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**SOME REPRODUCING KERNELS FOR THE UNIT DISK**

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# SOME REPRODUCING KERNELS FOR THE UNIT DISK

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**Introduction.** Let  $S(t)$  denote the class of functions  $\varphi$  analytic in the unit disk  $U$  with center 0 and satisfying

$$(1) \quad \int_U |\varphi(z)| (1 - |z|^2)^t dx dy < \infty \quad (z = x + iy)$$

for  $t$  real. In this paper we shall prove that for  $\lambda$  and  $\nu$  properly restricted,  $|\zeta| < 1$  and  $\varphi \in S(t)$ , the following formulas are valid:

$$(2) \quad \varphi(\zeta) = \frac{(\lambda + 1)^\nu}{\Gamma(\nu) \pi} \int_U \int \frac{\varphi(z) (1 - |z|^2)^\lambda}{(1 - \bar{z}\zeta)^{\lambda+2}} \ln^{\nu-1} \left( \frac{1 - \bar{z}\zeta}{1 - |z|^2} \right) dx dy ,$$

and

$$(3) \quad \varphi^{(m)}(\zeta) = \frac{\lambda + 1}{\pi} \iint \bar{z}^m \frac{\varphi(z) (1 - |z|^2)^\lambda}{(1 - \bar{z}\zeta)^{\lambda+2+m}} \sum_{i=0}^m a_i \ln^{\nu-1-i} \left( \frac{1 - \bar{z}\zeta}{1 - |z|^2} \right) dx dy ,$$

where the  $a_i$  are suitably chosen constants (with respect to  $\varphi$  and the variables  $z$  and  $\zeta$ ). Finally, if

$$(4) \quad \begin{aligned} F_n(\zeta, \nu, \lambda) = & \frac{(-1)^{n+1}}{\pi} \iint \frac{\varphi(z) (1 - |z|^2)^\lambda}{\bar{z}^n (1 - \bar{z}\zeta)^{\lambda+2-n}} \\ & \cdot \left[ \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 1)} \ln^{\nu+n-2} \left( \frac{1 - \bar{z}\zeta}{1 - |z|^2} \right) \right. \\ & \left. + \frac{1}{\Gamma(n)} \ln^{n-1} \left( \frac{1 - \bar{z}\zeta}{1 - |z|^2} \right) \right] dx dy , \end{aligned}$$

then  $F_n(\zeta, \nu, \lambda)$  has the property that

$$(5) \quad \frac{d^n}{d\zeta^n} F_n(\zeta, \nu, \lambda) = \varphi(\zeta) .$$

Formula (2) reduces to the well known results of Ahlfors [1] and Bergman [2] for particular choices of the parameters  $t$ ,  $\lambda$ , and  $\nu$ . The author is indebted to Professor Ahlfors for suggesting this problem.

*Notation.* Define

$$\begin{aligned} N(z, \lambda) &= (1 - |z|^2)^\lambda , \\ D(z, \zeta, \lambda) &= (1 - \bar{z}\zeta)^\lambda , \\ L(z, \zeta, \nu) &= \ln^{\nu-1} \left( \frac{1 - \bar{z}\zeta}{1 - |z|^2} \right) \end{aligned}$$

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where the principal values of the functions on the right are used.

*Reproducing Kernels.* In this section we shall prove

**THEOREM 1.** *If  $\varphi \in S(t)$  for some  $t$ , then*

- (a) *for  $Re \nu \geq 1$  and  $Re \lambda > t$ , (2) is satisfied and*  
 (b) *for  $Re \nu = 1$  and  $Re \lambda \geq t$ , (2) is satisfied.*

**REMARKS.** If

$$K_1(z, \zeta, \nu, \lambda) = \frac{(\lambda + 1)^\nu}{\Gamma(\nu)\pi} N(z, \lambda) D(z, \zeta, -\lambda, -2) L(z, \zeta, \nu),$$

then because  $|z| < 1$ ,  $|\zeta| < 1$  and principal values were used in defining  $N$ ,  $D$  and  $L$ ,  $K_1$  is unambiguously defined. Thus (2) can be written

$$(2') \quad \varphi(\zeta) = \int_{\sigma} \int \varphi(z) K_1(z, \zeta, \nu, \lambda) dx dy.$$

Also, if  $\varphi \in S(t)$  and  $\varphi \neq 0$ , then  $t > -1$  as is easily seen by considering (1) in polar coordinates.

The proof of Theorem 1 will be preceded by the statement and proof of three lemmas.

**LEMMA 1.** *For  $\varphi \in S(t)$ , and for  $Re \lambda \geq t$ , (1) implies*

$$\lim_{r \rightarrow 1} (1 - r^2)^{Re \lambda + 1} \int_0^{2\pi} |\varphi(re^{i\theta})| d\theta = 0.$$

*Proof.* If  $f(r) = \int_0^{2\pi} |\varphi(re^{i\theta})| d\theta$ , then  $f$  is a nondecreasing function of  $r$  for  $0 < r < 1$  (the trivial case of  $\varphi \equiv 0$  is excluded in the sequel). Suppose now that  $\limsup (1 - r^2)^{Re \lambda + 1} f(r) = a > 0$  ( $a$  may be infinite). Let  $0 < b < a$ . Then there exists a sequence  $\{r_i\}$  of real numbers,  $0 < r_{i-1} < r_i < 1$ , converging to 1 such that  $f(r) \geq b(1 - r^2)^{-(Re \lambda + 1)}$  for  $r > r_i$  and  $1 - r_i^2 < (1 - r_{i-1}^2)/2$ . Then (1) becomes

$$\begin{aligned} \int_0^1 \int_0^{2\pi} r(1 - r^2)^{Re \lambda} |\varphi(re^{i\theta})| dr d\theta &\geq \sum_{i=2}^{\infty} f(r_{i-1}) \int_{r_{i-1}}^{r_i} r(1 - r^2)^{Re \lambda} dr \\ &= \sum_{i=2}^{\infty} \frac{b}{Re \lambda + 1} \left[ 1 - \left( \frac{1 - r_i^2}{1 - r_{i-1}^2} \right)^{Re \lambda + 1} \right] \\ &\geq \sum_{i=2}^{\infty} \frac{b}{Re \lambda + 1} 1^i \left[ 1 - \left( \frac{1}{2} \right)^{Re \lambda + 1} \right] = \infty. \end{aligned}$$

This contradiction implies

$$\lim_{r \rightarrow 1} (1 - r^2)^{Re \lambda + 1} \int_0^{2\pi} |\varphi(re^{i\theta})| d\theta = 0.$$

LEMMA 2. If  $\varphi \in S(t)$  for some  $t$ ,  $Re\lambda > t$  and  $Re\nu \geq 1$ , then

$$(6) \quad \int_{\sigma} \int \varphi(z) K_1(z, \zeta, \nu, \lambda) dx dy = \int_{\sigma} \int \varphi(z) K_1(z, \zeta, \nu + 1, \lambda) dx dy .$$

*Proof.* Let  $K_1(\nu) = K_1(z, \zeta, \nu, \lambda)$ . Then

$$K_1(\nu) = [K_1(\nu) - K_1(\nu + 1)] + K_1(\nu + 1)$$

and if

$$f(z, \zeta, \nu, \lambda) = \frac{(\lambda + 1)^\nu}{\Gamma(\nu + 1) \pi} \frac{\varphi(z)}{z - \zeta} N(z, \lambda + 1) D(z, \zeta, -\lambda - 1) L(z, \zeta, \nu + 1),$$

then

$$\frac{\partial f}{\partial \bar{z}} = (K_1(\nu) - K_1(\nu + 1)) \varphi(z) .$$

We are, therefore, in a position to apply Green's formula since the singularity of  $f$  at  $z = \zeta$  is only apparent ( $\lim_{z \rightarrow \zeta} (z - \zeta)^{-1} L(z, \zeta, \nu + 1) = 0$ ). Thus for  $0 < r < 1$ ,

$$(7) \quad \int_{|z| < r} \int \varphi(z) K_1(\nu) dx dy = \frac{1}{2i} \int_{|z|=r} f(z, \zeta, \nu, \lambda) dz \\ + \int_{|z| < r} \int \varphi(z) K_1(\nu + 1) dx dy ,$$

and the lemma will be proved if we establish that the line integral in (7) vanishes as  $r \rightarrow 1$ . To show that this is the case, let  $\varepsilon > 0$  and  $t + \varepsilon < Re\lambda$ . Then

$$(8) \quad I_r = \frac{1}{2i} \int_{|z|=r} f(z, \zeta, \nu, \lambda) dz \\ = C \int_0^{2\pi} \frac{\varphi(re^{i\theta})}{re^{i\theta} - \zeta} N(r, \lambda + 1) D(re^{i\theta}, \zeta, -\lambda - 1) L(re^{i\theta}, \zeta, \nu + 1) re^{i\theta} d\theta ,$$

and for  $r$  near 1,

$$(9) \quad |I_r| \leq C_1 (1 - r^2)^{Re\lambda + 1 - \varepsilon/2} \int_0^{2\pi} |\varphi(re^{i\theta})| d\theta$$

where the factor  $(1 - r^2)^{\varepsilon/2}$  was used to suppress the logarithm near  $r = 1$ . On applying Lemma 1 in (9) we get

$$|I_r| \leq C_2 (1 - r^2)^{\varepsilon/2} ,$$

and the result follows.

LEMMA 2'. Lemma 2 is valid for  $Re\lambda \geq t$  if  $Re\nu = 1$ .

*Proof.* The proof of this lemma is similar to that of Lemma 2 except that the factor of  $(1 - r^2)^{\epsilon/2}$  is not needed to suppress the logarithm and, therefore, the range of  $\lambda$  can be extended.

LEMMA 3. *If  $Re\nu \geq k$ ,  $Re\lambda > -1$  and  $p$  is a positive integer, then*

$$(10) \quad \int_0^1 r^{2p-1} N(r, \lambda) L(r, 0, \nu - k + 1) dr \\ = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \frac{\Gamma(\nu - k + 1)}{2(\lambda + i + 1)^{\nu-k+1}}.$$

*Proof.* Induction on  $p$  will be used. If  $p = 1$ , (10) reads

$$\int_0^1 r N(r, \lambda) L(r, 0, \nu - k + 1) dr = \frac{\Gamma(\nu - k + 1)}{2(\lambda + 1)^{\nu-k+1}}.$$

Substituting

$$t = (\lambda + 1)L(r, 0, 2), \quad dt = (\lambda + 1) \frac{2r}{1 - r^2} dr$$

in the left hand side, we get

$$\int_0^1 r N(r, \lambda) L(r, 0, \nu - k + 1) dr = \frac{1}{2(\lambda + 1)^{\nu-k+1}} \int_0^\infty e^{-t} t^{\nu-k} dt$$

where the path of integration in the right hand member is the half line through the origin inclined at the angle  $\arg(\lambda + 1)$ . That integral is  $\Gamma(\nu - k + 1)$ , and the result is established for  $p = 1$ . Suppose that (10) has been proved for  $p - 1$ . The left hand side of (10) can be written in the form

$$\int_0^1 r^{2p-3} N(r, \lambda) L(r, 0, \nu - k + 1) dr - \int_0^1 r^{2p-3} N(r, \lambda + 1) L(r, 0, \nu - k + 1) dr \\ = \frac{\Gamma(\nu - k + 1)}{2(\lambda + 1)^{\nu-k+1}} + \sum_{i=1}^{p-2} (-1)^i \left[ \binom{p-2}{i+1} + \binom{p-2}{i} \right] \frac{\Gamma(\nu - k + 1)}{2(\lambda + 1 + i)^{\nu-k+1}} \\ + (-1)^{p-1} \frac{\Gamma(\nu - k + 1)}{2(\lambda + p)^{\nu-k+1}} = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} \frac{\Gamma(\nu - k + 1)}{2(\lambda + i + 1)^{\nu-k+1}}.$$

*Proof of Theorem 1.* This proof will be accomplished by showing that the  $m$ th derivative of  $\varphi$  evaluated at 0 is given by the  $m$ th derivative of (2) evaluated at 0. Induction will be used.

It is clear that (1) implies the absolute convergence of (2), and that if  $Re\lambda$  is large enough, differentiation with respect to  $\zeta$ ,  $\lambda$ , and  $\nu$  will commute with integration. Differentiating (2)  $m$  times with respect to  $\zeta$ , one gets

$$(11) \quad \varphi^{(m)}(\zeta) = \frac{\lambda + 1}{\pi} \int_{\sigma} \int_{\sigma} \bar{z}^m \varphi(z) N(z, \lambda) D(z, \zeta, -\lambda - 2 - m) \sum_{i=0}^m a_i L(z, \zeta, \nu - i) dx dy$$

if  $Re \nu \geq m + 1$  and the  $a_i$  are properly chosen constants.

Let  $F(\zeta) = \int_{\sigma} \int_{\sigma} \varphi(z) K_1(\nu) dx dy$ . Then  $F(0) = \int_{\sigma} \int_{\sigma} \varphi(z) K_1(z, 0, \nu, \lambda) dx dy$  which by (1) can be written

$$F(0) = \frac{(\lambda + 1)^{\nu}}{\Gamma(\nu)\pi} \int_0^1 r N(r, \lambda) L(r, 0, \nu) dr \int_0^{2\pi} \varphi(re^{i\theta}) d\theta \\ = \frac{2(\lambda + 1)^{\nu}}{\Gamma(\nu)} \varphi(0) \int_0^1 r N(r, \lambda) L(r, 0, \nu) dr .$$

By Lemma 3 this last integral is  $\Gamma(\nu)/2(\lambda + 1)^{\nu}$ , and the desired result follows.

Suppose now that  $Re \nu > 1$ . Because of a complication in the inductive hypothesis, it will also be necessary to show that  $F'(0) = \varphi'(0)$ . Notice, however, that if we differentiate  $F$  with respect to  $\zeta$  two terms arise, and in one of these the exponent of  $\ln$  is  $\nu - 2$ . If  $Re \nu < 2$ , this would cause trouble. This difficulty is avoided if we first apply Lemma 2 to  $F$  to write it in a form for which  $Re \nu \geq 2$ . Then

$$F'(0) = \frac{(\lambda + 1)^{\nu}}{\Gamma(\nu)\pi} \iint \bar{z} \varphi(z) N(z, \lambda) [(\lambda + 2) L(z, 0, \nu) - (\nu - 1) L(z, 0, \nu - 1)] dx dy .$$

By splitting this into two integrals and proceeding just as above, we derive

$$F'(0) = \varphi'(0) .$$

Suppose now that it has been established that  $F^{(p-1)}(0) = \varphi^{(p-1)}(0)$ . Use Lemma 2 to write  $F$  in a form for which  $Re \nu \geq p + 1$ .

Let the following be taken as the inductive hypothesis:

$$(12a) \quad F^{(p-1)}(0) = \varphi^{(p-1)}(0) ,$$

$$(12b) \quad a_0 + \sum_{i=1}^{p-1} a_i \frac{(\lambda + 1)^i}{(\nu - 1)(\nu - 2) \dots (\nu - i)} = (p - 1)! ,$$

and

$$(12c) \quad a_0 + \sum_{i=1}^{p-1} a_i \frac{(\lambda + k)^i}{(\nu - 1)(\nu - 2) \dots (\nu - i)} = 0$$

for  $k = 2, 3, \dots, p$ . When  $p = 2$ , (12a) was proved above. In this

case  $a_0 = \lambda + 2$  and  $a_1 = -(\nu - 1)$  so that both (12b) and (12c) are satisfied. Consider now  $F^{(p)}(0)$  when  $F^{(p-1)}(\zeta)$  is given by the right hand side of (11) with  $m = p - 1$ .

$$(13) \quad F^{(p)}(0) = \frac{(\lambda + 1)^\nu}{\Gamma(\nu)\pi} \int_{\sigma} \int \bar{z}^p \varphi(z) N(z, \lambda) \\ \left[ (\lambda + 1 + p) \sum_{i=0}^{p-1} a_i L(z, 0, \nu - i) - \sum_{i=0}^{p-1} a_i (\nu - i) L(z, 0, \nu - i - 1) \right] dx dy .$$

After some algebra (13) becomes

$$F^{(p)}(0) = \frac{2(\lambda + 1)^\nu}{p! \Gamma(\nu)} \varphi^{(p)}(0) \left[ b_0 \frac{\Gamma(\nu)}{2(\lambda + 1)^\nu} - b_1 \binom{p}{1} \frac{\Gamma(\nu)}{2(\lambda + 2)^\nu} \right. \\ \left. + \dots (-1)^p b_p \frac{\Gamma(\nu)}{2(\lambda + p + 1)^\nu} \right]$$

where

$$b_0 = a_0(\lambda + 1 + p) + \frac{\lambda + 1}{\nu - 1} [a_1(\lambda + 1 + p) - a_0(\nu - 1)] \\ + \frac{(\lambda + 1)^2}{(\nu - 1)(\nu - 2)} [a_2(\lambda + 1 + p) - a_1(\nu - 2)] \\ + \dots - a_{p-1}(\nu - p) \frac{(\lambda + 1)^p}{(\nu - 1)(\nu - 2) \dots (\nu - p)} \\ = (\lambda + 1 + p)(p - 1)! - (\lambda + 1)(p - 1)! \\ = p! \quad \text{by (12b)}$$

and

$$b_k = a_0(\lambda + 1 + p) + \frac{\lambda + k + 1}{\nu - 1} [a_1(\lambda + 1 + p) - a_0(\nu - 1)] \\ + \frac{(\lambda + k + 1)^2}{(\nu - 1)(\nu - 2)} [a_2(\lambda + 1 + p) - a_1(\nu - 2)] \\ + \dots - a_{p-1}(\nu - p) \frac{(\lambda + k + 1)^p}{(\nu - 1)(\nu - 2) \dots (\nu - p)} \\ = (\lambda + 1 + p)0 + (\lambda + k + 1)0 = 0 \text{ by (12c) for}$$

$k = 2, 3, \dots, p$ . It follows immediately that

$$F^{(p)}(0) = \varphi^{(p)}(0)$$

as was to be shown.

The case  $Re \nu = 1$ ,  $Re \lambda \geq t$  is treated as above except that Lemma 2' is used in place of Lemma 2. The proof is omitted.

**REMARKS.** Notice that in proving Theorem 1 we have also established

that (11) is a correct formula for the  $m$ th derivative of  $\varphi$ .

As mentioned above we are also at liberty to differentiate (2) with respect to  $\nu$  and  $\lambda$ . It is readily verified that differentiating (2) with respect to  $\lambda$  and using the results of Theorem 1 yields

$$\varphi(\zeta) = \int_{\sigma} \int \varphi(z) K_1(\nu + 1) dx dy$$

which is nothing new. However, differentiating (2) with respect to  $\nu$  and using Theorem 1 we derive the new formula,

$$(14) \quad \varphi(\zeta) = \frac{(\lambda + 1)^\nu}{\Gamma'(\nu)\pi - \ln(\lambda + 1)\Gamma(\nu)\pi} \int_{\sigma} \int \varphi(z) N(z, \lambda) D(z, \zeta, -\lambda - 2) L(z, \zeta, \nu) \ln(L(z, \zeta, 2)) dx dy .$$

The integral in (14) is absolutely convergent in spite of the apparent difficulties with  $\ln(L)$ . Further derivations with respect to  $\zeta$ ,  $\nu$ , and  $\lambda$  are, of course, possible.

An interesting formula results from (11) for the case in which  $\lambda$  is an integer and  $\nu = 1$ . Here,  $a_0 = \Gamma(n + m + 1)/\Gamma(n + 1)$  and the rest of the  $a$ 's are zero. The  $\theta$  integral is

$$\int_0^{2\pi} (re^{-i\theta})^m \frac{\varphi(re^{i\theta})}{(1 - re^{-i\theta}\zeta)^{m+n+2}} d\theta = 2\pi \frac{r^{2m}}{(m + n + 1)!} [z^{n+2}\varphi(z)]_{z=r^2\zeta}^{(m+n+1)},$$

and (11) becomes

$$\varphi^{(m)}(\zeta) = \frac{2}{n!} \int_0^1 r^{2m+1} (1 - r^2)^n [z^{n+2}\varphi(z)]_{z=r^2\zeta}^{(m+n+1)} dr .$$

This expression is readily checked for  $\varphi(z) = z^k$  and, thereby, for any  $\varphi \in S(n)$ .

*Primitive Kernels.* In this section we shall prove

**THEOREM 2.** *If  $\varphi \in S(t)$  and*

$$K_2^n(z, \zeta, \nu, \lambda) = \frac{(-1)^{n+1}}{\bar{z}^n \pi} N(z, \lambda) D(z, \zeta, -\lambda - 2 + n) \left[ \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 1)} L(z, \zeta, \nu + n - 1) + \frac{1}{\Gamma(n)} L(z, \zeta, n) \right],$$

then for  $Re \nu = 2$  and  $Re \lambda \geq t$  or  $Re \nu \geq 2$  and  $Re \lambda > t$ ,

$$(15) \quad F_n(\zeta, \nu, \lambda) = \int_{\sigma} \int \varphi(z) K_2^n(z, \zeta, \nu, \lambda) dx dy$$

has the property that  $F_n^{(n)}(\zeta, \nu, \lambda) = \varphi(\zeta)$  (differentiation is with respect to  $\zeta$ ). If  $Re \lambda \geq t$  and  $\nu = 1$ , then



$$(16) \quad H_n(\zeta, \lambda) = \iint \varphi(z) K_2^n(z, \zeta, 1, \lambda) dx dy$$

has the property that  $H_n^{(n)}(\zeta, \lambda) = 2\varphi(\zeta)$ .

*Proof.* The proof will be by induction. Consider  $F_1(\zeta)$ . To differentiate under the integral sign in (15) it is sufficient to show that the given and resulting integrals are absolutely convergent. However,

$$\int_{\sigma} \int |\varphi(z) K_2^1(z, \zeta, \nu, \lambda)| dx dy = \int_{|z| \leq r} + \int_{r < |z| < 1}.$$

The integral over the annulus offers no difficulty and for small  $r$ ,

$$|\varphi(z) K_2^1(z, \zeta, \nu, \lambda)| \leq C \frac{1}{r}$$

where  $C$  is constant. Thus

$$\int_{|z| \leq r} |\varphi(z) K_2^1(z, \zeta, \nu, \lambda)| dx dy \leq 2\pi r C.$$

Because  $Re \nu \geq 2$ , all of the integrals occurring after differentiation are absolutely convergent and, hence,

$$\begin{aligned} F_1'(\zeta, \nu, \lambda) &= \int_{\sigma} \int \varphi(z) \frac{\partial}{\partial \zeta} K_2^1(z, \zeta, \nu, \lambda) dx dy \\ &= \int_{\sigma} \int \varphi(z) [K_1(\nu) + K_1(1) - K_1(\nu - 1)] dx dy \\ &= \varphi(\zeta). \end{aligned}$$

Similarly  $H_1'(\zeta, \lambda) = 2\varphi(\zeta)$  and thus

$$H_1(\zeta, \lambda) = 2F_1(\zeta, \nu, \lambda) + C.$$

Suppose now that it has been established that for some  $n \geq 2$ ,

- (a)  $F_{n-1}(\zeta, \nu, \lambda)$  is an  $(n - 1)$ st primitive and
- (b)  $H_{n-1}(\zeta, \lambda) = 2F_{n-1}(\zeta, \nu, \lambda) + P(\zeta, \nu, \lambda)$  where

$P$  is a polynomial of degree  $n - 2$  in  $\zeta$ . The absolute convergence of the needed integrals can be established as above. Therefore, from (15) we get

$$\begin{aligned} (17) \quad F_n''(\zeta, \nu, \lambda) &= \frac{(-1)^{n+1}}{\pi} \int_{\sigma} \int \frac{\varphi(z)}{z^{n-1}} N(z, \lambda) D(z, \zeta, -\lambda - 1 + n) \\ &\quad \left[ (\lambda + 2 - n) \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 1)} L(z, \zeta, \nu + n - 1) \right. \\ &\quad + (\lambda + 2 - n) \frac{1}{\Gamma(n)} L(z, \zeta, n) \\ &\quad - \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 2)} L(z, \zeta, \nu + n - 2) \\ &\quad \left. - \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1) \right] dx dy. \end{aligned}$$

The last two terms in this square bracket yield  $F_{n-1}(\zeta, \nu, \lambda)$ . Now let us add and subtract  $2(\lambda + 2 - n)L(z, \zeta, n - 1)/[(\lambda + 1)\Gamma(n - 1)]$  to the first two terms to write them as

$$\begin{aligned} & \frac{\lambda + 2 - n}{\lambda + 1} \left[ \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 2)} L(z, \zeta, \nu + n - 2) + \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1) \right. \\ & \quad + \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 2)} L(z, \zeta, \nu + n - 2) + \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1) \\ & \quad \left. - \frac{2}{\Gamma(n - 1)} L(z, \zeta, n - 1) \right] \end{aligned}$$

where the first term comes from the first term of (17) with  $\nu$  replaced by  $\nu + 1$  and the third term comes from the second term of (17) with  $\nu = 2$ . Thus (17) yields

$$\begin{aligned} F'_n(\zeta, \nu, \lambda) &= F_{n-1}(\zeta, \nu, \lambda) - \frac{\lambda + 2 - n}{\lambda + 1} [F_{n-1}(\zeta, \nu + 1, \lambda) \\ & \quad + F_{n-1}(\zeta, 2, \lambda) - H_{n-1}(\zeta, \lambda)] \\ &= F_{n-1}(\zeta, \nu, \lambda) + Q(\zeta, \nu, \lambda) \end{aligned}$$

where  $Q$  is a polynomial of degree  $(n - 2)$  in  $\zeta$ .

To complete the inductive argument, it is necessary to show that  $H'_n(\zeta, \lambda) = 2 F_{n-1}(\zeta, \nu, \lambda) + P(\zeta, \nu, \lambda)$ .

$$(18) \quad \begin{aligned} H'_n(\zeta, \lambda) &= 2(-1)^{n+1} \int_U \int \frac{\mathcal{P}(z)}{\bar{z}^{n-1}} N(z, \lambda) D(z, \zeta, -\lambda - 1 + n) \\ & \quad \left[ \frac{\lambda + 2 - n}{\Gamma(n)} L(z, \zeta, n) - \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1) \right] dx dy. \end{aligned}$$

Using the same techniques as above, the square brackets can be written

$$\begin{aligned} & \frac{\lambda + 2 - n}{\lambda + 1} \left[ \frac{(\lambda + 1)^{\nu-1}}{\Gamma(\nu + n - 2)} L(z, \zeta, \nu + n - 2) + \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1) \right] \\ & \quad - \left( \frac{\lambda + 2 - n}{\lambda + 1} + 1 \right) \frac{1}{\Gamma(n - 1)} L(z, \zeta, n - 1) \end{aligned}$$

where  $\nu = 2$  in the first term. On placing this expression in (18), we get

$$H'_n(\zeta, \lambda) = -2 \left( \frac{\lambda + 2 - n}{\lambda + 1} \right) F_{n-1}(\zeta, 2, \lambda) + \left( \frac{\lambda + 2 - n}{\lambda + 1} + 1 \right) H_{n-1}(\zeta, \lambda).$$

By the inductive hypothesis,  $H_{n-1}(\zeta, \lambda) = 2 F_{n-1}(\zeta, \nu, \lambda) + R(\zeta, \nu, \lambda)$  where  $R$  is of degree  $(n - 2)$  in  $\zeta$ . We have then that

$$H'_n(\zeta, \lambda) = 2 F_{n-1}(\zeta, \nu, \lambda) + P(\zeta, \nu, \lambda)$$

where  $P$  is of degree  $(n - 2)$  in  $\zeta$ . This proves Theorem 2.

It is interesting to note that  $F_n$  and  $H_n$  depend analytically on  $\nu$  and  $\lambda$  and are not necessarily constants (with respect to these two variables).

It is easy to prove

**THEOREM 3.** *If (a)  $\varphi \in S(Re \lambda)$  and has a zero of order at least  $n$  at 0, (b) either  $\lambda$  is not an integer or  $\lambda$  is an integer greater than  $n - 2$ , (c)*

$$K_3^n = \frac{\lambda + 1}{\pi} \frac{\Gamma(\lambda + 3 - n)}{\Gamma(\lambda + 3)} \bar{z}^{-n} N(z, \lambda) D(z, \zeta, -\lambda - 2 + n)$$

and (d)

$$(19) \quad G_n(\zeta) = \int_{\sigma} \int \varphi(z) K_3^n(z, \zeta, \lambda) dx dy ,$$

then

$$G_n^{(n)}(\zeta) = \varphi(\zeta) .$$

The conditions imposed on  $\lambda$  are sufficient to guarantee that the integral (19) converges absolutely. The proof of the theorem is just a matter of differentiating and is omitted. If, however,  $\varphi \in S(Re \lambda)$ , then for each positive integer  $n$ ,  $z^n \varphi(z)$  is also in  $S(Re \lambda)$ , and, therefore, if we define

$$(20) \quad E_n(\zeta) = \int_{\sigma} \int z^n \varphi(z) K_3^n(z, \zeta, \lambda) dx dy ,$$

$E_n(\zeta)$  is well defined, absolutely convergent and has the property that

$$E_n^{(n)}(\zeta) = \zeta^n \varphi(\zeta) .$$

The simplicity of (20) may make it more useful than either (15) or (16) in some cases.

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