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1. Introduction. A function f defined on a set H of real numbers is *subadditive* on H if $f(x+y) \leqq f(x) + f(y)$ for all $x, y \in H$ such that $x + y \in H$. If the inequality is reversed, f is *superadditive.* This paper considers several problems in the extension of subadditive and monotone subadditive functions to domains of which the given H is a proper subset, and some related problems. First, extensions of a function from the set J of all nonnegative integers of the set $E = [0, \infty)$ will be discussed. Then extensions of functions from E to the set of all real numbers and extensions from an interval $[0, a]$ to $[0, b]$ $(b > a)$ and to E will be discussed. This last discussion will emphasize the *maximal extension* first defined by Bruckner [1] in the superadditive case, will treat the problem of convergence of sequences of such extensions, and will study the operator properties of the extension. Finally, an example will be considered which is relevant to a problem in extremal elements of cones of functions.

2. Extensions from the integers. Let f be a subadditive function defined on J. It has been shown that the *polygonal extension, F,* of f to E, obtained by joining consecutive points $(n, f(n))$ of the graph of f by straight line segments, is subadditive on E. [1]. It is easy to show that the left-continuous step function extension G, defined by $G(0) = f(0)$ and $G(x) = f(n)$ for all $x \in (n-1, n]$ $(n = 1, 2, \cdots)$ is subadditive on E when f is nondecreasing. These two extensions appear as the extreme cases of the class of extensions described in the following result.

THEOREM 1. *Let f be a nondecreasing subadditive function on J. Let g be a nondecreasing concave function on $[0, 1]$ with $g(0) = 0$ and $g(1) = 1$. The function F defined on E by*

$$F(x) = f([x]) + \{f([x+1]) - f([x])\}g(x - [x]) ,$$

where $[x]$ is the integer $x - 1 < [x] \leqq x$, is subadditive and nondecreasing on E.

Proof. The function F is obviously nondecreasing. To show subadditivity, let $x = m + u$ and $y = n + v$, where $m, n \in J$ and

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$u, v \in [0, 1)$. If $u + v = h \leqq 1$ and $g(u) + g(v) \leqq 1$, then g is subadditive on $[0, 1]$ since $g(x)/x$ is nonincreasing there, and

$$\begin{aligned} F(x + y) &\leqq f(m + n) + \{f(m + n + 1) - f(m + n)\}\{g(u) + g(v)\} \\ &\leqq \{f(m) + f(n)\}\{1 - g(u) - g(v)\} \\ &\quad + \{f(m + 1) + f(n)\}g(u) + \{f(m) + f(n + 1)\}g(v) \\ &\leqq f(m) + \{f(m + 1) - f(m)\}g(u) + f(n) \\ &\quad + \{f(n + 1) - f(n)\}g(v) = F(x) + F(y) . \end{aligned}$$

If $g(u) + g(v) > 1$, assume notation such that $f(n + 1) - f(n) \leqq f(m + 1) - f(m)$. Then $\{f(n + 1) - f(n)\}\{g(u) + g(v)\} + f(n) \geqq f(n + 1)$, and it follows that

$$\begin{aligned} F(x + y) &\leqq f(m + n + 1) \leqq f(m) \\ &\quad + \{f(n + 1) - f(n)\}\{g(u) + g(v)\} + f(n) \\ &\leqq f(m) + \{f(n + 1) - f(n)\}g(u) + f(n) \\ &\quad + \{f(n + 1) - f(n)\}g(v) \leqq F(x) + F(y) . \end{aligned}$$

If $u + v = 1 + h > 1$, the concavity of g yields the inequality $g(h) \leqq g(u) + g(v) - 1$. Then

$$\begin{aligned} F(x + y) &\leqq f(m + n + 1) \\ &\quad + \{f(m + n + 2) - f(m + n + 1)\}\{g(u) + g(v) - 1\} \\ &\leqq \{f(m) + f(n + 1)\}\{1 - g(u)\} + \{f(m + 1) + f(n)\}\{1 - g(v)\} \\ &\quad + \{f(m + 1) + f(n + 1)\}\{g(u) + g(v) - 1\} = F(x) + F(y) . \end{aligned}$$

Simple examples show that the theorem fails if either hypothesis on f or g is removed.

A similar but much simpler proof can be given for the following construction of periodic subadditive functions, which is suggested by the subadditivity of $|\sin x|$.

THEOREM 2. *Let g be concave and nonnegative on* $[0, 1)$. *The extension, F, of g to E defined by $F(x) = g(x - [x])$ is subadditive on E.*

The concavity of g is not necessary, even if $g(0) = g(1) = 0$, since the polygonal extension of the function defined on $\{0, 1, 2, 3, 4\}$ by $f(0) = f(4) = 0$, $f(1) = f(3) = 2$, and $f(2) = 1$ can be extended to E as a periodic subadditive function.

3. Extensions from E to R. It is the purpose of this section to mention some results on the extension of a subadditive function f, defined on $E = [0, \infty)$, to the whole line R. An idea of what not to expect is provided by theorems of Hille and Phillips [2], who show that a finite-valued subadditive function defined on $(0, \infty)$ has no finite subadditive extension to R if either $f(x) \to \infty$ as $x \to 0$ or if $f(x)/x \to -\infty$ as $x \to \infty$, and of Cooper [3], who has noted that every even $(f(x) = f(-x))$ subadditive function is nowhere negative. The following theorem completely characterizes even subadditive functions.

THEOREM 3. *Let f be subadditive on E. Then f can be extended to a subadditive even function on R if, and only if, $f(x-y) \leqq f(x) + f(y)$ for all $x \geqq y$ in E.*

THEOREM 4. *If f is nondecreasing and subadditive on E and nonincreasing and subadditive on $(-\infty, 0)$, then f is subadditive on R.*

COROLLARY. *Every nondecreasing subadditive function defined on E can be extended to R as an even subadditive function. Every nondecreasing subadditive function f on E can be extended as a subadditive function to R by $f(x) = 0$, or by $f(x) = f(0)$, for all $x < 0$.*

On the other hand, a nonincreasing subadditive function on E can be extended to R as an even subadditive function if, and only if, $\sup\{f(x) : x \in E\} \leqq 2(\inf\{f(x) : x \in E\})$.

4. Extensions from $[0, a]$ to E. The following discussion concerns the extension of a subadditive function f defined on the interval $[0, a]$, $a > 0$, to $E = [0, \infty)$ or to an interval $[0, b]$, $b > a$. The inclusion of the origin is sometimes convenient and often a nuisance. There is an obviously parallel theory for extensions from $J_k = \{0, 1, 2, \cdots, k\}$ to J which, together with Bruckner's theorem on polygonal extensions, provides a fruitful collection of examples in the continuous case. Two simple ways of extending monotone functions will be mentioned first.

THEOREM 5. *Let f be a nondecreasing subadditive function on $[0, a]$. Extend f by $F(x) = f(x)$ if $x \in [0, a]$ and by $F(x) = f(a)$ if $x > a$. Then F is subadditive on E.*

THEOREM 6. *Let f be nondecreasing and subadditive on $[0, a]$. Let g be defined by $g(x) = f(x)$ if $x \in [0, a]$, $g(x) = f(a)$ if $x \in (a, 2a]$, and $g(x) = f(a) + f(x - 2a)$ if $x \in (2a, 3a]$. Then g is subadditive on $[0, 3a]$.*

Proof. Note that g is also nondecreasing (which means that this construction can be repeated as often as desired). If $x, y, x + y \in [0, 2a]$, then g is subadditive by Theorem 5. If $x + y \in (2a, 3a]$, then, by cases, (1) if $x, y \in (a, 2a]$,

$$g(x+y) \leqq g(3a) = 2f(a) = g(x) + g(y) ;$$

(2) if $x \in [0, a]$ and $y \in (a, 2a]$, then $y - 2a \leqq 0$ and

$$g(x+y) = f(a) + f(x+y-2a) \leqq f(a) + f(x) = g(x) + g(y) ;$$

and (3) if $x \in [0, a]$ and $y \in (2a, 3a]$, then

$$\begin{aligned} g(x+y) &= f(a) + f(x+y-2a) \\ &\leqq f(a) + f(x) + f(y-2a) \leqq g(x) + g(y) . \end{aligned}$$

Attention now turns to the topic of maximal extensions. Let f be a subadditive function on the interval $[0, a]$. The function Sf defined at each $x \in E$ by $Sf(x) = \inf \Sigma f(x_i)$, where the infimum is taken over all finite collections $\{x_1, \cdots, x_n\}$ such that $0 \leqq x_i \leqq a$ $(i = 1, \cdots, n)$ and $x_1 + \cdots + x_n = x$ is called the *maximal subadditive extension* of f to E. Each collection $\{x_1, \cdots, x_n\}$ is called an *a-partition* of x. It is verifiable that Sf is subadditive and $Sf(x) \geqq F(x)$ for all x and all subadditive functions F on E which are extensions of f. For any given a-partition of x it can be shown, using $f(x_i + x_j) \leqq f(x_i) + f(x_j)$, that there exists an a-partition (called a *refinement* of the given one) which does not contain 0 and does contain at most one element $v \leqq a/2$—providing an approximation to $Sf(x)$ at least as good as the original with the additional feature of an upper bound on n. These ideas have been discussed by Bruckner [1] for the analogous case of minimal extensions of superadditive functions. Contrary to the spirit of that paper, assumptions of continuity are avoided in the following discussion.

THEOREM 7. *Let f be subadditive on $[0, a]$. Then f is nondecreasing on $[0, a]$ if, and only if, Sf is nondecreasing on E.*

Proof. Since f is the restriction of Sf to $[0, a]$ (denoted hereafter "$f = Sf \mid [0, a]$"), the monotonicity of f follows from that of Sf. Conversely, if Sf decreases, then there exist $x, y \in E$ such that $0 < y - x < a/2$ and $Sf(y) < Sf(x)$. If $y \in [0, a]$, then the argument is complete. If $y > a$, let $\varepsilon > 0$ be given. Let $\{y_1, \cdots, y_n\}$ be a refined a-partition of y such that $y_1 > a/2$ and $Sf(y) + \varepsilon > f(y_1) + \cdots + f(y_n)$. Let $z = y_1 - (y - x)$. Then $\{z, y_2, \cdots, y_n\}$ is an a-partition of x, so that $Sf(x) \leqq f(z) + f(y_2) + \cdots + f(y_n)$. Subtraction yields $Sf(y) - Sf(x) + \varepsilon > f(y_1) - f(z)$, implying that f decreases on $[0, a]$. A slight amendment of this argument verifies that "strictly increasing" may be substituted for "nondecreasing" in the theorem.

THEOREM 8. *If f is subadditive on $[0, a]$, if $0 < c < a$, and if $g = f \mid [0, c]$, then $Sg(x) \geqq Sf(x)$ for all $x \in E$. Also, $Sg = Sf$ if, and only if, $Sg \mid [0, a] = f$.*

The somewhat tedious proof of this theorem is omitted. These two theorems have served to emphasize the regularity of behavior of Sf. Treated as an operator on the set T of all subadditive functions on $[0, a]$, S is a monotone, positive-homogeneous, superadditive operator, and is additive on certain subsets of T. In particular, $S(f + g) = Sf + Sg$ if g is a nonnegative scalar multiple of f, or if f and g are concave and nonnegative at 0. The concave functions satisfy the condition of the following theorem, and it should be noted that the set of all functions satisfying the condition is closed under addition and that S is additive on this set.

THEOREM 9. *Let f be a subadditive function on $[0, a]$ such that $Sf(a + x) = f(a) + f(x)$ for all $x \in (0, a]$. Then $Sf(ma + x) = mf(a) + f(x)$ for all $m \in J$ and all $x \in (0, a]$.*

The proof of this theorem involves generating an a-partition of $y \in (a, \infty)$ of the form $\{a, a, \cdots, a, x\}$ from an arbitrary a-partition and using the hypothesis to show that it yields $Sf(ma + x)$. A similar method can be used to show that, if $Sf((n + 1)a + x) = f(a) + Sf(na + x)$ for some $n \in J$ and all $x \in (0, a]$, then $Sf(ma + x) = (m - n)f(a) + Sf(na + x)$ for all $m \geqq n$ and all $x \in (0, a]$.

5. Boundedness and convergence of maximal extensions.

The following theorem generalizes a result of Bruckner [1], who usually assumes continuity or differentiability.

THEOREM 10. *If f is a bounded subadditive function on $(0, a]$, if $m = \inf\{f(x)/x : x \in (0, a]\}$, and if $b = \sup\{f(x) - mx : x \in (0, a]\}$, then $mx \leqq Sf(x) \leqq mx + b$ for all $x \in (0, \infty)$.*

Proof. Since $f(2x) \leqq 2f(x)$ implies $f(2x)/2x \leqq f(x)/x$ for $x \in (0, a/2]$, only values of $x \in (a/2, a]$ need to be considered in finding a lower bound of $f(x)/x$. Since f is bounded, both m and b are easily shown to exist. Consider $\varepsilon > 0$ and $y \in (a, \infty)$. Let $\{x_1, \cdots, x_n\}$ be a refined a-partition for y such that $Sf(y) + \varepsilon \geqq f(x_1) + \cdots + f(x_n)$. Since $m \leqq f(x_i)/x_i$ $(i = 1, \cdots, n)$, $m \leqq (\Sigma f(x_i))/\Sigma x_i \leqq (Sf(y) + \varepsilon)/y$, or $my \leqq Sf(y) + \varepsilon$. Since ε is arbitrary, $my \leqq Sf(y)$.

There exists a unique integer p such that $y = pa/2 + z$, $0 \leqq z < a/2$. Let $t \in (a/2, a]$ such that $f(t)/t < m + \varepsilon/pa$. Then the integer r is determined such that $y = rt + z'$, where $0 \leqq z' < t$. Note that

$r \leqq p$. By definition, $Sf(y) \leqq rf(t) + f(z')$. Since $f(t) < tm + t\varepsilon/pa$ and $f(z') \leqq mz' + b$,

$$Sf(y) < r(tm + t\varepsilon/pa) + mz' + b$$
$$= m(rt + z') + b + tr\varepsilon/pa \leqq my + b + \varepsilon .$$

Thus $Sf(y) \leqq my + b$.

The proof of Bruckner's Theorem 3 can be used to show that, if $\{f_n\}$ is a sequence of continuous subadditive functions converging uniformly to the function f on $[0, a]$, then $\lim Sf_n = Sf$. (That f is subadditive follows, even for pointwise convergence, from a result stated in [2].) His proof makes use of the monotonicity of nonnegative superadditive functions to establish the uniform convergence. The statement, "f_n subadditive and $f_n - f$ imply $Sf_n \rightarrow Sf$," is false, even for continuous nonnegative functions. For example, if f_n is the polygonal extension to $[0, 1]$ of the function g_n defined by $g_n(1/2^n) = g_n(1 - 1/2^n) = 1/2$ and $g_n(k/2^n) = k/2^n$ $(k = 0, 2, 3, \cdots, 2^n - 2, 2^n)$, and if $f(x) = x$ on $[0, 1]$, then $f_n \rightarrow f$, $Sf(x) = x$ on E, but Sf_n is tending in the direction of $y = (x + 1)/2$ by Theorem 10. However, other conditions which imply $Sf_n \rightarrow Sf$ can be given.

THEOREM 11. *If $\{f_n\}$ is a sequence of subadditive functions on $[0, a]$ converging to f there, and if $f_n \geqq f$ for all n, then $Sf_n \rightarrow Sf$ on E.*

It is also noteworthy that the usual kinds of conditions implying uniform convergence can be modified for sequences of subadditive functions. In fact, a "classical" example, $nx/(1 + n^2x^2)$, of nonuniform convergence on E provides an example of a sequence of subadditive functions pertinent to Theorems 11 and 12.

THEOREM 12. *Let $\{f_n\}$ be a sequence of subadditive functions (not necessarily continuous) converging to the continuous function f on $[0, a]$ and such that there exists a real number $m \geqq 0$ such that $f_n(x) \leqq mx$ for all n and all $x \in [0, a]$. Then the convergence $f_n \rightarrow f$ is uniform on $[0, a]$.*

6. The Cantor function. Let K be the function defined on the complement of the Cantor "middle-third" set in $[0, 1]$ by $K(x) = 1/2$ if $x \in (1/3, 2/3)$, $K(x) = 1/4$ if $x \in (1/9, 2/9)$, $K(x) = 3/4$ if $x \in (7/9, 8/9)$, etc., and by the limit at points of the Cantor set. The function K is a frequently-used example in connection with continuity properties. To show that K is subadditive, let $K_n(x) = K(x)$ if x is in an interval which has been deleted from $[0, 1]$ at the nth stage in the formation of the Cantor set and extend K_n polygonally to $[0, 1]$ with $K_n(0) = 0$ and $K_n(1) = 1$. Theorem 6 may be applied to show that K_n is subadditive on $[0, 1]$. Since $\lim K_n = K$, K is subadditive. This example helps to illuminate the unsolved problem of characterizing the extremal elements of the convex cone of all nondecreasing subadditive functions on $[0, 1]$, for K is extremal. The other known extremal elements are of much simpler character.

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