Pacific Journal of Mathematics

ON THE SPECTRUM OF A TOEPLITZ OPERATOR

HAROLD WIDOM

Vol. 14, No. 1

May 1964

ON THE SPECTRUM OF A TOEPLITZ OPERATOR

HAROLD WIDOM

Given a function $\phi \in L_{\infty}(-\pi, \pi)$, the Toeplitz operator T_{ϕ} is the operator on H_2 (the set of $f \in L_2$ with Fourier series of the form $\sum_{0}^{\infty} c_n e^{in\theta}$) which consists of multiplication by ϕ followed by P, the natural projection of L_2 onto H_2 : if $f \sim \sum_{-\infty}^{\infty} c_n e^{in\theta}$ then $Pf \sim \sum_{0}^{\infty} c_n e^{in\theta}$. Succinctly,

$$T_{\phi}f = P(\phi f)$$
 $f \in H_2.$

In [5] a necessary and sufficient condition on ϕ was given for the invertibility of T_{ϕ} . This will be stated below. (The paper [5] is needlessly complicated. In a recent paper of Devinatz [1], however, all results of [5] and more are proved without undue complication in a general Dirichlet algebra setting.) Halmos [2] has posed the following as a test question for any theory of invertibility of Toeplitz operators: Is the spectrum of a Toeplitz operator necessarily connected? We shall shown here that the answer is Yes.

The proof consists mainly of applications of Theorem I of [5], which says the following.

A necessary and sufficient condition for the invertibility of T_{ϕ} is the existence of function ϕ_+ and ϕ_- belonging respectively to H_2 and \overline{H}_2 (the set of complex conjugates of H_2 functions) such that

(a) $\phi = \phi_+ \phi_-$,

(b) $\phi_{+}^{-1} \in H_2$ and $\phi_{-}^{-1} \in \overline{H}_2$,

(c) for $f \in L_{\infty}$, $Sf = \phi_+^{-1}P\phi_-^{-1}f \in L_2$, and $f \rightarrow Sf$ extends to a bounded operator on L_2 .

We don't want to prove the theorem here but we do have to say where the functions ϕ_{\pm} come from under the assumption that T_{ϕ} is ivertible. If we set

$$\psi_{+}=T_{\phi}^{-1}\mathbf{1}, \ \overline{\psi}_{-}=T_{\phi}^{*-1}\mathbf{1}$$

then it can be shown that $\phi \psi_+ \psi_- = c$, a constant. We must have $c \neq 0$ since ψ_{\pm} can vanish only on sets of measure zero and ϕ is not identically zero. One then defines

$$\phi_+=1/\psi_+, \hspace{1em} \phi_-=c/\psi_-$$

and (a) and (b) hold.

As for the relevance of condition (c), it turns out that the ex-Received April 15, 1963. Sloan Foundation fellow. tension of S, restricted to H_2 , is exactly T_{ϕ}^{-1} . It follows that

$$(1) || (Pf) \phi_- ||_2 \leq || \phi ||_\infty || T_{\phi}^{-1} || || f \phi_- ||_2 f \in L_\infty.$$

Conversely, suppose there exists an M such that

$$\| (Pf) \phi_- \|_2 \leq M \| f \phi_- \|_2 \qquad \qquad f \in L_\infty.$$

Then we can deduce

$$\| \, \phi_+^{-1} P \phi_-^{-1} f \, \|_{_2} \leq M \, \| \, \phi^{-1} \, \|_{_\infty} \, \| \, f \, \|_{_2} \qquad \qquad f \in L_{_\infty}.$$

It is a simple consequence of (c) that $||\phi^{-1}||_{\infty} < \infty$. (See [5], Theorem I, corollary, or [1], Lemma 2.) Thus (c) may be replaced by

(c') $\phi^{-1} \in L_{\infty}$ and the map $f \rightarrow Pf$ is bounded in the space $L_2(|\phi_-|^2 d\theta)$.

We shall need this fact.

To begin the proof of the connectedness of $\sigma(T_{\phi})$, the spectrum of T_{ϕ} , let Λ be a compact set disjoint from $\sigma(T_{\phi})$. (Think of Λ as being a simple closed curve surrounding a portion of $\sigma(T_{\phi})$.) For each $\lambda \in \Lambda$ the operator $T_{\phi} - \lambda = T_{\phi-\lambda}$ is invertible, so we have the corresponding functions

 $\psi_+(\lambda)=(\,T_\phi-\lambda)^{-1}\mathbf{1},\ \ ar\psi_-(\lambda)=(\,T_\phi-\lambda)^{st-1}\mathbf{1}$

and the constant $c(\lambda)$ as described above, and

(2)
$$\phi - \lambda = \phi_+(\lambda)\phi_-(\lambda)$$

where

$$\phi_+(\lambda)=1/\psi_+(\lambda)$$
 , $\phi_-(\lambda)=c(\lambda)/\psi_-(\lambda)$.

Let us consider the continuity of these various function of λ . It follows from the definition of $\psi_{\pm}(\lambda)$ and the continuity, in the uniform operator topology, of the mappings $\lambda \to (T_{\phi} - \lambda)^{-1}$ and $\lambda \to (T_{\phi} - \lambda)^{*-1}$, that $\lambda \to \psi_{\pm}(\lambda)$ are continuous functions from Λ to L_2 . This implies that $\lambda \to c(\lambda)/(\phi - \lambda)$ is continuous from Λ to L_1 . Since $\lambda \to \phi - \lambda$ is continuous from Λ to L_{∞} we conclude that $\lambda \to c(\lambda)$ is continuous from Λ to L_1 , so $c(\lambda)$ is a continuous complex valued function. Since $c(\lambda) \neq 0$ it follows also that $\lambda \to \phi_+(\lambda) = (\phi - \lambda)\psi_-(\lambda)/c(\lambda)$ and $\lambda \to \phi_-(\lambda) = (\phi - \lambda)\psi_+(\lambda)$ are continuous from Λ to L_2 . To recapitulate, the four functions $\phi_{\pm}(\lambda)^{\pm 1}$ are L_2 continuous.

The next step is to take logarithms. Since both $\phi_+(\lambda)$ and $1/\phi_+(\lambda)$ belong to H_2 , $\phi_+(\lambda)$ is an outer function. Recall that this means it has the representation

$$\phi_+(\lambda) = lpha_+(\lambda) e^{\log |\phi_+(\lambda)| + i [\log |\phi_+(\lambda)|]}$$

where the tilde denotes conjugate function and

$$lpha_+(\lambda) = \mathrm{sgn}\int \phi_+(\lambda) d heta$$

is a constant of absolute value 1. Since $\phi_+(\lambda)^{\pm 1}$ are L_2 continuous so is $\log |\phi_+(\lambda)|$, and therefore also $[\log |\phi_+(\lambda)|]^{\sim}$ (since $u \to \tilde{u}$ is L_2 continuous). The continuity of the complex valued function $\alpha_+(\lambda)$ follows from the fact that $\int \phi_+(\lambda) d\theta$ is continuous and nonzero.

Similarly we can write

$$\phi_{-}(\lambda) = lpha_{-}(\lambda) e^{\log |\phi_{-}(\lambda)| - i [\log |\phi_{-}(\lambda)|]^{\sim}}$$

with $\alpha_{-}(\lambda)$ continuous and nonzero. Putting our representations together and using (2) we have

$$(3) \qquad \phi - \lambda = \alpha(\lambda) e^{\log|\phi_{+}(\lambda)| + i [\log|\phi_{+}(\lambda)|]^{\sim}} e^{\log|\phi_{-}(\lambda)| - i [\log|\phi_{-}(\lambda)|]^{\sim}}$$

where $\alpha(\lambda) = \alpha_+(\lambda)\alpha_-(\lambda)$ is a continuous nowhere vanishing complex valued function.

The sum of the two exponents in (3), which we shall call $l(\lambda, \theta)$, is for each λ an element of L_2 , and the map $\lambda \rightarrow l(\lambda, \cdot)$ is L_2 continuous. It is important that we be able to say that for each θ (or almost every θ), $l(\lambda, \theta)$ is a continuous function of λ . This is false for general L_2 valued functions but in our situation something as good is true.

LEMMA 1. There is a null set $N \subset (-\pi, \pi)$ and a function $L(\lambda, \theta)$ defined on $\Lambda \times N'$ such that for each λ

$$L(\lambda, \theta) = l(\lambda, \theta) a.e.,$$

for each $\theta \in N'$

 $L(\lambda, \theta)$ is continuous in λ .

and for all $\lambda \in \Lambda$, $\theta \in N'$

$$\phi(\theta) - \lambda = \alpha(\lambda) e^{L(\lambda,\theta)}$$

Proof. First we make sure that ϕ is defined everywhere and that its range has positive distance from Λ . This we can do since Λ is a compact set disjoint from $R(\phi)$, the essential range of ϕ . (Recall that $T_{\phi-\lambda}$ invertible implies $(\phi - \lambda)^{-1} \in L_{\infty}$.)

Take $\lambda_0 \in A$ and let $L_0(\lambda_0, \theta)$ be a function of θ which equals $l(\lambda_0, \theta)$ a.e. and for which

$$\phi(heta) - \lambda_0 = lpha(\lambda_0) e^{L_0(\lambda_0, heta)}$$

everywhere. Let $U = \{\lambda \in \Lambda : |\lambda - \lambda_0| < \delta\}$ be a neighborhood of λ_0 so small that $\lambda \in U$ implies

$$igg|rac{lpha(\lambda)}{lpha(\lambda_0)}-1igg|<1 \ , \ igg|rac{\phi(heta)-\lambda}{\phi(heta)-\lambda_0}-1igg|<1 \ , \ \ \ ext{ all } heta.$$

We extend $L_0(\lambda_0 \theta)$ to a function defined on $U \times (-\pi, \pi)$ by

$$(4) L_0(\lambda, \theta) = L_0(\lambda_0, \theta) + \log \frac{\phi(\theta) - \lambda}{\phi(\theta) - \lambda_0} - \log \frac{\alpha(\lambda)}{\alpha(\lambda_0)}$$

where the logarithms are defined by the usual power series. Clearly $L_0(\lambda, \theta)$ is continuous on U for each θ and $\phi(\theta) - \lambda = \alpha(\lambda)e^{L_0(\lambda,\theta)}$ everywhere on $U \times (-\pi, \pi)$. We shall show $L_0(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U$, at least if δ is small enough. Let us set

$$egin{aligned} u_+(\lambda) &= rac{\phi_+(\lambda)}{lpha_+(\lambda)} = e^{\log|\phi_+(\lambda)|+i\lceil\log|\phi_+(\lambda)|]} & \ u_-(\lambda) &= rac{\phi_-(\lambda)}{lpha_-(\lambda)} = e^{\log|\phi_-(\lambda)|-i\lceil\log|\phi_-(\lambda)|]} & \end{aligned}$$

and

$$v_{\pm}(\lambda) = e^{1/2L_0(\lambda \mid oldsymbol{ heta}) \pm i/2\widetilde{L}_0(\lambda, oldsymbol{ heta})}$$
 .

We know $u_+(\lambda)^{\pm 1} \in L_2$. Actually for each λ , $u_+(\lambda)^{\pm 1} \in L_p$ for some p > 2 (the *p* depending on λ). The reason is the following. Condition (c') in the criterion given above for invertibility implies that the map $f \to Pf$ is bounded in the space $L_2(|u_-(\lambda)|^2 d\theta)$. Helson and Szegö have determined ([3], Theorem 1) all measures $d\mu$ such that $f \to Pf$ is bounded in $L_2(d\mu)$. They are measures of the form

$$d\mu=e^{
ho+\widetilde{\sigma}}d heta$$

with $\rho \in L_{\infty}$ and $||\sigma||_{\infty} < \pi/2$. However

$$||\sigma||_{\infty} < rac{\pi}{2} ext{ implies } e^{\widetilde{\sigma}} \in L_1$$
 .

This is a theorem of Zygmund. (See [6], p. 257.) A statement which is only at first glance stronger is

$$||\,\sigma\,||_{\infty}<rac{\pi}{2} ext{ implies } e^{\pm \widetilde{\sigma}} \in L_{\scriptscriptstyle 1+arepsilon} ext{ for some } arepsilon>0$$
 .

Putting these things together we can conclude that $u_{-}(\lambda)^{\pm 1} \in L_{p}$ for

some p>2, and so also $u_+(\lambda)^{\pm 1} \in L_p$.

Since $L_0(\lambda_0, \theta) = l(\lambda_0, \theta)$ a.e., a routine check shows $|v_+(\lambda_0)| = c |u_+(\lambda_0)|$ a.e., where c is a nonzero constant, so we have $v_+(\lambda_0)^{\pm 1} \in L_{p_0}$. We shall show from this that $v_+(\lambda)^{\pm 1} \in L_2$ for all $\lambda \in U$ is δ is sufficiently small. We have

$$rac{v_+(\lambda)}{v_+(\lambda_0)}=e^{1/2[{\it L}_0(\lambda, heta)-{\it L}_0(\lambda_0, heta)]}e^{i/2[\widetilde{\it L}_0(\lambda, heta)-\widetilde{\it L}_0(\lambda_0, heta)]}\;.$$

It follows from (4) that

$$\lim_{\lambda o\lambda_0}||\,L_{\scriptscriptstyle 0}\!(\lambda,\, heta)-L_{\scriptscriptstyle 0}\!(\lambda_{\scriptscriptstyle 0},\, heta)\,||_{\scriptscriptstyle \infty}=0$$
 .

Therefore, from Zygmund's theorem again, we can say this: given any $q_0 < \infty$ there exists a δ so that $v_+(\lambda)/v_+(\lambda_0) \in L_{q_0}$ whenever $|\lambda - \lambda_0| < \delta$. If we choose q_0 so that $p_0^{-1} + q_0^{-1} = 1/2$ then we shall have $v_+(\lambda) \in L_2$. In fact me shall have $v_+(\lambda) \in H_2$. (Any function of the form $\exp(\sigma + i\tilde{\sigma})$, $\sigma \in L_2$, which belongs to L_2 also belongs to H_2 ; see [6], pp. 282-3.) Similarly

$$v_+(\lambda)^{-1} \in H_2$$
 and $v_-(\lambda)^{\pm 1} \in \overline{H}_2$.

Now almost everywhere

$$u_+(\lambda)u_-(\lambda)=v_+(\lambda)v_-(\lambda)\left(=rac{\phi-\lambda}{lpha(\lambda)}
ight)$$

so

$$rac{u_+(\lambda)}{v_+(\lambda)}=rac{v_-(\lambda)}{u_-(\lambda)}\;.$$

The left side belongs to H_1 and the right to \overline{H}_1 so both sides must be a constant $\beta = \beta(\lambda)$, and

$$rac{v_-(\lambda)}{v_+(\lambda)}=eta(\lambda)^2rac{u_-(\lambda)}{u_+(\lambda)}\;.$$

If we take the logarithm of the absolute value of both sides we obtain

$$[\mathscr{F} L_{\scriptscriptstyle 0}(\lambda, heta)]^{\sim} = 2 \log |eta(\lambda)| + \log |\phi_{\scriptscriptstyle -}(\lambda)| - \log |\phi_{\scriptscriptstyle +}(\lambda)|$$

and so

$$\mathscr{F}L_{0}(\lambda, \theta) = [\log |\phi_{+}(\lambda)|]^{\sim} - [\log |\phi_{-}(\lambda)|]^{\sim} + \gamma(\lambda)$$

where $\gamma(\lambda)$ is, for each λ , a constant. Since

$$\mathscr{R}L_{\scriptscriptstyle 0}(\lambda, heta) = \log \left|rac{\phi(heta)-\lambda}{lpha(\lambda)}
ight| = \log |\phi_+(\lambda)| + \log |\phi_-(\lambda)|$$

we have upon adding,

$$L_0(\lambda, \theta) = l(\lambda, \theta) + i\gamma(\lambda)$$
 a.e.

Given a sequence $\lambda_n \to \lambda(\lambda_n, \lambda \in U)$ there is a subsequence $\lambda_{n'}$ for which $l(\lambda_{n'}, \theta) \to l(\lambda, \theta)$ a.e. (This follows from the L_2 continuity of l.) Since $L_0(\lambda_{n'}, \theta) \to L_0(\lambda, \theta)$ everywhere we have $\gamma(\lambda_{n'}) \to \gamma(\lambda)$. This shows that γ is a continuous function of λ . Since $\gamma(\lambda_0) = 0$ (recall that by definition, $L_0(\lambda_0, \theta) = l(\lambda_0, \theta)$ a.e.) and γ is for each λ an integral multiple of 2π , we must have $\gamma(\lambda) = 0$. Thus $L_0(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U$.

Because of what we have done and the compactness of Λ we can find a finite open covering $\{U_k\}$ of Λ and for each k a function $L_k(\lambda, \theta)$ defined on $U_k \times (-\pi, \pi)$ so that $L_k(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each $\lambda \in U_k$, $L_k(\lambda, \theta)$ is continuous on U_k for each θ , and $\phi(\theta) - \lambda =$ $\alpha(\lambda)e^{L_k(\lambda,\theta)}$ on $U_k \times (-\pi, \pi)$. Consider a pair of these open sets U_j and U_k , and let $\lambda_1, \lambda_2, \cdots$ be dense in $U_j \cap U_k$. For each λ_n there is a θ -set E_n of measure zero outside of which both $L_j(\lambda_n, \theta)$ and $L_k(\lambda_n, \theta)$ equal $l(\lambda_n, \theta)$. Thus if θ does not belong to $\bigcup E_n$ we have $L_j(\lambda_n, \theta) = L_k(\lambda_n, \theta)$ for all n. By the continuity of L_j and L_k in λ and the density of $\{\lambda_n\}$ we conclude that $L_j(\lambda, \theta) = L_k(\lambda, \theta)$ for all $\lambda \in U_j \cap U_k$ as long as θ does not belong to the set $F_{j,k} = \bigcup E_n$. Thus as long as θ does not belong to the set $N = \bigcup_{j,k} F_{j,k}$ any two of the functions $L_k(\lambda, \theta)$ agree where they are both defined. We can therefore combine all the L_k to define a single function $L(\lambda, \theta)$ on $\Lambda \times N'$ which has all the required properties.

LEMMA 2. If Λ is a simple closed curve disjoint from $\sigma(T_{\phi})$ then $R(\phi)$, the essential range of ϕ , lies entirely inside or entirely outside Λ .

Proof. Lemma 1 says that $\phi(\theta) - \lambda = \alpha(\lambda)e^{L(\lambda,\theta)}$ where $L(\lambda, \theta)$ is continuous in λ for each $\theta \in N'$. For each θ the index (winding number) of Λ with respect to $\phi(\theta)$ is the index of $-\alpha(\lambda)$ with respect to the origin, and so is independent of θ . But the index is 1 if $\phi(\theta)$ is inside Λ and 0 if $\phi(\theta)$ is outside Λ , and this establishes the lemma.

LEMMA 3. If Λ is a simple closed curve disjoint from $\sigma(T_{\phi})$ and such that $R(\phi)$ lies entirely outside Λ , then $\sigma(T_{\phi})$ lies entirely outside Λ .

Proof. Write

$$\phi(heta) - \lambda = e^{L(\lambda, \theta) + \log lpha(\lambda)}$$
 $\lambda \in A, \ \theta \in N'$

where $\log \alpha(\lambda)$ denotes a continuous logarithm of $\alpha(\lambda)$. This exists since $\alpha(\lambda)$ has index zero. Let $d\mu_z$ be the Borel measure on Δ which solves the interior Dirichlet problem, i.e., if f is a continuous function on Δ then $\int f(\lambda) d\mu_z(\lambda)$ is the value at the point z inside Δ of the function harmonic inside Δ , continuous on the union of Δ and its inside, and equal to f on Δ . Now $L(\lambda, \theta) + \log \alpha(\lambda)$ is (for fixed $\theta \in N'$) a continuous logarithm of $\phi(\theta) - \lambda$. Since $\phi(\theta)$ is outside Δ this can be extended to a continuous logarithm of $\phi(\theta) - z$ for zinside Λ . The extension is a harmonic function, so

$$\int [L(\lambda, heta) + \log lpha(\lambda)] d\mu_z(\lambda)$$

is the value of the extension at z. Consequently

(5)
$$\phi(\theta) - z = e^{\int [L(\lambda,\theta) + \log \alpha(\lambda)] d\mu_z(\lambda)}.$$

The integral $I(\theta) = \int L(\lambda, \theta) d\mu_z(\lambda)$ is a pointwise integral, i.e., for each θ , $L(\lambda, \theta)$ is a Borel measurable function of λ and $I(\theta)$ is its integral. We prefer to think of it as a weak integral, i.e., I is the unique L_2 function which satisfies, for all $u \in L_2$,

$$(I, u) = \int (L(\lambda, \theta), u(\theta)) d\mu_z(\lambda)$$
.

This identity follows from Fubini's theorem. If we use the fact that $L(\lambda, \theta) = l(\lambda, \theta)$ a.e. for each λ , we can write (5) as

$$\phi(\theta) - z = e^{\int \log \alpha(\lambda) d\mu_z(\lambda)} e^{\int \log |\phi_+(\lambda)| d\mu_z(\lambda) + i \int [\log |\phi_+(\lambda)|]^{\sim} d\mu_z(\lambda)} \cdot e^{\int \log |\phi_-(\lambda)| d\mu_z(\lambda) - i \int [\log |\phi_-(\lambda)|]^{\sim} d\mu_z(\lambda)}$$

where all integrals are weak integrals. Now \sim commutes with integration respect to $d\mu_z(\lambda)$; this follows from the definition of \sim in terms of Fourier coefficients. Thus if we set

$$egin{aligned} A &= e \int^{\log lpha(\lambda) d\mu_{z}(\lambda)} \ t_{+} &= \int \log | \, \phi_{+}(\lambda) \, | \, d\mu_{z}(\lambda) \ t_{-} &= \int \log | \, \phi_{-}(\lambda) \, | \, d\mu_{z}(\lambda) \end{aligned}$$

we have

$$\phi-z=Ae^{t_++i\widetilde{ extsf{t}}_+}e^{t_--i\widetilde{ extsf{t}}_-}$$
 .

We shall show that this factorization exhibits the invertibility of $T_{\phi} - z$. Set

$$\phi_+ = A e^{t_+ + i \widetilde{ extsf{t}}_+}$$
 , $\phi_- = e^{t_- - i \widetilde{ extsf{t}}_-}$.

We must verify that $\phi_{+}^{\pm 1} \in H_2$, that $\phi_{-}^{\pm 1} \in \overline{H}_2$, and that the map $f \to Pf$ is bounded in $L_2(|\phi_-|^2 d\theta)$.

The following fact is crucial. If $w_1, w_2 \ge 0$ satisfy

$$\int |Pf|^2 w_i d\theta \leq M \int |f|^2 w_i d\theta \qquad (i = 1, 2)$$

for all $f \in L_{\infty}$, and $w = w_1^{\alpha} w_2^{1-\alpha} (0 \leq \alpha \leq 1)$, then also

$$\int \mid Pf \mid^{_{2}} w \, d heta \leq M \int \mid f \mid^{_{2}} w \, d heta$$
 .

This follows from an interpolation theorem first proved for general operators and weight functions by Stein ([4], Theorem 2). We shall need an extension of this theorem to families of weight functions, and for convenience we state this extension together with another little fact as,

SUBLEMMA. Assume $\lambda \rightarrow r(\lambda, \theta)$ is continuous from the compact set Λ to real L_2 and such that for all λ

$$\int e^{r(\lambda, heta)}d heta \leq K$$
 .

Let μ be a nonnegative Borel measure on Λ with $\mu(\Lambda) = 1$. Then

$$\int e^{\int r(\lambda, heta)d\mu(\lambda)}d heta \leq K$$
 .

If in addition

$$\int |Pf|^2 e^{r(\lambda, heta)} d heta \leq M \int |f|^2 e^{r(\lambda, heta)} d heta$$

for all $f \in L_{\infty}$, then also

$$\int \mid Pf \mid^2 e^{\int r(\lambda, heta) d\mu(\lambda)} d heta \leq M \int \mid f \mid^2 e^{\int r(\lambda, heta) d\mu(\lambda)} d heta$$
 .

Suppose for the moment that this has been established. If we apply the first part of the sublemma to the four functions $\pm \log |\phi_{\pm}(\lambda)|^2$ and recall that by continuity the norms $||\phi_{\pm}(\lambda)^{\pm 1}||_2$ are uniformly bounded on Λ , we conclude that

$$e^{\pm t_{\pm}} = e \int^{\log|\phi_{\pm}(\lambda)| \pm 1} d\mu_{z}(\lambda)$$

belong to L_2 , and so $\phi_+^{\pm 1} \in H_2$ and $\phi_-^{\pm 1} \in \overline{H}_2$. Next it follows from (c')

of the criterion for invertibility and the fact that $T_{\varphi} - \lambda$ is invertible for each $\lambda \in A$ that

$$\int \mid Pf \mid^2 \mid \phi_-(\lambda) \mid^2 d heta \leq M \int \mid f \mid^2 \mid \phi_-(\lambda) \mid^2 d heta$$

for all $f \in L_{\infty}$; *M* can be chosen independently of λ since Λ is bounded away from $\sigma(T_{\phi})$. (See (1).) Therefore, by the sublemma again,

$$\int \mid Pf \mid^2 e^{2t} - d heta \, \leq M \int \mid f \mid^2 e^{2t} - d heta$$
 ,

i.e., $f \to Pf$ is bounded in $L_2(|\phi_-|^2 d\theta)$. This concludes the proof of invertibility of $T_{\phi} - z$. Since $T_{\phi} - z$ is invertible for any zinside Λ we conclude that $\sigma(T_{\phi})$ lies entirely outside Λ .

It remains to prove the sublemma. For each integer n let $E_{n,i}$ $(i = 1, 2, \dots)$ be a finite partition of Λ into Borel sets so that

$$|| r(\lambda, \theta) - r(\lambda', \theta) ||_2 < \frac{1}{n}$$

if λ, λ' belong to the same $E_{n,i}$. Choose points $\lambda_{n,i} \in E_{n,i}$ and set

$$egin{aligned} &w_n = \exp\left\{\sum\limits_i r(\lambda_{n,i},\, heta)\mu(E_{n,i})
ight\} \ &w = \exp\left\{\int\!\!\!\!\int\!\!r(\lambda,\, heta)d\mu(\lambda)
ight\}\,. \end{aligned}$$

It follows from (6) that $\log w_n \to \log w$ in L_2 and our problem is to justify various passages to the limit under the integral sign. It follows from Hölder's inequality that for each n we have $||w_n||_1 \leq K$. There is a sequence n' so that $w_{n'} \to w$ a.e., so Fatou's lemma gives $||w||_1 \leq K$. This is the first part of the sublemma.

The unextended interpolation theorem has a trivial generalization to arbitrary finite logarithmically convex combinations of weight functions. Since $0 \leq \mu(E_{n,i}) \leq 1$ and $\sum_i \mu(E_{n,i}) = \mu(A) = 1$ we can conclude that for each n

$$\int |Pf|^2 w_n d heta \leq M \int |f|^2 w_n d heta$$
 .

A slight modification of this which also follows from the unextended interpolation theorem is

(7)
$$\int |Pf|^2 w_n^{1-\varepsilon} w_1^\varepsilon d\theta \leq M \int |f|^2 w_n^{1-\varepsilon} w_1^\varepsilon d\theta$$

for all $\varepsilon(0 < \varepsilon < 1/2)$, n, f. (Here w_1 is just w_n with n = 1.) By Hölder's inequality $||w_n^{1-\varepsilon}w_1^{\varepsilon}||_1 \leq K$. This implies that $w_n^{1-2\varepsilon}$ have uniformly bounded norm in $L_p(w_1^{\varepsilon}d\theta)$, where $p = (1 - \varepsilon)/(1 - 2\varepsilon)$. Since $f \in L_{\infty}$ the functions $|f|^2 w_n^{1-2\varepsilon}$ also have uniformly bounded norm. Since p > 1 we can find a sequence n' so that $|f|^2 w_{n'}^{1-2\varepsilon}$ converge weakly to a function in $L_p(w_1^{\varepsilon}d\theta)$. But n' has a subsequence n'' so that $|f|^2 w_{n''}^{1-2\varepsilon}$ converges a.e. to $|f|^2 w^{1-2\varepsilon}$. It follows that

$$|f|^2 w_{n'}^{1-2arepsilon}
ightarrow |f|^2 w^{1-2arepsilon}$$

weakly. The conjugate space of $L_p(w_i^{\varepsilon}d\theta)$ is $L_q(w_i^{\varepsilon}d\theta)$ where $q = (1-\varepsilon)/\varepsilon$. Since $w_i^{\varepsilon} \in L_q(w_i^{\varepsilon}d\theta)$ it follows from the weak convergence that

(8)
$$\int |f|^2 w_{n'}^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \longrightarrow \int |f|^2 w^{1-2\varepsilon} w_1^{2\varepsilon} d\theta \ .$$

This holds of course if n' is replaced by any subsequence, in particular one such that $w_{n''} \to w$ a.e. Then (7) with ε replaced by 2ε , (8), and Fatou's lemma give

$$\int \mid Pf \mid^2 w^{1-2arepsilon} w_1^{2arepsilon} d heta \leq \int \mid f \mid^2 w^{1-2arepsilon} w_1^{2arepsilon} d heta$$
 .

Since $w^{1-2\varepsilon}w_1^{2\varepsilon} \leq \max(w, w_1) \in L_1$ we can take the limit as $\varepsilon \to 0$ under the integral on the right, and apply Fatou's lemma to the integral on the left, to obtain the final conclusion of the sublemma.

Now we are in a position to prove, without much more difficulty, that $\sigma(T_{\phi})$ is connected. Suppose not. Then we can find a simple closed curve Λ , disjoint from $\sigma(T_{\phi})$, so that a non-empty portion of $\sigma(T_{\phi})$ lies inside Λ and a non-empty portion of $\sigma(T_{\phi})$ lies outside Λ . Call these portions σ_1 and σ_2 respectively. By Lemmas 2 and 3, $R(\phi)$ lies entirely inside Λ . Let Γ_{ε} be a simple closed curve surrounding a non-empty portion σ_3 of σ_2 and such that each point of Γ_{ε} is within ε of σ . Since σ_2 is contained in the convex hull of $R(\phi)$ (in fact all of $\sigma(T_{\phi})$ is; this will be explained in a moment) Γ_{ε} will be contained in the convex hull of Λ if ε is sufficiently small. Thus of the three possibilities for disjoint simple closed curves (Λ and Γ_{ε} will be disjoint is ε is small enough),

$$\Lambda \text{ inside } \Gamma_{\varepsilon}$$
$$\Gamma_{\varepsilon} \text{ inside } \Lambda$$
$$\Gamma_{\varepsilon} \text{ there } \Lambda$$

 Γ_{ε} , Λ have disjoint insides,

the first is eliminated since Γ_{ε} is contained in the convex hull of Λ , the second is eliminated since σ_3 lies entirely outside Λ , and the third is eliminated by Lemma 3: since $R(\phi)$ lies outside Γ_3 so does $\sigma(T_{\phi})$. The assumption that $\sigma(T_{\phi})$ is disconnected has led to a contradiction.

It remains to see why $\sigma(T_{\phi})$ is contained in the convex hull of $R(\phi)$. It suffices to show that T_{ϕ} is invertible if $R(\phi)$ is contained in an open angle of opening less than π with vertex 0, and since

invertibility of T_{ϕ} is not destroyed by multiplying ϕ by a nonzero constant we may assume that this angle has the positive real axis as bisector. But then for sufficiently small ε we shall have $||1 - \varepsilon \phi||_{\infty} < 1$, i.e. $||I - \varepsilon T_{\phi}|| < 1$, and this implies T_{ϕ} is invertible.

References

1. A. Devinatz, Toeplitz operators on H^2 spaces, Trans. Amer. Math. Soc., to appear.

2. P. R. Halmos, A glimpse into Hilbert space, article in "A Survey of Modern Mathematics," Wiley, 1963.

3. H. Helson and G. Szegö, A problem in prediction theory, Annali di Mat., 41 (1960), 107-138.

4. E. M. Stein, Interpolation of linear operators, Trans. Amer. Math. Soc., 83 (1956), 482-492.

5. H. Widom, Inversion of Toeplitz matrices II, Ill. J. Math., 4 (1960), 88-99.

6. A. Zygmund, Trigonometric Series, vol. I, Cambridge, 1959.

CORNELL UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

ROBERT OSSERMAN

Stanford University Stanford, California

M. G. ARSOVE University of Washington Seattle 5, Washington J. DUGUNDJI University of Southern California Los Angeles 7, California

LOWELL J. PAIGE University of California Los Angeles 24, California

ASSOCIATE EDITORS

B. H. NEUMANN

E. F. BECKENBACH

F. Wolf

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA CALIFORNIA INSTITUTE OF TECHNOLOGY UNIVERSITY OF CALIFORNIA MONTANA STATE UNIVERSITY UNIVERSITY OF NEVADA NEW MEXICO STATE UNIVERSITY OREGON STATE UNIVERSITY UNIVERSITY OF OREGON OSAKA UNIVERSITY UNIVERSITY OF SOUTHERN CALIFORNIA STANFORD UNIVERSITY UNIVERSITY OF TOKYO UNIVERSITY OF UTAH WASHINGTON STATE UNIVERSITY UNIVERSITY OF WASHINGTON * * * *

AMERICAN MATHEMATICAL SOCIETY CALIFORNIA RESEARCH CORPORATION SPACE TECHNOLOGY LABORATORIES NAVAL ORDNANCE TEST STATION

Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics Vol. 14, No. 1 May, 1964

Diskand America Namuel Gran for a Dfaffan	1
Richard Arens, <i>Normal form for a Pfaffian</i>	1
Charles Vernon Coffman, Non-linear differential equations on cones in Banach	9
spaces	9 17
Ralph DeMarr, Order convergence in linear topological spaces	
Peter Larkin Duren, On the spectrum of a Toeplitz operator	21
Robert E. Edwards, <i>Endomorphisms of function-spaces which leave stable all</i>	31
translation-invariant manifolds	49
Erik Maurice Ellentuck, <i>Infinite products of isols</i>	
William James Firey, Some applications of means of convex bodies	53
Haim Gaifman, Concerning measures on Boolean algebras	61
Richard Carl Gilbert, <i>Extremal spectral functions of a symmetric operator</i>	75
Ronald Lewis Graham, On finite sums of reciprocals of distinct nth powers	85
Hwa Suk Hahn, On the relative growth of differences of partition functions	93
Isidore Isaac Hirschman, Jr., Extreme eigen values of Toeplitz forms associated	
with Jacobi polynomials	107
Chen-jung Hsu, <i>Remarks on certain almost product spaces</i>	163
George Seth Innis, Jr., <i>Some reproducing kernels for the unit disk</i>	177
Ronald Jacobowitz, <i>Multiplicativity of the local Hilbert symbol</i>	187
Paul Joseph Kelly, <i>On some mappings related to graphs</i>	191
William A. Kirk, On curvature of a metric space at a point	195
G. J. Kurowski, On the convergence of semi-discrete analytic functions	199
Richard George Laatsch, <i>Extensions of subadditive functions</i>	209
V. Marić, On some properties of solutions of $\Delta \psi + A(r^2) X \nabla \psi + C(r^2) \psi = 0$	217
William H. Mills, <i>Polynomials with minimal value sets</i>	225
George James Minty, Jr., On the monotonicity of the gradient of a convex	
function	243
George James Minty, Jr., On the solvability of nonlinear functional equations of	
'monotonic' type	249
J. B. Muskat, On the solvability of $x^e \equiv e \pmod{p}$	257
Zeev Nehari, On an inequality of P. R. Bessack	261
Raymond Moos Redheffer and Ernst Gabor Straus, <i>Degenerate elliptic</i>	
equations	265
Abraham Robinson, On generalized limits and linear functionals	269
Bernard W. Roos, On a class of singular second order differential equations with a	
non linear parameter	285
Tôru Saitô, Ordered completely regular semigroups	295
Edward Silverman, A problem of least area	309
Robert C. Sine, Spectral decomposition of a class of operators	333
Jonathan Dean Swift, Chains and graphs of Ostrom planes	353
John Griggs Thompson, 2-signalizers of finite groups	363
Harold Widom, On the spectrum of a Toeplitz operator	365