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1. Introduction. Let X be a locally compact Hausdorff space and μ a positive Radon measure on X. Let $\mathscr{H}$ be a separable Hibert space and let $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq +\infty)$ denote the space of $\mathscr{H}$-valued functions on X which are weakly measurable and whose norms are in scalar $L^p(d\mu)$. Call P a *measurable range function* if P is a function on X defined a.e. $(d\mu)$ to the space of orthogonal projections on $\mathscr{H}$ which is weakly measurable. We shall regard two range functions P, P' to be the same if $P(x) = P'(x)$ l.a.e., i.e. $P(x) = P'(x)$ a.e. on every compact subset of X. We shall denote by $\hat{P}$ the operator on $L^p_{\mathscr{H}}$ defined by $(\hat{P}f)(x) = P(x)f(x)$ l.a.e. Let A be a subalgebra of the algebra $C(X)$ of bounded continuous functions on X such that $A \cup \bar{A}$ (where the bar denotes complex conjugation) is weakly* dense in $L^\infty(d\mu)$. Say that a subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ is *doubly invariant* if

(i) $\mathscr{M}$ is closed in $L^p_{\mathscr{H}}$ if $1 \leqq p < \infty$ and weakly* closed if $p = \infty$,

(ii) $\mathscr{M}$ is invariant under multiplication by functions in $A \cup \bar{A}$.

We shall refer to the following theorem as Wiener's theorem for $L^p_{\mathscr{H}}$:

THEOREM. *Every doubly invariant subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq \infty)$ is of the form $\hat{P}L^p_{\mathscr{H}}$ for some measurable range function P (and trivially conversely); $\mathscr{M}$ determines P uniquely.*

For compact spaces X, Wiener's theorem was proved in [4] for arbitrary $\mathscr{H}$ for $p = 2$ and for the scalar $\mathscr{H}$ (the space of complex numbers) for arbitrary p. It was pointed out in [4] that the $L^2_{\mathscr{H}}$ theorem is true for locally compact spaces and the proof was outlined considering the real line as an example. It was also mentioned in [4] that the $L^2_{\mathscr{H}}$ theorem is a special case of a known theorem on rings of operators [2; p. 167, Théorème 1]. But the proof in [4] and the proof of the more general theorem in [2] implicitly assume the σ-finiteness of μ or at least of the separability of $L^2_{\mathscr{H}}$ (as opposed to the separability of $\mathscr{H}$). The theorem itself is true without this restriction not only for $p = 2$ but for all p and all (separable) $\mathscr{H}$ (not necessarily the scalar $\mathscr{H}$). Indeed the general $L^p_{\mathscr{H}}$ theorem is true even under the weaker assumption that the restriction of $A \cup \bar{A}$ to every compact subset K of X is L^2-dense in $L^2(d\mu \mid K)$, instead of being weakly* dense in L^∞. In this paper we prove this theorem

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(Theorem 4) in its full generality (with the above weaker assumption). This is done as follows: Using the techniques employed in [5] we first show in § 2 (Theorem 2) that a general class of subalgebras dense in L^2 is weakly* dense, which seems to be of independent interest. This enables us to reduce the L^2-density case to that of weak* density. To overcome the difficulties caused by the (possible) non-separability of $L^2_{\mathscr{H}}$ we extend in § 3 (Theorem 3) a theorem of Dunford-Pettis [1; p. 46, Corollaire 2] to apply to our setup. We finally use the $L^2_{\mathscr{H}}$ theorem for compact X in [4] and the broad techniques in [4] to complete the proof. As pointed out in [4], the $L^p_{\mathscr{H}}$ theorem for $p \neq 2$ is of special interest as it shows that the doubly invariant subspaces of $L^p_{\mathscr{H}}$ admit projections of norm 1 commuting with bounded (scalar) functions; as is well known, a closed linear subspace of a Banach space does not in general have any bounded projection at all. In the final section of the paper we extend a known theorem [2] on operators in $L^p_{\mathscr{H}}$ which commute with multiplication by bounded (scalar) functions (Theorem 5).

2. Weak* density of certain subalgebras of L^∞.

THEOREM 1. *Let (X, m) be a finite measure space. Any subalgebra $\mathscr{A}$ of $L^\infty(dm)$ which is conjugate-closed and dense in $L^2(dm)$ is weakly* dense in $L^\infty(dm)$.*[1]

The following three lemmas will lead to the proof of the theorem.

LEMMA 1. *Let $\mathscr{B}$ be a conjugate-closed subalgebra of $L^\infty(dm)$ which contains constants and is closed in $L^\infty(dm)$. Then $\mathscr{B}$ is closed for absolute values.*

Proof. Let $f \in \mathscr{B}$, $0 \leqq f \leqq 1/2$, say. Then $f^{\frac{1}{2}} = (1 - (1 - f))^{\frac{1}{2}}$ can be expressed as the sum of a convergent series in $L^\infty(dm)$ whose terms come from $\mathscr{B}$; it follows that $f^{\frac{1}{2}} \in \mathscr{B}$ for all non-negative $f \in \mathscr{B}$. Since $\mathscr{B}$ is conjugate-closed, the lemm follows.

LEMMA 2. *Let (X, m) be a finite measure space and A a subalgebra of $L^\infty(dm)$ such that $A \cup \bar{A}$ is dense in $L^2(dm)$. Then every closed subspace $\mathscr{M}$ of $L^2(dm)$ which is invariant under multiplication by functions in $A \cup \bar{A}$ is of the form $C_S L^2(dm)$ for some measurable subset S of X (where C_S denotes the characteristic function of S).*

Proof. Let $\mathscr{B}$ be the closed subalgebra of $L^\infty(dm)$ generated by $A \cup \bar{A}$ and the constants. Then $\mathscr{M}$ is clearly invariant under multi-

[1] A weaker result was proved in [5].

plication by functions in $\mathscr{B}$. By Lemma 1, $\mathscr{B}$ is closed for absolute values. Let q be the orthogonal projection of the constant function 1 on $\mathscr{M}$. Then $1 - q \perp \mathscr{M}$. Since $\mathscr{M}$ is invariant under multiplication by function in $\mathscr{B}$, it follows that

$$\int fqdm = \int f \mid q \mid^2 dm \tag{2.1}$$

for all $f \in \mathscr{B}$. Let Y be any measurable subset of X and let $\{f_n\}$ be a sequence of functions from $\mathscr{B}$ which converges to C_Y in $L^2(dm)$. Since $|f_m - f_n| \in \mathscr{B}$, we have from (2.1)

$$\int |f_m - f_n| \mid q \mid^2 dm = \int |f_m - f_n| qdm$$

and the last integral is less than $\left(\int |f_m - f_n|^2 dm\right)^{\frac{1}{2}} \times \left(\int |q|^2 dm\right)^{\frac{1}{2}}$. It follows that $\{f_n \mid q \mid^2\}$ is a Cauchy sequence in $L^1(dm)$. Hence $f_n \mid q \mid^2 \rightarrow C_Y \mid q \mid^2$ in $L^1(dm)$; in particular,

$$\int f_n \mid q \mid^2 dm \rightarrow \int_Y \mid q \mid^2 dm\ . \tag{2.2}$$

Since $f_n \rightarrow C_Y$ in $L^2(dm)$, $f_n q \rightarrow C_Y q$ in $L^1(dm)$ and thus

$$\int f_n qdm \rightarrow \int_Y qdm\ . \tag{2.3}$$

It follows from (2.1)–(2.3) that $\int_Y \mid q \mid^2 dm = \int_Y qdm$ for all measurable subsets Y; hence $|q|^2 = q$ a.e. Thus $q = C_S$ a.e. for some $S \subset X$.

Because of invariance, $C_S L^2(dm) \subset \mathscr{M}$. If the inclusion were strict, let $g \in \mathscr{M} \ominus C_S L^2(dm)$. Then $g \perp C_S \mathscr{B}$ also $C_{S'} \in \mathscr{M}^\perp$ (where $S' = X - S$) and $\mathscr{M}^\perp$ is also invariant along with $\mathscr{M}$ under multiplications by functions in $\mathscr{B}$. So $g \perp C_{S'} \mathscr{B}$. It follows that $g \perp \mathscr{B}$ and because of density of $\mathscr{B}$ in $L^2(dm)$, we have $g = 0$ a.e. Thus $\mathscr{M} = C_S L^2(dm)$.

LEMMA 3. *Let (X, m) and A be as in Lemma 2. Then every closed subspace of $L^1(dm)$ which is invariant under multiplication by functions in $A \cup \bar{A}$ is of the form $C_S L^1(dm)$ for some measurable subset S.*

Proof. This follows from Lemma 2 above and Theorem 7 in [4].

Proof of Theorem 1. Let $\mathscr{M} = \left\{f \in L^1(dm) \colon \int fgdm = 0 \text{ for all } g \in \mathscr{A}\right\}$. Then $\mathscr{M}$ is $\mathscr{A}$-invariant, meaning invariant under multiplication by functions in $\mathscr{A}$ and Lemma 3 applies for $\mathscr{M}$ (with $\mathscr{A}$

replacing A). Thus $\mathscr{M} = C_S L^1(dm)$ for some S, so $\mathscr{M} \cap L^2(dm) = C_S L^2(dm)$. But $\mathscr{M} \cap L^2(dm) = L^2(dm) \ominus \mathscr{A}$. Since $\mathscr{A}$ is dense in $L^2(dm)$ by assumption, it follows that $C_S = 0$ a.e. Therefore $\mathscr{M} = \{0\}$ and the theorem follows.

REMARK. One of the corollaries of Theorem 1 is the "uniqueness" of the Fourier coefficients of any function in $L^1(G)$, for a compact Abelian group G. The characters are dense in $L^2(G)$ so that the subspace $\mathscr{A}$ of their finite linear combinations is weakly* dense in $L^\infty(dm)$ by Theorem 1 and the uniqueness follows.

We now extend Theorem 1 to infinite measure spaces. For convenience we state the result in terms of Radon measures on locally compact spaces. We have

THEOREM 2. *Let X be a locally compact Hausdorff space and μ a positve Radon measure on X. Let $\mathscr{A}$ be a subalgebra of the algebra of bounded continuous functions on X such that*

(i) *$\mathscr{A}$ is conjugate-closed,*

(ii) *$\mathscr{A} \mid K$ is dense in $L^2(d\mu \mid K)$ for every compact subset K of X. Then $\mathscr{A}$ is weakly* dense in $L^\infty(d\mu)$.*

Proof. Let $\mathscr{M} = \left\{f \in L^1(d\mu) \colon \int fg d\mu = 0 \text{ for all } g \in \mathscr{A}\right\}$. If we show that $\mathscr{M} = \{0\}$, the theorem is proved. Now $\mathscr{M}$ is clearly a closed subspace of $L^1(d\mu)$ and is $\mathscr{A}$-invariant. We need the following lemma which will be proved below.

LEMMA 4. *Every closed $\mathscr{A}$-invariant subspace $\mathscr{M}$ of $L^1(d\mu)$ is of the form $C_S L^1(d\mu)$ for some measurable subset S* (where $\mathscr{A}$ is as in Theorem 2).

Assuming Lemma 4, the main theorem follows at once. For, since $\mathscr{M} = C_S L^1(d\mu)$, $\mathscr{A} \subset \mathscr{M}^\perp = C_{S'} L^\infty(d\mu)$. If $\mu(S) > 0$, then S contains a compact subset K of positive measure. Since $\mathscr{A} \subset C_{S'} L^\infty(d\mu)$, $\mathscr{A} \mid K = \{0\}$, contradicting the density of $\mathscr{A} \mid K$ in $L^2(d\mu \mid K)$. Hence $\mu(S) = 0$, so $\mathscr{M} = \{0\}$, completing the proof of the theorem.

Proof of Lemma 4. Let $\mathscr{M}_K = C_K \mathscr{M}$, $\mathscr{A}_K = C_K \mathscr{A}$ and $\mu_K = C_K \mu$. We shall identify $L^p(d\mu \mid K)$, $L^p(d\mu_K)$ and $C_K L^p(d\mu)$ which are clearly mutually isometrically isomorphic. Each $\mathscr{M}_K$ is closed and $\mathscr{A}_K$-invariant in $L^1(d\mu_K)$, so by Lemma 3, $\mathscr{M}_K = C_{S(K)} L^1(d\mu_K)$ for some $S(K) \subset K$. If $K' \supset K$, compact, then

$$\begin{aligned} C_{S(K)}L^1(d\mu) &= C_{S(K)}L^1(d\mu_K) = \mathscr{M}_K = C_K C_{K'}\mathscr{M} \\ &= C_K C_{S(K')}L^1(d\mu_{K'}) = C_{S(K')\cap K}L^1(d\mu_{K'}) \\ &= C_{S(K')\cap K}L^1(d\mu)\ , \end{aligned}$$

so that $S(K) = S(K') \cap K$ (modulo null sets).

Let $\mathscr{K}$ denote the set of all continuous functions with compact support and let σ be the linear functional on $\mathscr{K}$ defined by

$$\sigma(\varphi) = \int_{S(K)} \varphi d\mu \tag{2.4}$$

for $\varphi \in \mathscr{K}$ where K is any compact subset containing the support of φ. Then σ is well-defined and is continuous in the L^1-norm, so can be uniquely extended to a bounded linear functional on $L^1(d\mu)$, which we again denote by σ. Let σ be realized by the L^∞-function g so that

$$\sigma(f) = \int fgd\mu \tag{2.5}$$

for all $f \in L^1(d\mu)$. From (2.4) and (2.5) it is easy to see that $g \mid K = C_{S(K)}$ a.e. for every compact subset K; so we may assume $g = C_S$ for some measurable S with $S \cap K = S(K)$ (modulo null sets). Now

$$C_K C_S L^1(d\mu) = C_{S\cap K}L^1(d\mu) = C_{S(K)}L^1(d\mu) = \mathscr{M}_K = C_K\mathscr{M}$$

for all compact K. Since for any $f \in L^1(d\mu)$, $C_K f \to f$ in $L^1(d\mu)$, it follows from the above that $C_S L^1(d\mu) = \mathscr{M}$.

REMARK. The assumption that $\mathscr{A}$ is an algebra is crucial in both Theorems 1 and 2; the conclusion would be false if $\mathscr{A}$ were merely a linear subspace satisfying the rest of the assumptions. The following example shows that, in the locally compact case for instance, a conjugate-closed linear subspace of $L^\infty(d\mu)$ may be weakly* dense on every compact subset but not on the whole space.

Let X be a locally compact space and μ a non-finite Radon measure on X. Let $f \in L^1(d\mu)$ be real and have a support of infinite μ-measure. Then the support is non-compact. Let $\mathscr{A} = \left\{g \in L^\infty(d\mu)\colon \int gfd\mu = 0\right\}$. Then $\mathscr{A}$ is clearly not weakly* dense in $L^\infty(d\mu)$. But if g is any continuous function with compact support which is "orthogonal" to $\mathscr{A}$, then g must be in the linear span of f in $L^1(d\mu)$. It follows from our assumption on f that g is the zero function. Hence $\mathscr{A}$ is weakly* dense on every compact subset.

3. Dunford-Pettis theorem. Let X denote a locally compact Hausdorff space and μ a positive Radon measure on X. Let E be a separable Banach space and $\mathscr{K}_E$ denote the space of continuous functions from X into E with compact support. For $1 \leqq p < \infty$, let $\mathscr{F}_E^p$ be the space of all functions f from X into E with

$$N_p(f) = \left(\int_X^* \|f(x)\|^p \, d\mu(x)\right)^{1/p} < \infty$$

where $\int^*$ denotes the upper integral. $\mathscr{F}_E^p$ is then a locally convex space with respect to the seminorm N_p. Let $\mathscr{L}_E^p$ denote the closure of $\mathscr{K}_E$ in $\mathscr{F}_E^p$ and let $L_E^p = \mathscr{L}_E^p / \mathscr{N}_E^p$ where $\mathscr{N}_E^p$ is the set of all functions $f \in \mathscr{L}_E^p$ with $N_p(f) = 0$. Then L_E^p is a Banach space with the norm induced by N_p in the obvious way.

Denote by $\mathscr{L}_{E^*}^\infty$ the space of all weakly* measurable functions f on X to the dual E^* of E such that $\|f(x)\| \leqq A < \infty$ l.a.e. ($\|f(x)\| \leqq A$ a.e. on every compact subset). For $f \in \mathscr{L}_{E^*}^\infty$ let

$$N_\infty(f) = \sup_K (\text{ess.}\sup_{x \in K} \|f(x)\|)$$

where K ranges over all compact subsets of X. Then N_∞ is a seminorm which makes $\mathscr{L}_{E^*}^\infty$ a locally convex space. Let $L_{E^*}^\infty$ be the quotient of $\mathscr{L}_{E^*}^\infty$ by the space of all functions in $\mathscr{L}_{E^*}^\infty$ which vanish l.a.e. Then $L_{E^*}^\infty$ is a Banach space.

The following theorem is well-known (cf. for instance [1; p. 46, Corollaire 2]):

THEOREM (Dunford-Pettis). *Let F be a separable Banach space. For $f \in L_{F^*}^\infty$ and $g \in L^1(d\mu)$, let*

$$w_f(g) = \int_X g f d\mu \, .$$

Then $w_f(g) \in F^*$ and the mapping $f \to w_f$ induces an isometric isomorphism from $L_{F^*}^\infty$ onto $\mathscr{L}(L^1, F^*)$, the space of bounded linear maps from $L^1(d\mu)$ to F^*.

We need the following variant of the Dunford-Pettis theorem:

THEOREM 3. *Let E, F be separable Banach spaces. For any bounded linear map u of L_E^1 into F^* there exists a function Φ from X into $\mathscr{L}(E, F^*)$ such that*

(i) *$\langle \Phi(x)s, t \rangle$ is measurable for every $s \in E$, $t \in F$,*

(ii) *$N_\infty(\Phi) < \infty$, and*

(iii) *$u(f) = \int_X \Phi(x) f(x) d\mu(x)$ for every $f \in L_E^1$ with $\|u\| = N_\infty(\Phi)$.*

Conversely, any function Φ satisfying (i) *and* (ii) *defines a bounded linear map u satisfying* (iii).

Proof. Only the direct part needs a proof. First we note that $\mathscr{L}(E, F^*)$ can be regarded as the strong dual of the projective tensor product $E \hat{\otimes} F$. Indeed, the strong dual of $E \hat{\otimes} F$ is canonically identified with the space $B(E, F)$ of bounded bilinear forms on $E \times F$ and $\mathscr{L}(E, F^*)$ is canonically isomorphic with $B(E, F)$. Since E, F are separable, so is $E \hat{\otimes} F$ and therefore $\mathscr{L}(E, F^*)$ can be regarded as the strong dual of a separable Banach space.

Let u be a bounded linear map of L^1_E into F^*. Then u induces a bounded bilinear form $\tilde{u}$ on $L^1 \times E$ into F^* by $\tilde{u}(f, s) = u(f \otimes s)$ for $f \in L^1$, $s \in E$. For any fixed $f \in L^1$, $s \to \tilde{u}(f, s)$ is a bounded linear map of E into F^* which we shall denote by u_f. Then u_1: $f \to u_f$ is a bounded linear map from L^1 into $\mathscr{L}(E, F^*)$ with $\|u_1\| = \|u\|$. By the Dunford-Pettis theorem, there exists a function Φ: $X \to \mathscr{L}(E, F^*)$ such that

(i) $\langle \Phi(x)s, t\rangle$ is measurable for each $s \in E$, $t \in F$

(ii) $N_\infty(\Phi) = \|u_1\|$, and

(iii) $u_1(f) = u_f = \int_X f(x)\Phi(x)d\mu(x)$.

Hence

$$u(f \otimes s) = \tilde{u}(f, s) = u_f(s) = \int_X f\Phi s d\mu$$
$$= \int_X \Phi(f \otimes s)d\mu \, .$$

Because of the continuity of u, the theorem follows.

4. Doubly invariant subspaces. In this section we prove Wiener's theorem in the general setup. Let as usual X denote a locally compact Hausdorff space, μ a positive Radon measure on X, $\mathscr{H}$ a separable Hilbert space and $\mathscr{K}_{\mathscr{H}}$ the space of continuous functions from X into $\mathscr{H}$ with compact support. Let A be a subalgebra of the algebra of bounded continuous functions on X and $\mathscr{A}$ denote the algebra generated by $A \cup \bar{A}$ and the constants. A subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ is clearly invariant under multiplication by functions is $A \cup \bar{A}$ if and only if it is $\mathscr{A}$-invariant. We recall that $\mathscr{M}$ is *doubly invariant* if

(i) $\mathscr{M}$ is closed in $L^p_{\mathscr{H}}$ if $1 \leqq p < \infty$ and weakly* closed if $p = \infty$,

(ii) $\mathscr{M}$ is $\mathscr{A}$-invariant.

Then we have

THEOREM 4. *If $\mathscr{A} | K$ is dense in $L^2(d\mu | K)$ for every compact subset K, then every doubly invariant subspace $\mathscr{M}$ of $L^p_{\mathscr{H}}$ $(1 \leqq p \leqq \infty)$ is of the form $\hat{P}L^p_{\mathscr{H}}$ for some measurable range function P; $\mathscr{M}$ determines P uniquely.*

Proof. We divide the proof into three parts; in the first and the second we assume $\mu(X) < \infty$ and the proof is an imitation of that of the scalar case in [4]. In the last part we treat the case of arbitrary measure spaces and an indication of the proof in this case was given in the proof of Theorem 2.

(i) $\mu(X) < \infty$, $1 \leqq p \leqq 2$. By Theorem 2, $\mathscr{A}$ is weakly* dense in $L^\infty(d\mu)$ and in this case the theorem has been proved in [4] for $p = 2$. Let $1 \leqq p < 2$ and $\mathscr{N} = \mathscr{M} \cap L^2_{\mathscr{H}}$. Then $\mathscr{N}$ is a doubly invariant subspace of $L^2_{\mathscr{H}}$ and so $\mathscr{N} = \hat{P}L^2_{\mathscr{H}}$ for some measurable range function P. We wish to show that $\mathscr{M} = \hat{P}L^p_{\mathscr{H}}$.

For any $f \in \mathscr{M}$ let $f_1(x) = \|f(x)\|^{1-(p/2)}$ and $f_2(x) = f_1(x)^{-1}f(x)$ (of course $f_2(x) = 0$ if $f_1(x) = 0$). Then $f_1 \in L^s(d\mu)$ where $(1/s) + (1/2) = (1/p)$ and $f_2 \in L^2_{\mathscr{H}}$. Let $\mathscr{N}_2$ be the doubly invariant subspace of $L^2_{\mathscr{H}}$ generated by f_2. Then $\mathscr{N}_2 = \hat{P}_2L^2_{\mathscr{H}}$ for a measurable range function P_2. Here we may assume that $P_2(x) = 0$ for those x for which $f_1(x) = 0$. For any $\varphi \in \mathscr{K}_{\mathscr{H}}$

$$f_1\hat{P}_2\varphi \in f_1\hat{P}_2L^2_{\mathscr{H}} = f_1\mathscr{N}_2 \subset \mathscr{M} .$$

On the other hand, since $s > 2$,

$$f_1\hat{P}_2\varphi \in L^s_{\mathscr{H}} \subset L^2_{\mathscr{H}}$$

as $f_1 \in L^s$, $\hat{P}_2\varphi$ is bounded and $\mu(X) < \infty$. Hence

$$f_1\hat{P}_2\varphi \in \mathscr{M} \cap L^2_{\mathscr{H}} = \mathscr{N} = \hat{P}L^2_{\mathscr{H}} .$$

This means that $\hat{P}\hat{P}_2f_1\varphi = \hat{P}_2f_1\varphi$ for all $\varphi \in \mathscr{K}_{\mathscr{H}}$. So, $P_2(x) \leqq P(x)$ l.a.e. Thus we have $\mathscr{N}_2 = \hat{P}_2L^2_{\mathscr{H}} \subset \hat{P}L^2_{\mathscr{H}}$. Hence

$$f = f_1f_2 \in f_1\mathscr{N}_2 \subset f_1\hat{P}L^2_{\mathscr{H}} \subset \hat{P}L^p_{\mathscr{H}} ;$$

the last inclusion resulting from the fact that $f_1 \in L^s$ where $(1/s) + (1/2) = (1/p)$. This shows that $\mathscr{M} \subset PL^p_{\mathscr{H}}$.

Since $\mathscr{M} \supset \mathscr{N} = \hat{P}L^2_{\mathscr{H}}$, we have $\mathscr{M} \supset \hat{P}\mathscr{K}_{\mathscr{H}}$. But $\mathscr{K}_{\mathscr{H}}$ is dense in $L^p_{\mathscr{H}}$ and $\hat{P}$ is L^p-continuous. So $\mathscr{M} \supset \hat{P}L^p_{\mathscr{H}}$ and we have $\mathscr{M} = PL^p_{\mathscr{H}}$.

(ii) $\mu(X) < \infty$, $2 < p \leqq \infty$. Let $\mathscr{M}' = \{f \in L^q_{\mathscr{H}} : f \perp \mathscr{M}\}$ where $(1/q) + (1/p) = 1$. Then $1 \leqq q < 2$ and $\mathscr{M}'$ is doubly invariant in $L^q_{\mathscr{H}}$. Hence by (i) $\mathscr{M}' = \hat{P}'L^q_{\mathscr{H}}$ for some measurable range function P'. Then it is easy to see that $\mathscr{M} = \hat{P}L^p_{\mathscr{H}}$ where $P(x) = I - P'(x)$, I denoting the identity operator on $\mathscr{H}$.

(iii) $\mu(X)$ not necessarily finite, $1 \leqq p \leqq \infty$. Consider any compact subset K of X. Let $\mathscr{M}_K = C_K\mathscr{M}$, $\mathscr{A}_K = C_K\mathscr{A}$ and $\mu_K = C_K\mu$. We shall identify $L^p_{\mathscr{H}}(d\mu\,|\,K)$, $L^p_{\mathscr{H}}(d\mu_K)$ and $C_KL^p_{\mathscr{H}}(d\mu)$ which are obviously mutually isometrically isomorphic and denote any of them by $L^p_{\mathscr{H}}(K)$. Now $\mathscr{M}_K$ is a doubly invariant subspace of $L^p_{\mathscr{H}}(d\mu_K)$ (with $\mathscr{A}_K$ replacing $\mathscr{A}$) and $\mathscr{A}_K$ is dense in $L^2(d\mu_K)$. Hence by (i) and (ii) above, $\mathscr{M}_K = \hat{P}_K L^p_{\mathscr{H}}(K)$. We extend P_K to the whole of X by defining $P_K(x) = 0$ outside of K.

For any two compact subsets K_1, K_2 with $K_1 \supset K_2$ we have

$$\begin{aligned}\hat{P}_{K_2} L^p_{\mathscr{H}} &= \hat{P}_{K_2} L^p_{\mathscr{H}}(K_2) = \mathscr{M}_{K_2} = C_{K_2} C_{K_1} \mathscr{M} = C_{K_2} \hat{P}_{K_1} L^p_{\mathscr{H}}(K_1) \\ &= \hat{P}_{K_1} C_{K_2} L^p_{\mathscr{H}}(K_1) = \hat{P}_{K_1} C_{K_2} L^p_{\mathscr{H}} .\end{aligned}$$

Hence $P_{K_2} = P_{K_1} C_{K_2}$ a.e. It follows from this that the map σ: $\mathscr{K}_{\mathscr{H}} \to \mathscr{H}$ given by

$$\sigma(\varphi) = \int_X P_K(x)\varphi(x) d\mu(x) ,$$

where K is any compact subset containing the support of φ, is well-defined. σ is clearly continuous with respect to the $L^1_{\mathscr{H}}$-norm and so can be uniquely extended to the whole of $L^1_{\mathscr{H}}$ to be continuous. We shall denote the extended map by $\tilde{\sigma}$. By Theorem 3 there exists a weakly measurable bounded operator-valued function Φ: $X \to \mathscr{L}(\mathscr{H}, \mathscr{H})$ such that

$$\tilde{\sigma}(f) = \int_X \Phi(x) f(x) d\mu(x)$$

for all $f \in L^1$. Then, since $\tilde{\sigma}$ entends σ, it is obvious that

$$\Phi \mid K = P_K \text{ a.e.}$$

for every compact set K; so there exists a measurable range function P such that $\Phi = P$ l.a.e.

We assert that $\mathscr{M} = \hat{P} L^p_{\mathscr{H}}$. This follows from the fact that $C_K \mathscr{M} = C_K \hat{P} L^p_{\mathscr{H}}$ for every compact set K and every $f \in \mathscr{M}$ is the L^p-limit (or the weak* limit if $p = \infty$) of $C_K f$. This completes the proof.

The uniqueness of P (for a given $\mathscr{M}$) follows from the uniqueness established in [4] for finite measure spaces.

5. **Decomposable operators.** Let X, μ, A and $\mathscr{A}$ be as in § 4 and let T be an operator in $L^p_{\mathscr{H}}$ bounded if $1 \leqq p < \infty$ and in addition weakly* continuous if $p = \infty$. Clearly T commutes with multiplication by functions in $A \cup \bar{A}$ if and only if it commutes with functions in $\mathscr{A}$, and any operator T which operates pointwise (l.a.e.), meaning

$$(Tf)(x) = T(x) f(x) \text{ l.a.e.}$$

for an operator-valued function $T(x)$, clearly has this property. We wish to prove the following converse.

THEOREM 5. *If T is a bounded (and weakly* continuous, if*

$p=\infty$) linear map from $L^p_{\mathscr{H}}$ into $L^p_{\mathscr{H}}$ ($1 \leqq p \leqq \infty$) which commutes with multiplication by functions in $\mathscr{A}$, then there exists an operator-valued function $T(x)$ defined a.e. *with $T(x) \in \mathscr{L}(\mathscr{H}, \mathscr{H})$ which is weakly measurable and uniformly bounded such that*

$$(Tf)(x) = T(x)f(x) \text{ a.e. } ((Tf)(x) = T(x)f(x) \text{ l.a.e. } if\ p = \infty)$$

This theorem is usually stated for $L^2_{\mathscr{H}}$ [2; p. 162, Theoreme 1] and as far as we are aware, the existing proofs require $L^2_{\mathscr{H}}$ to be separable. We use the variant of Dunford-Pettis theorem established by us in § 3 to get around the difficulties that may be caused by non-separability (we of course assume that the Hilbert space $\mathscr{H}$ is separable).

Proof of Theorem 5. We first consider the case $1 \leqq p < \infty$, for convenience we assume that T is bounded by 1. Let $f \in L^p_{\mathscr{H}}$. Then

$$\int_X \|(Tf)(x)\|^p \, d\mu(x) \leqq \int_X \|f(x)\|^p \, d\mu(x) .$$

Since T commutes with multiplication by functions in $\mathscr{A}$, this yields

$$\int_X |\alpha(x)|^p \, \|(Tf)(x)\|^p \, d\mu(x) \leqq \int_X |\alpha(x)|^p \, \|f(x)\|^p \, d\mu(x)$$

for all $\alpha \in \mathscr{H}$. From the weak* density of $\mathscr{A}$ in L^∞, it follows that

$$\|(Tf)(x)\| \leqq \|f(x)\| \text{ a.e.}$$

If $L^p_{\mathscr{H}}$ is separable, we can obtain $T(x)$ by an explicit construction. In the general case we argue as follows:

Define a map u: $\mathscr{K}_{\mathscr{H}} \to \mathscr{H}$ by setting

$$u(\varphi) = \int_X (T\varphi)(x) d\mu(x) , \qquad \varphi \in \mathscr{K}_{\mathscr{H}} .$$

Then u is continuous with respect to the $L^1_{\mathscr{H}}$-norm on $\mathscr{K}_{\mathscr{H}}$ because

$$\left\| \int_X (T\varphi)(x) d\mu(x) \right\| \leqq \int_X \|(T\varphi)(x)\| \, d\mu(x)$$
$$\leqq \int_X \|\varphi(x)\| \, d\mu(x) .$$

Since $\mathscr{K}_{\mathscr{H}}$ is dense in $L^1_{\mathscr{H}}$, u can be extended by continuity to the whole $L^1_{\mathscr{H}}$ without increasing its norm. We denote the extended map also by u. By Theorem 3 there exists a function $\Phi(x)$ from X into $\mathscr{L}(\mathscr{H}, \mathscr{H})$ such that Φ is weakly measurable, uniformly bounded with $\|\Phi(x)\| \leqq \|u\| \leqq 1$ and

$$u(f) = \int_X \Phi(x) f(x) d\mu(x)$$

for every $f \in L^1_{\mathscr{H}}$. Thus for any $\varphi \in \mathscr{K}_{\mathscr{H}}$

$$\int_X (T\varphi)(x)d\mu(x) = u(\varphi) = \int_X \Phi(x)\varphi(x)d\mu(x) .$$

Since T commutes with multiplication by functions in $\mathscr{A}$ and every $\alpha \in \mathscr{A}$ is continuous, we get

$$\begin{aligned}\int_X \alpha(x)\Phi(x)\varphi(x)d\mu(x) &= \int_X \Phi(x)\alpha(x)\varphi(x)d\mu(x) \\ &= \int_X (T\alpha\varphi)(x)d\mu(x) = \int_X \alpha(x)(T\varphi)(x)d\mu(x) .\end{aligned}$$

By the weak* density of $\mathscr{A}$ in L^∞, this implies

$$(T\varphi)(x) = \Phi(x)\varphi(x) \text{ a.e.}$$

for all $\varphi \in \mathscr{K}_{\mathscr{H}}$. If $\hat{\Phi}$ denotes the operator in $L^p_{\mathscr{H}}$ defined by

$$(\hat{\Phi}f)(x) = \hat{\Phi}(x)f(x) \text{ a.e.,}$$

then we have $T\varphi = \hat{\Phi}\varphi$ for all $\varphi \in \mathscr{K}_{\mathscr{H}}$. Since both T and $\hat{\Phi}$ are bounded in $L^p_{\mathscr{H}}$ and $\mathscr{K}_{\mathscr{H}}$ is dense in $L^p_{\mathscr{H}}$, it follows that $T = \hat{\Phi}$. Now we have only to put $\Phi(x) = T(x)$ in order to get the theorem.

If $p = \infty$ and T is bounded and weakly* continuous, then the transposed map T^* of T maps $L^1_{\mathscr{H}}$ into $L^1_{\mathscr{H}}$. Since T^* commutes with multiplication by functions in $\mathscr{A}$, T^* is expressed by an operator-valued function which is weakly measurable and uniformly bounded. Therefore T is also a uniformly bounded and weakly measurable operator-valued function $T(x)$. In this case, we clearly have

$$(Tf)(x) = T(x)f(x) \text{ l.a.e.}$$

for all $f \in L^\infty_{\mathscr{H}}$.

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