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Given a function $\phi \in L_{\infty}(-\pi, \pi)$, the Toeplitz operator T_{ϕ} is the operator on H_2 (the set of $f \in L_2$ with Fourier series of the form $\sum_{0}^{\infty} c_n e^{in\theta}$) which consists of multiplication by ϕ followed by P, the natural projection of L_2 onto H_2 : if $f \sim \sum_{-\infty}^{\infty} c_n e^{in\theta}$ then $Pf \sim \sum_{0}^{\infty} c_n e^{in\theta}$. Succinctly,

$$T_{\phi}f = P(\phi f)$$
 $f \in H_2.$

In [5] a necessary and sufficient condition on ϕ was given for the invertibility of T_{ϕ} . This will be stated below. (The paper [5] is needlessly complicated. In a recent paper of Devinatz [1], however, all results of [5] and more are proved without undue complication in a general Dirichlet algebra setting.) Halmos [2] has posed the following as a test question for any theory of invertibility of Toeplitz operators: Is the spectrum of a Toeplitz operator necessarily connected? We shall shown here that the answer is Yes.

The proof consists mainly of applications of Theorem I of [5], which says the following.

A necessary and sufficient condition for the invertibility of T_{ϕ} is the existence of function ϕ_+ and ϕ_- belonging respectively to H_2 and \overline{H}_2 (the set of complex conjugates of H_2 functions) such that

(a) $\phi = \phi_+ \phi_-$,

(b) $\phi_{+}^{-1} \in H_2 \ and \ \phi_{-}^{-1} \in \overline{H}_2$,

(c) for $f \in L_{\infty}$, $Sf = \phi_{+}^{-1}P\phi_{-}^{-1}f \in L_2$, and $f \to Sf$ extends to a bounded operator on L_2 .

We don't want to prove the theorem here but we do have to say where the functions ϕ_{\pm} come from under the assumption that T_{ϕ} is ivertible. If we set

$$\psi_{+}=\,T_{\phi}^{_{-1}}$$
1, $ar{\psi}_{-}=\,T_{\phi}^{*-1}$ 1

then it can be shown that $\phi \psi_+ \psi_- = c$, a constant. We must have $c \neq 0$ since ψ_{\pm} can vanish only on sets of measure zero and ϕ is not identically zero. One then defines

$$\phi_+=1/\psi_+, \hspace{1em} \phi_-=c/\psi_-$$

and (a) and (b) hold.

As for the relevance of condition (c), it turns out that the ex-Received April 15, 1963. Sloan Foundation fellow. tension of S, restricted to H_2 , is exactly T_{ϕ}^{-1} . It follows that

$$(1) || (Pf)\phi_{-} ||_{2} \leq || \phi ||_{\infty} || T_{\phi}^{-1} || || f \phi_{-} ||_{2} f \in L_{\infty}.$$

Conversely, suppose there exists an M such that

$$\| (Pf) \phi_- \|_2 \leq M \, \| f \phi_- \|_2 \qquad \qquad f \in L_\infty.$$

Then we can deduce

$$\| \, \phi_{+}^{-1} P \phi_{-}^{-1} f \, \|_{_{2}} \leq M \, \| \, \phi_{-}^{-1} \, \|_{_{\infty}} \, \| f \, \|_{_{2}} \qquad \qquad f \in L_{\infty}.$$

It is a simple consequence of (c) that $||\phi^{-1}||_{\infty} < \infty$. (See [5], Theorem I, corollary, or [1], Lemma 2.) Thus (c) may be replaced by

(c') $\phi^{-1} \in L_{\infty}$ and the map $f \rightarrow Pf$ is bounded in the space $L_2(|\phi_-|^2 d\theta)$.

We shall need this fact.

To begin the proof of the connectedness of $\sigma(T_{\phi})$, the spectrum of T_{ϕ} , let Δ be a compact set disjoint from $\sigma(T_{\phi})$. (Think of Λ as being a simple closed curve surrounding a portion of $\sigma(T_{\phi})$.) For each $\lambda \in \Delta$ the operator $T_{\phi} - \lambda = T_{\phi-\lambda}$ is invertible, so we have the corresponding functions

$$\psi_+(\lambda)=(T_\phi-\lambda)^{-1}\mathbf{1},\ \ ar\psi_-(\lambda)=(T_\phi-\lambda)^{st-1}\mathbf{1}$$

and the constant $c(\lambda)$ as described above, and

(2)
$$\phi - \lambda = \phi_{+}(\lambda)\phi_{-}(\lambda)$$

where

$$\phi_+(\lambda)=1/\psi_+(\lambda)$$
 , $\phi_-(\lambda)=c(\lambda)/\psi_-(\lambda)$,

Let us consider the continuity of these various function of λ . It follows from the definition of $\psi_{\pm}(\lambda)$ and the continuity, in the uniform operator topology, of the mappings $\lambda \to (T_{\phi} - \lambda)^{-1}$ and $\lambda \to (T_{\phi} - \lambda)^{*-1}$, that $\lambda \to \psi_{\pm}(\lambda)$ are continuous functions from Λ to L_2 . This implies that $\lambda \to c(\lambda)/(\phi - \lambda)$ is continuous from Λ to L_1 . Since $\lambda \to \phi - \lambda$ is continuous from Λ to L_{∞} we conclude that $\lambda \to c(\lambda)$ is continuous from Λ to L_1 , so $c(\lambda)$ is a continuous complex valued function. Since $c(\lambda) \neq 0$ it follows also that $\lambda \to \phi_+(\lambda) = (\phi - \lambda)\psi_-(\lambda)/c(\lambda)$ and $\lambda \to \phi_-(\lambda) = (\phi - \lambda)\psi_+(\lambda)$ are continuous from Λ to L_2 . To recapitulate, the four functions $\phi_{\pm}(\lambda)^{\pm 1}$ are L_2 continuous.

The next step is to take logarithms. Since both $\phi_+(\lambda)$ and $1/\phi_+(\lambda)$ belong to H_2 , $\phi_+(\lambda)$ is an outer function. Recall that this means it has the representation

$$\phi_+(\lambda) = lpha_+(\lambda) e^{\log |\phi_+(\lambda)| + i [\log |\phi_+(\lambda)|]}$$

where the tilde denotes conjugate function and

$$lpha_+(\lambda) = \mathrm{sgn}\int \phi_+(\lambda) d heta$$

is a constant of absolute value 1. Since $\phi_+(\lambda)^{\pm 1}$ are L_2 continuous so is $\log |\phi_+(\lambda)|$, and therefore also $[\log |\phi_+(\lambda)|]^{\sim}$ (since $u \to \tilde{u}$ is L_2 continuous). The continuity of the complex valued function $\alpha_+(\lambda)$ follows from the fact that $\int \phi_+(\lambda) d\theta$ is continuous and nonzero.

Similarly we can write

$$\phi_{-}(\lambda) = \alpha_{-}(\lambda) e^{\log|\phi_{-}(\lambda)| - i [\log|\phi_{-}(\lambda)|]^{-1}}$$

with $\alpha_{-}(\lambda)$ continuous and nonzero. Putting our representations together and using (2) we have

$$(3) \qquad \phi - \lambda = \alpha(\lambda) e^{\log|\phi_{+}(\lambda)| + i [\log|\phi_{+}(\lambda)|]^{\sim}} e^{\log|\phi_{-}(\lambda)| - i [\log|\phi_{-}(\lambda)|]^{\sim}}$$

where $\alpha(\lambda) = \alpha_+(\lambda)\alpha_-(\lambda)$ is a continuous nowhere vanishing complex valued function.

The sum of the two exponents in (3), which we shall call $l(\lambda, \theta)$, is for each λ an element of L_2 , and the map $\lambda \rightarrow l(\lambda, \cdot)$ is L_2 continuous. It is important that we be able to say that for each θ (or almost every θ), $l(\lambda, \theta)$ is a continuous function of λ . This is false for general L_2 valued functions but in our situation something as good is true.

LEMMA 1. There is a null set $N \subset (-\pi, \pi)$ and a function $L(\lambda, \theta)$ defined on $\Lambda \times N'$ such that for each λ

$$L(\lambda, \theta) = l(\lambda, \theta) a.e.,$$

for each $\theta \in N'$

 $L(\lambda, \theta)$ is continuous in λ .

and for all $\lambda \in \Lambda$, $\theta \in N'$

$$\phi(heta) - \lambda = lpha(\lambda) e^{L(\lambda, heta)}$$
 .

Proof. First we make sure that ϕ is defined everywhere and that its range has positive distance from Λ . This we can do since Λ is a compact set disjoint from $R(\phi)$, the essential range of ϕ . (Recall that $T_{\phi-\lambda}$ invertible implies $(\phi - \lambda)^{-1} \in L_{\infty}$.)

Take $\lambda_0 \in \Lambda$ and let $L_0(\lambda_0, \theta)$ be a function of θ which equals $l(\lambda_0, \theta)$ a.e. and for which

$$\phi(\theta) - \lambda_0 = \alpha(\lambda_0) e^{L_0(\lambda_0, \theta)}$$

everywhere. Let $U = \{\lambda \in \Lambda : |\lambda - \lambda_0| < \delta\}$ be a neighborhood of λ_0 so small that $\lambda \in U$ implies

$$igg|rac{lpha(\lambda)}{lpha(\lambda_0)}-1igg|<1\ , \ igg|rac{\phi(heta)-\lambda}{\phi(heta)-\lambda_0}-1igg|<1\ , \ \ \ ext{all}\ heta.$$

We extend $L_0(\lambda_0 \theta)$ to a function defined on $U \times (-\pi, \pi)$ by

$$(\ 4\) \qquad \qquad L_{\scriptscriptstyle 0}(\lambda,\,\theta) = L_{\scriptscriptstyle 0}(\lambda_{\scriptscriptstyle 0},\,\theta) + \log \frac{\phi(\theta) - \lambda}{\phi(\theta) - \lambda_{\scriptscriptstyle 0}} - \log \frac{\alpha(\lambda)}{\alpha(\lambda_{\scriptscriptstyle 0})}$$

where the logarithms are defined by the usual power series. Clearly $L_0(\lambda, \theta)$ is continuous on U for each θ and $\phi(\theta) - \lambda = \alpha(\lambda)e^{L_0(\lambda,\theta)}$ everywhere on $U \times (-\pi, \pi)$. We shall show $L_0(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U$, at least if δ is small enough. Let us set

$$u_{+}(\lambda) = \frac{\phi_{+}(\lambda)}{\alpha_{+}(\lambda)} = e^{\log|\phi_{+}(\lambda)| + i[\log|\phi_{+}(\lambda)|]^{\sim}}$$
$$u_{-}(\lambda) = \frac{\phi_{-}(\lambda)}{\alpha_{-}(\lambda)} = e^{\log|\phi_{-}(\lambda)| - i[\log|\phi_{-}(\lambda)|]^{\sim}}$$

and

$$v_{\pm}(\lambda) = e^{1/2L_0(\lambda \cdot \theta) \pm i/2\widetilde{L}_0(\lambda, \theta)}$$

We know $u_+(\lambda)^{\pm 1} \in L_2$. Actually for each λ , $u_+(\lambda)^{\pm 1} \in L_p$ for some p > 2 (the *p* depending on λ). The reason is the following. Condition (c') in the criterion given above for invertibility implies that the map $f \to Pf$ is bounded in the space $L_2(|u_-(\lambda)|^2 d\theta)$. Helson and Szegö have determined ([3], Theorem 1) all measures $d\mu$ such that $f \to Pf$ is bounded in $L_2(d\mu)$. They are measures of the form

$$d\mu=e^{
ho+\widetilde{\sigma}}d heta$$

with $\rho \in L_{\infty}$ and $||\sigma||_{\infty} < \pi/2$. However

$$||\sigma||_{\infty} < rac{\pi}{2} ext{ implies } e^{\widetilde{\sigma}} \in L_1$$
 .

This is a theorem of Zygmund. (See [6], p. 257.) A statement which is only at first glance stronger is

$$||\,\sigma\,||_{\scriptscriptstyle \infty} < rac{\pi}{2} ext{ implies } e^{\pm \widetilde{\sigma}} \in L_{\scriptscriptstyle 1+arepsilon} ext{ for some } arepsilon > 0$$
 .

Putting these things together we can conclude that $u_{-}(\lambda)^{\pm 1} \in L_{p}$ for

some p>2, and so also $u_+(\lambda)^{\pm 1} \in L_p$.

Since $L_0(\lambda_0, \theta) = l(\lambda_0, \theta)$ a.e., a routine check shows $|v_+(\lambda_0)| = c |u_+(\lambda_0)|$ a.e., where c is a nonzero constant, so we have $v_+(\lambda_0)^{\pm 1} \in L_{p_0}$. We shall show from this that $v_+(\lambda)^{\pm 1} \in L_2$ for all $\lambda \in U$ is δ is sufficiently small. We have

$$rac{v_+(\lambda)}{v_+(\lambda_0)}=e^{1/2[L_0(\lambda, heta)-L_0(\lambda_0, heta)]}e^{i/2[\widetilde{L}_0(\lambda, heta)-\widetilde{L}_0(\lambda_0, heta)]}\;.$$

It follows from (4) that

$$\lim_{\lambda o\lambda_0}||\,L_{\scriptscriptstyle 0}\!(\lambda,\, heta)-L_{\scriptscriptstyle 0}\!(\lambda_{\scriptscriptstyle 0},\, heta)\,||_{\scriptscriptstyle \infty}=0$$
 .

Therefore, from Zygmund's theorem again, we can say this: given any $q_0 < \infty$ there exists a δ so that $v_+(\lambda)/v_+(\lambda_0) \in L_{q_0}$ whenever $|\lambda - \lambda_0| < \delta$. If we choose q_0 so that $p_0^{-1} + q_0^{-1} = 1/2$ then we shall have $v_+(\lambda) \in L_2$. In fact me shall have $v_+(\lambda) \in H_2$. (Any function of the form $\exp(\sigma + i\tilde{\sigma})$, $\sigma \in L_2$, which belongs to L_2 also belongs to H_2 ; see [6], pp. 282-3.) Similarly

$$v_+(\lambda)^{-1} \in H_2$$
 and $v_-(\lambda)^{\pm 1} \in \overline{H}_2$.

Now almost everywhere

$$u_+(\lambda)u_-(\lambda)=v_+(\lambda)v_-(\lambda)\left(=rac{\phi-\lambda}{lpha(\lambda)}
ight)$$

so

$$rac{u_+(\lambda)}{v_+(\lambda)} = rac{v_-(\lambda)}{u_-(\lambda)}$$
.

The left side belongs to H_1 and the right to H_1 so both sides must be a constant $\beta = \beta(\lambda)$, and

$$rac{v_-(\lambda)}{v_+(\lambda)}=eta(\lambda)^2rac{u_-(\lambda)}{u_+(\lambda)}\;.$$

If we take the logarithm of the absolute value of both sides we obtain

$$[\mathscr{F}L_{\mathtt{0}}(\lambda, heta)]^{\sim}=2\log|eta(\lambda)|+\log|\phi_{-}(\lambda)|-\log|\phi_{+}(\lambda)|$$

and so

$$\mathscr{I}L_{0}(\lambda,\, heta)=[\log |\,\phi_{+}(\lambda)\,|]^{\sim}-[\log |\,\phi_{-}(\lambda)\,|]^{\sim}+\gamma(\lambda)$$

where $\gamma(\lambda)$ is, for each λ , a constant. Since

$$\mathscr{R}L_{0}(\lambda, heta) = \log \left|rac{\phi(heta)-\lambda}{lpha(\lambda)}
ight| = \log |\phi_{+}(\lambda)| + \log |\phi_{-}(\lambda)|$$

we have upon adding,

$$L_0(\lambda, \theta) = l(\lambda, \theta) + i\gamma(\lambda)$$
 a.e.

Given a sequence $\lambda_n \to \lambda(\lambda_n, \lambda \in U)$ there is a subsequence $\lambda_{n'}$ for which $l(\lambda_{n'}, \theta) \to l(\lambda, \theta)$ a.e. (This follows from the L_2 continuity of l.) Since $L_0(\lambda_n, \theta) \to L_0(\lambda, \theta)$ everywhere we have $\gamma(\lambda_{n'}) \to \gamma(\lambda)$. This shows that γ is a continuous function of λ . Since $\gamma(\lambda_0) = 0$ (recall that by definition, $L_0(\lambda_0, \theta) = l(\lambda_0, \theta)$ a.e.) and γ is for each λ an integral multiple of 2π , we must have $\gamma(\lambda) = 0$. Thus $L_0(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U$.

Because of what we have done and the compactness of Λ we can find a finite open covering $\{U_k\}$ of Λ and for each k a function $L_k(\lambda, \theta)$ defined on $U_k \times (-\pi, \pi)$ so that $L_k(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U_k$, $L_k(\lambda, \theta)$ is continuous on U_k for each θ , and $\phi(\theta) - \lambda = \alpha(\lambda)e^{L_k(\lambda,\theta)}$ on $U_k \times (-\pi, \pi)$. Consider a pair of these open sets U_j and U_k , and let $\lambda_1, \lambda_2, \cdots$ be dense in $U_j \cap U_k$. For each λ_n there is a θ -set E_n of measure zero outside of which both $L_j(\lambda_n, \theta)$ and $L_k(\lambda_n, \theta)$ equal $l(\lambda_n, \theta)$. Thus if θ does not belong to $\bigcup E_n$ we have $L_j(\lambda_n, \theta) = L_k(\lambda_n, \theta)$ for all n. By the continuity of L_j and L_k in λ and the density of $\{\lambda_n\}$ we conclude that $L_j(\lambda, \theta) = L_k(\lambda, \theta)$ for all $\lambda \in U_j \cap U_k$ as long as θ does not belong to the set $F_{j,k} = \bigcup E_n$. Thus as long as θ does not belong to the set $N = \bigcup_{j,k} F_{j,k}$ any two of the functions $L_k(\lambda, \theta)$ agree where they are both defined. We can therefore combine all the L_k to define a single function $L(\lambda, \theta)$ on $\Lambda \times N'$ which has all the required properties.

LEMMA 2. If Λ is a simple closed curve disjoint from $\sigma(T_{\phi})$ then $R(\phi)$, the essential range of ϕ , lies entirely inside or entirely outside Λ .

Proof. Lemma 1 says that $\phi(\theta) - \lambda = \alpha(\lambda)e^{L(\lambda,\theta)}$ where $L(\lambda, \theta)$ is continuous in λ for each $\theta \in N'$. For each θ the index (winding number) of Λ with respect to $\phi(\theta)$ is the index of $-\alpha(\lambda)$ with respect to the origin, and so is independent of θ . But the index is 1 if $\phi(\theta)$ is inside Λ and 0 if $\phi(\theta)$ is outside Λ , and this establishes the lemma.

LEMMA 3. If Λ is a simple closed curve disjoint from $\sigma(T_{\phi})$ and such that $R(\phi)$ lies entirely outside Λ , then $\sigma(T_{\phi})$ lies entirely outside Λ .

Proof. Write

$$\phi(heta) - \lambda = e^{L(\lambda, heta) + \log lpha(\lambda)}$$

 $\lambda \in \Lambda, \ \theta \in N'$

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where $\log \alpha(\lambda)$ denotes a continuous logarithm of $\alpha(\lambda)$. This exists since $\alpha(\lambda)$ has index zero. Let $d\mu_z$ be the Borel measure on Λ which solves the interior Dirichlet problem, i.e., if f is a continuous function on Λ then $\int f(\lambda) d\mu_z(\lambda)$ is the value at the point z inside Λ of the function harmonic inside Λ , continuous on the union of Λ and its inside, and equal to f on Λ . Now $L(\lambda, \theta) + \log \alpha(\lambda)$ is (for fixed $\theta \in N'$) a continuous logarithm of $\phi(\theta) - \lambda$. Since $\phi(\theta)$ is outside Λ this can be extended to a continuous logarithm of $\phi(\theta) - z$ for zinside Λ . The extension is a harmonic function, so

$$\int [L(\lambda, heta) + \log lpha(\lambda)] d\mu_z(\lambda)$$

is the value of the extension at z. Consequently

(5)
$$\phi(\theta) - z = e^{\int [L(\lambda,\theta) + \log \alpha(\lambda)] d\mu_z(\lambda)}$$

The integral $I(\theta) = \int L(\lambda, \theta) d\mu_z(\lambda)$ is a pointwise integral, i.e., for each θ , $L(\lambda, \theta)$ is a Borel measurable function of λ and $I(\theta)$ is its integral. We prefer to think of it as a weak integral, i.e., I is the unique L_2 function which satisfies, for all $u \in L_2$,

$$(I, u) = \int (L(\lambda, \theta), u(\theta)) d\mu_z(\lambda)$$
.

This identity follows from Fubini's theorem. If we use the fact that $L(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each λ , we can write (5) as

$$\phi(\theta) - z = e^{\int \log \alpha(\lambda) d\mu_z(\lambda)} e^{\int \log |\phi_+(\lambda)| d\mu_z(\lambda) + i \int [\log |\phi_+(\lambda)|]^{\sim} d\mu_z(\lambda)}$$
$$\cdot e^{\int \log |\phi_-(\lambda)| d\mu_z(\lambda) - i \int [\log |\phi_-(\lambda)|]^{\sim} d\mu_z(\lambda)}$$

where all integrals are weak integrals. Now \sim commutes with integration respect to $d\mu_z(\lambda)$; this follows from the definition of \sim in terms of Fourier coefficients. Thus if we set

$$egin{aligned} A &= e \int^{\log a(\lambda) d\mu_x(\lambda)} \ t_+ &= \int \log | \, \phi_+(\lambda) \, | \, d\mu_z(\lambda) \ t_- &= \int \log | \, \phi_-(\lambda) \, | \, d\mu_z(\lambda) \end{aligned}$$

we have

$$\phi-z=Ae^{t_++i\widetilde{ extsf{t}}_+}e^{t_--i\widetilde{ extsf{t}}_-}$$
 .

We shall show that this factorization exhibits the invertibility of $T_{\phi} - z$. Set

$$\phi_+ = A e^{t_+ + \widetilde{ ext{it}}_+}$$
 , $\phi_- = e^{t_- - \widetilde{ ext{it}}_-}$

We must verify that $\phi_{+}^{\pm 1} \in H_2$, that $\phi_{-}^{\pm 1} \in \overline{H}_2$, and that the map $f \to Pf$ is bounded in $L_2(|\phi_-|^2 d\theta)$.

The following fact is crucial. If $w_1, w_2 \ge 0$ satisfy

$$\int |Pf|^2 w_i d\theta \leq M \int |f|^2 w_i d\theta \qquad (i = 1, 2)$$

for all $f \in L_{\infty}$, and $w = w_1^{\alpha} w_2^{1-\alpha} (0 \leq \alpha \leq 1)$, then also

$$\int \mid Pf \mid^{_{2}} w \, d heta \leq M \int \mid f \mid^{_{2}} w \, d heta$$
 .

This follows from an interpolation theorem first proved for general operators and weight functions by Stein ([4], Theorem 2). We shall need an extension of this theorem to families of weight functions, and for convenience we state this extension together with another little fact as,

SUBLEMMA. Assume $\lambda \rightarrow r(\lambda, \theta)$ is continuous from the compact set Λ to real L_2 and such that for all λ

$$\int e^{r(\lambda, heta)}d heta \leq K$$
 .

Let μ be a nonnegative Borel measure on Λ with $\mu(\Lambda) = 1$. Then

$$\int e^{\int r(\lambda, heta)d\mu(\lambda)}d heta\leq K$$
 .

If in addition

$$\int |Pf|^2 e^{r(\lambda, heta)} d heta \leq M \int |f|^2 e^{r(\lambda, heta)} d heta$$

for all $f \in L_{\infty}$, then also

$$\int |Pf|^2 e^{\int r(\lambda, heta)d\mu(\lambda)}d heta \leq M \int |f|^2 e^{\int r(\lambda, heta)d\mu(\lambda)}d heta$$
 .

Suppose for the moment that this has been established. If we apply the first part of the sublemma to the four functions $\pm \log |\phi_{\pm}(\lambda)|^2$ and recall that by continuity the norms $||\phi_{\pm}(\lambda)^{\pm 1}||_2$ are uniformly bounded on Λ , we conclude that

$$e^{\pm t_{\pm}} = e^{\int \log |\phi_{\pm}(\lambda)|^{\pm 1} d\mu_{z}(\lambda)}$$

belong to L_2 , and so $\phi_+^{\pm 1} \in H_2$ and $\phi_-^{\pm 1} \in \overline{H}_2$. Next it follows from (c')

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of the criterion for invertibility and the fact that $T_{\varphi} - \lambda$ is invertible for each $\lambda \in A$ that

$$\int \mid Pf \mid^2 \mid \phi_-(\lambda) \mid^2 d heta \, \leq M \int \mid f \mid^2 \mid \phi_-(\lambda) \mid^2 d heta$$

for all $f \in L_{\infty}$; *M* can be chosen independently of λ since Λ is bounded away from $\sigma(T_{\phi})$. (See (1).) Therefore, by the sublemma again,

$$\int \mid Pf \mid^2 e^{2t} - d heta \, \leq \, M \int \mid f \mid^2 e^{2t} - d heta$$
 ,

i.e., $f \to Pf$ is bounded in $L_2(|\phi_-|^2 d\theta)$. This concludes the proof of invertibility of $T_{\phi} - z$. Since $T_{\phi} - z$ is invertible for any zinside Λ we conclude that $\sigma(T_{\phi})$ lies entirely outside Λ .

It remains to prove the sublemma. For each integer n let $E_{n,i}$ $(i = 1, 2, \dots)$ be a finite partition of Λ into Borel sets so that

$$|| r(\lambda, \theta) - r(\lambda', \theta) ||_2 < \frac{1}{n}$$

if λ, λ' belong to the same $E_{n,i}$. Choose points $\lambda_{n,i} \in E_{n,i}$ and set

$$egin{aligned} &w_n = \exp\left\{\sum\limits_i r(\lambda_{n,i},\, heta)\mu(E_{n,i})
ight\} \ &w = \exp\left\{\int\!\!\!\!\int\!\!r(\lambda,\, heta)d\mu(\lambda)
ight\}\,. \end{aligned}$$

It follows from (6) that $\log w_n \to \log w$ in L_2 and our problem is to justify various passages to the limit under the integral sign. It follows from Hölder's inequality that for each n we have $||w_n||_1 \leq K$. There is a sequence n' so that $w_{n'} \to w$ a.e., so Fatou's lemma gives $||w||_1 \leq K$. This is the first part of the sublemma.

The unextended interpolation theorem has a trivial generalization to arbitrary finite logarithmically convex combinations of weight functions. Since $0 \leq \mu(E_{n,i}) \leq 1$ and $\sum_i \mu(E_{n,i}) = \mu(A) = 1$ we can conclude that for each n

$$\int |Pf|^2 w_n d heta \leq M \int |f|^2 w_n d heta$$
 .

A slight modification of this which also follows from the unextended interpolation theorem is

(7)
$$\int |Pf|^2 w_n^{1-\varepsilon} w_1^{\varepsilon} d\theta \leq M \int |f|^2 w_n^{1-\varepsilon} w_1^{\varepsilon} d\theta$$

for all $\varepsilon(0 < \varepsilon < 1/2)$, n, f. (Here w_1 is just w_n with n = 1.) By Hölder's inequality $||w_n^{1-\varepsilon}w_1^{\varepsilon}||_1 \leq K$. This implies that $w_n^{1-2\varepsilon}$ have uniformly bounded norm in $L_p(w_1^{\varepsilon}d\theta)$, where $p = (1 - \varepsilon)/(1 - 2\varepsilon)$. Since $f \in L_{\infty}$ the functions $|f|^2 w_n^{1-2\varepsilon}$ also have uniformly bounded norm. Since p > 1 we can find a sequence n' so that $|f|^2 w_{n'}^{1-2\varepsilon}$ converge weakly to a function in $L_p(w_1^{\varepsilon}d\theta)$. But n' has a subsequence n'' so that $|f|^2 w_{n''}^{1-2\varepsilon}$ converges a.e. to $|f|^2 w^{1-2\varepsilon}$. It follows that

$$|f|^2 w_{n'}^{1-2arepsilon}
ightarrow |f|^2 w^{1-2arepsilon}$$

weakly. The conjugate space of $L_p(w_i^{\varepsilon}d\theta)$ is $L_q(w_i^{\varepsilon}d\theta)$ where $q = (1-\varepsilon)/\varepsilon$. Since $w_i^{\varepsilon} \in L_q(w_i^{\varepsilon}d\theta)$ it follows from the weak convergence that

$$(8) \qquad \qquad \int |f|^2 w_{n'}^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \longrightarrow \int |f|^2 w^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \ .$$

This holds of course if n' is replaced by any subsequence, in particular one such that $w_{n''} \to w$ a.e. Then (7) with ε replaced by 2ε , (8), and Fatou's lemma give

$$\int \mid Pf \mid^2 w^{1-2arepsilon} w_1^{2arepsilon} d heta \leqq \int \mid f \mid^2 w^{1-2arepsilon} w_1^{2arepsilon} d heta$$
 .

Since $w^{1-2\varepsilon}w_1^{2\varepsilon} \leq \max(w, w_1) \in L_1$ we can take the limit as $\varepsilon \to 0$ under the integral on the right, and apply Fatou's lemma to the integral on the left, to obtain the final conclusion of the sublemma.

Now we are in a position to prove, without much more difficulty, that $\sigma(T_{\phi})$ is connected. Suppose not. Then we can find a simple closed curve Λ , disjoint from $\sigma(T_{\phi})$, so that a non-empty portion of $\sigma(T_{\phi})$ lies inside Λ and a non-empty portion of $\sigma(T_{\phi})$ lies outside Λ . Call these portions σ_1 and σ_2 respectively. By Lemmas 2 and 3, $R(\phi)$ lies entirely inside Λ . Let Γ_{ε} be a simple closed curve surrounding a non-empty portion σ_3 of σ_2 and such that each point of Γ_{ε} is within ε of σ . Since σ_2 is contained in the convex hull of $R(\phi)$ (in fact all of $\sigma(T_{\phi})$ is; this will be explained in a moment) Γ_{ε} will be contained in the convex hull of Λ if ε is sufficiently small. Thus of the three possibilities for disjoint simple closed curves (Λ and Γ_{ε} will be disjoint is ε is small enough),

$$\begin{array}{l} \Lambda \text{ inside } \Gamma_{\varepsilon} \\ \Gamma_{\varepsilon} \text{ inside } \Lambda \end{array}$$

 Γ_{ε} , Λ have disjoint insides,

the first is eliminated since Γ_{ϵ} is contained in the convex hull of Λ , the second is eliminated since σ_{3} lies entirely outside Λ , and the third is eliminated by Lemma 3: since $R(\phi)$ lies outside Γ_{3} so does $\sigma(T_{\phi})$. The assumption that $\sigma(T_{\phi})$ is disconnected has led to a contradiction.

It remains to see why $\sigma(T_{\phi})$ is contained in the convex hull of $R(\phi)$. It suffices to show that T_{ϕ} is invertible if $R(\phi)$ is contained in an open angle of opening less than π with vertex 0, and since

invertibility of T_{ϕ} is not destroyed by multiplying ϕ by a nonzero constant we may assume that this angle has the positive real axis as bisector. But then for sufficiently small ε we shall have $||1 - \varepsilon \phi||_{\infty} < 1$, i.e. $||I - \varepsilon T_{\phi}|| < 1$, and this implies T_{ϕ} is invertible.

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