ denote a sequence such that $x_n = hk^{-1}(x_{n-1})$ for $n \geqq 1$, and let $f_1, f_2, \cdots$ denote a sequence of functions defined on $[a, b]$ such that for $n \geqq 1$ (1) $f_n(x) = f_0(x)$, for $a \leqq x \leqq f_0^{-1}(u_1)$, and (2) $f_n(x) = hk^{-1}f_{n-1}(x)$, for $f_0^{-1}(u_1) < x \leqq b$. For each n, f_n is a homeomorphism from $[a, b]$ onto $[c, x_n]$, but, according to Lemma 2, $x_n < u$ for some n. This yields a contradiction, so $u = c$.

Case 2. If the second conclusion of Lemma 3 holds, then it follows, by an argument similar to the one in Case 1, that $v = c$. Let u_1 denote a point between c and u, and g an element of H_0 such that $g(c) = u_1$. There is a point u_2 such that $c < u_2 < u_1$ and an element h of I_0 mapping $[a, b]$ onto $[g^{-1}(u_2), c]$. The function $g(h)$ is an element of I_0 mapping $[a, b]$ onto $[u_2, u_1]$. Let k denote an element of H_0 such that $k(a) = c$. Since $k(b) \geqq u$, there is a point t such that $k(t) = gh(t)$. The function f defined by

$$f(x) = \begin{Bmatrix} k(x)\ , & a \leqq x \leqq t \\ gh(x)\ , & t < x \leqq b \end{Bmatrix}$$

is an element of I_0 which maps $[a, b]$ onto $[c, u_1]$, so in this case also, the assumption $c < u$ leads to a contradiction.

The proof of the main result now follows easily. Suppose $[a, b]$ and $[c, d]$ are closed intervals and g an element of H_0 such that $g(b) = d$.

Case 1. $g(a) \leqq c$. There is a point e such that $c < e < d$ and an element h of I_0 mapping $[a, b]$ onto $[c, e]$. As in case 2 of Lemma 4, a homeomorphism from $[a, b]$ onto $[c, d]$ may be constructed from g and h.

Case 2. $g(a) > c$. There is a point e such that $a < e < b$ and an element h of I_0 mapping $[c, d]$ onto $[a, e]$. However, h^{-1} is an element of I_0 mapping $[a, e]$ onto $[c, d]$, and a homeomorphism from $[a, b]$ onto $[c, d]$ may be easily constructed from g and h^{-1}.

In order to establish the next theorem it is helpful to use a result of Richard Arens'. A linear homogeneous continuum (LHC) has been defined by G. D. Birkhoff as any set of elements which 1. is simply ordered 2. provides a limit for any monotonely increasing (or decreasing) sequence 3. is isomorphic to every nondegenerate closed subinterval of itself. In [1] Arens shows, among other results, the following (reworded by the author).

THEOREM A. *If I is an LHC and for each positive integer p, I_p denotes I, then the space $I' = I_1 \times I_2 \times \cdots$ with the lexicographic order is also an LHC.*

THEOREM 3. *If L is homogeneous, $[a, b]$ is a closed interval, and for each positive integer p, I_p denotes $[a, b]$, then the space $x = L \times I_1 \times I_2 \times \cdots$ with the topology induced by the lexicographic order is also homogeneous.*

Proof. Let $[u_1, u_2, \cdots; v_1, v_2, \cdots]$ and $[x_1, x_2, \cdots; y_1, y_2, \cdots]$ denote closed subintervals of X. Let u and v denote elements of L such that $u < \min\{u_i, x_i\}$ and $v > \max\{v_i, y_i\}$ for $i = 1, 2, 3, \cdots$, and let g denote an element of I_0 which maps $[u, v]$ onto $[a, b]$. The function F defined by $F(t_0, t_1, t_2, \cdots) = [g(t_0), t_1, t_2, \cdots]$ is an order preserving homeomorphism from $[u, v] \times I_1 \times I_2 \times \cdots$ onto $[a, b] \times I_1 \times I_2 \times \cdots$. Theorem A shows that any two subintervals of the latter are homeomorphic, so it follows that $[x_1, x_2, \cdots; y_1, y_2, \cdots]$ and $[u_1, u_2, \cdots; v_1, v_2, \cdots]$ are homeomorphic. Therefore, by theorem 2, X is homogeneous.

Suppose $L_1, L_2, L_3, \cdots$ denotes a sequence of spaces such that (1) L_1 is the real line, and (2) for each n, L_{n+1} is constructed from L_n by a Theorem 3 type construction. The main theorem of Arens' paper [2] yields the result that if $i \neq j$, then L_i is not homeomorphic to L_j. Is it true that if a homogeneous space L' satisfies the axioms stated on the first page and also has the property that it can be covered by a countable collection of closed intervals, then L' is one of the spaces $L_1, L_2, L_3, \cdots$?

In part 2 of Theorem 2 the construction indicated gives an order preserving homeomorphism from $[a, b]$ onto $[c, d]$. This leads naturally to the following question: If L' satisfies the axioms of L, is homogeneous, and $[a, b]$ is a closed subinterval of L', then is there an order reversing homeomorphism from $[a, b]$ onto $[a, b]$?

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The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 13, No. 4 June, 1963