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**DOUBLY INVARIANT SUBSPACES. II**

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1. **Introduction.** Let  $X$  be a locally compact Hausdorff space and  $\mu$  a positive Radon measure on  $X$ . Let  $\mathcal{H}$  be a separable Hilbert space and let  $L^p_{\mathcal{H}}$  ( $1 \leq p \leq +\infty$ ) denote the space of  $\mathcal{H}$ -valued functions on  $X$  which are weakly measurable and whose norms are in scalar  $L^p(d\mu)$ . Call  $P$  a *measurable range function* if  $P$  is a function on  $X$  defined a.e. ( $d\mu$ ) to the space of orthogonal projections on  $\mathcal{H}$  which is weakly measurable. We shall regard two range functions  $P, P'$  to be the same if  $P(x) = P'(x)$  l.a.e., i.e.  $P(x) = P'(x)$  a.e. on every compact subset of  $X$ . We shall denote by  $\hat{P}$  the operator on  $L^p_{\mathcal{H}}$  defined by  $(\hat{P}f)(x) = P(x)f(x)$  l.a.e. Let  $A$  be a subalgebra of the algebra  $C(X)$  of bounded continuous functions on  $X$  such that  $A \cup \bar{A}$  (where the bar denotes complex conjugation) is weakly\* dense in  $L^\infty(d\mu)$ . Say that a subspace  $\mathcal{M}$  of  $L^p_{\mathcal{H}}$  is *doubly invariant* if

- (i)  $\mathcal{M}$  is closed in  $L^p_{\mathcal{H}}$  if  $1 \leq p < \infty$  and weakly\* closed if  $p = \infty$ ,
- (ii)  $\mathcal{M}$  is invariant under multiplication by functions in  $A \cup \bar{A}$ .

We shall refer to the following theorem as Wiener's theorem for  $L^p_{\mathcal{H}}$ :

**THEOREM.** *Every doubly invariant subspace  $\mathcal{M}$  of  $L^p_{\mathcal{H}}$  ( $1 \leq p \leq \infty$ ) is of the form  $\hat{P}L^p_{\mathcal{H}}$  for some measurable range function  $P$  (and trivially conversely);  $\mathcal{M}$  determines  $P$  uniquely.*

For compact spaces  $X$ , Wiener's theorem was proved in [4] for arbitrary  $\mathcal{H}$  for  $p = 2$  and for the scalar  $\mathcal{H}$  (the space of complex numbers) for arbitrary  $p$ . It was pointed out in [4] that the  $L^2_{\mathcal{H}}$  theorem is true for locally compact spaces and the proof was outlined considering the real line as an example. It was also mentioned in [4] that the  $L^2_{\mathcal{H}}$  theorem is a special case of a known theorem on rings of operators [2; p. 167, Théorème 1]. But the proof in [4] and the proof of the more general theorem in [2] implicitly assume the  $\sigma$ -finiteness of  $\mu$  or at least of the separability of  $L^2_{\mathcal{H}}$  (as opposed to the separability of  $\mathcal{H}$ ). The theorem itself is true without this restriction not only for  $p = 2$  but for all  $p$  and all (separable)  $\mathcal{H}$  (not necessarily the scalar  $\mathcal{H}$ ). Indeed the general  $L^p_{\mathcal{H}}$  theorem is true even under the weaker assumption that the restriction of  $A \cup \bar{A}$  to every compact subset  $K$  of  $X$  is  $L^2$ -dense in  $L^2(d\mu|_K)$ , instead of being weakly\* dense in  $L^\infty$ . In this paper we prove this theorem

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(Theorem 4) in its full generality (with the above weaker assumption). This is done as follows: Using the techniques employed in [5] we first show in § 2 (Theorem 2) that a general class of subalgebras dense in  $L^2$  is weakly\* dense, which seems to be of independent interest. This enables us to reduce the  $L^2$ -density case to that of weak\* density. To overcome the difficulties caused by the (possible) non-separability of  $L^2_{\mathcal{H}}$  we extend in § 3 (Theorem 3) a theorem of Dunford-Pettis [1; p. 46, Corollaire 2] to apply to our setup. We finally use the  $L^2_{\mathcal{H}}$  theorem for compact  $X$  in [4] and the broad techniques in [4] to complete the proof. As pointed out in [4], the  $L^p_{\mathcal{H}}$  theorem for  $p \neq 2$  is of special interest as it shows that the doubly invariant subspaces of  $L^p_{\mathcal{H}}$  admit projections of norm 1 commuting with bounded (scalar) functions; as is well known, a closed linear subspace of a Banach space does not in general have any bounded projection at all. In the final section of the paper we extend a known theorem [2] on operators in  $L^p_{\mathcal{H}}$  which commute with multiplication by bounded (scalar) functions (Theorem 5).

## 2. Weak\* density of certain subalgebras of $L^\infty$ .

**THEOREM 1.** *Let  $(X, m)$  be a finite measure space. Any subalgebra  $\mathcal{A}$  of  $L^\infty(dm)$  which is conjugate-closed and dense in  $L^2(dm)$  is weakly\* dense in  $L^\infty(dm)$ .<sup>1</sup>*

The following three lemmas will lead to the proof of the theorem.

**LEMMA 1.** *Let  $\mathcal{B}$  be a conjugate-closed subalgebra of  $L^\infty(dm)$  which contains constants and is closed in  $L^\infty(dm)$ . Then  $\mathcal{B}$  is closed for absolute values.*

*Proof.* Let  $f \in \mathcal{B}$ ,  $0 \leq f \leq 1/2$ , say. Then  $f^{\frac{1}{2}} = (1 - (1 - f))^{\frac{1}{2}}$  can be expressed as the sum of a convergent series in  $L^\infty(dm)$  whose terms come from  $\mathcal{B}$ ; it follows that  $f^{\frac{1}{2}} \in \mathcal{B}$  for all non-negative  $f \in \mathcal{B}$ . Since  $\mathcal{B}$  is conjugate-closed, the lemma follows.

**LEMMA 2.** *Let  $(X, m)$  be a finite measure space and  $A$  a subalgebra of  $L^\infty(dm)$  such that  $A \cup \bar{A}$  is dense in  $L^2(dm)$ . Then every closed subspace  $\mathcal{M}$  of  $L^2(dm)$  which is invariant under multiplication by functions in  $A \cup \bar{A}$  is of the form  $C_S L^2(dm)$  for some measurable subset  $S$  of  $X$  (where  $C_S$  denotes the characteristic function of  $S$ ).*

*Proof.* Let  $\mathcal{B}$  be the closed subalgebra of  $L^\infty(dm)$  generated by  $A \cup \bar{A}$  and the constants. Then  $\mathcal{M}$  is clearly invariant under multi-

<sup>1</sup> A weaker result was proved in [5].

plication by functions in  $\mathcal{B}$ . By Lemma 1,  $\mathcal{B}$  is closed for absolute values. Let  $q$  be the orthogonal projection of the constant function 1 on  $\mathcal{M}$ . Then  $1 - q \perp \mathcal{M}$ . Since  $\mathcal{M}$  is invariant under multiplication by function in  $\mathcal{B}$ , it follows that

$$(2.1) \quad \int f q d m = \int f |q|^2 d m$$

for all  $f \in \mathcal{B}$ . Let  $Y$  be any measurable subset of  $X$  and let  $\{f_n\}$  be a sequence of functions from  $\mathcal{B}$  which converges to  $C_Y$  in  $L^2(dm)$ . Since  $|f_m - f_n| \in \mathcal{B}$ , we have from (2.1)

$$\int |f_m - f_n| |q|^2 d m = \int |f_m - f_n| q d m$$

and the last integral is less than  $\left(\int |f_m - f_n|^2 d m\right)^{\frac{1}{2}} \times \left(\int |q|^2 d m\right)^{\frac{1}{2}}$ . It follows that  $\{f_n |q|^2\}$  is a Cauchy sequence in  $L^1(dm)$ . Hence  $f_n |q|^2 \rightarrow C_Y |q|^2$  in  $L^1(dm)$ ; in particular,

$$(2.2) \quad \int f_n |q|^2 d m \rightarrow \int_Y |q|^2 d m .$$

Since  $f_n \rightarrow C_Y$  in  $L^2(dm)$ ,  $f_n q \rightarrow C_Y q$  in  $L^1(dm)$  and thus

$$(2.3) \quad \int f_n q d m \rightarrow \int_Y q d m .$$

It follows from (2.1)–(2.3) that  $\int_Y |q|^2 d m = \int_Y q d m$  for all measurable subsets  $Y$ ; hence  $|q|^2 = q$  a.e. Thus  $q = C_S$  a.e. for some  $S \subset X$ .

Because of invariance,  $C_S L^2(dm) \subset \mathcal{M}$ . If the inclusion were strict, let  $g \in \mathcal{M} \ominus C_S L^2(dm)$ . Then  $g \perp C_S \mathcal{B}$  also  $C_{S'} \in \mathcal{M}^\perp$  (where  $S' = X - S$ ) and  $\mathcal{M}^\perp$  is also invariant along with  $\mathcal{M}$  under multiplications by functions in  $\mathcal{B}$ . So  $g \perp C_{S'} \mathcal{B}$ . It follows that  $g \perp \mathcal{B}$  and because of density of  $\mathcal{B}$  in  $L^2(dm)$ , we have  $g = 0$  a.e. Thus  $\mathcal{M} = C_S L^2(dm)$ .

**LEMMA 3.** *Let  $(X, m)$  and  $A$  be as in Lemma 2. Then every closed subspace of  $L^1(dm)$  which is invariant under multiplication by functions in  $A \cup \bar{A}$  is of the form  $C_S L^1(dm)$  for some measurable subset  $S$ .*

*Proof.* This follows from Lemma 2 above and Theorem 7 in [4].

*Proof of Theorem 1.* Let  $\mathcal{M} = \left\{ f \in L^1(dm) : \int f g d m = 0 \text{ for all } g \in \mathcal{A} \right\}$ . Then  $\mathcal{M}$  is  $\mathcal{A}$ -invariant, meaning invariant under multiplication by functions in  $\mathcal{A}$  and Lemma 3 applies for  $\mathcal{M}$  (with  $\mathcal{A}$

replacing  $A$ ). Thus  $\mathcal{M} = C_S L^1(dm)$  for some  $S$ , so  $\mathcal{M} \cap L^2(dm) = C_S L^2(dm)$ . But  $\mathcal{M} \cap L^2(dm) = L^2(dm) \ominus \mathcal{A}$ . Since  $\mathcal{A}$  is dense in  $L^2(dm)$  by assumption, it follows that  $C_S = 0$  a.e. Therefore  $\mathcal{M} = \{0\}$  and the theorem follows.

REMARK. One of the corollaries of Theorem 1 is the "uniqueness" of the Fourier coefficients of any function in  $L^1(G)$ , for a compact Abelian group  $G$ . The characters are dense in  $L^2(G)$  so that the subspace  $\mathcal{A}$  of their finite linear combinations is weakly\* dense in  $L^\infty(dm)$  by Theorem 1 and the uniqueness follows.

We now extend Theorem 1 to infinite measure spaces. For convenience we state the result in terms of Radon measures on locally compact spaces. We have

THEOREM 2. *Let  $X$  be a locally compact Hausdorff space and  $\mu$  a positive Radon measure on  $X$ . Let  $\mathcal{A}$  be a subalgebra of the algebra of bounded continuous functions on  $X$  such that*

- (i)  $\mathcal{A}$  is conjugate-closed,
- (ii)  $\mathcal{A}|_K$  is dense in  $L^2(d\mu|_K)$  for every compact subset  $K$  of  $X$ . Then  $\mathcal{A}$  is weakly\* dense in  $L^\infty(d\mu)$ .

*Proof.* Let  $\mathcal{M} = \left\{ f \in L^1(d\mu) : \int fgd\mu = 0 \text{ for all } g \in \mathcal{A} \right\}$ . If we show that  $\mathcal{M} = \{0\}$ , the theorem is proved. Now  $\mathcal{M}$  is clearly a closed subspace of  $L^1(d\mu)$  and is  $\mathcal{A}$ -invariant. We need the following lemma which will be proved below.

LEMMA 4. *Every closed  $\mathcal{A}$ -invariant subspace  $\mathcal{M}$  of  $L^1(d\mu)$  is of the form  $C_S L^1(d\mu)$  for some measurable subset  $S$  (where  $\mathcal{A}$  is as in Theorem 2).*

Assuming Lemma 4, the main theorem follows at once. For, since  $\mathcal{M} = C_S L^1(d\mu)$ ,  $\mathcal{A} \subset \mathcal{M}^\perp = C_S L^\infty(d\mu)$ . If  $\mu(S) > 0$ , then  $S$  contains a compact subset  $K$  of positive measure. Since  $\mathcal{A} \subset C_S L^\infty(d\mu)$ ,  $\mathcal{A}|_K = \{0\}$ , contradicting the density of  $\mathcal{A}|_K$  in  $L^2(d\mu|_K)$ . Hence  $\mu(S) = 0$ , so  $\mathcal{M} = \{0\}$ , completing the proof of the theorem.

*Proof of Lemma 4.* Let  $\mathcal{M}_K = C_K \mathcal{M}$ ,  $\mathcal{A}_K = C_K \mathcal{A}$  and  $\mu_K = C_K \mu$ . We shall identify  $L^p(d\mu|_K)$ ,  $L^p(d\mu_K)$  and  $C_K L^p(d\mu)$  which are clearly mutually isometrically isomorphic. Each  $\mathcal{M}_K$  is closed and  $\mathcal{A}_K$ -invariant in  $L^1(d\mu_K)$ , so by Lemma 3,  $\mathcal{M}_K = C_{S(K)} L^1(d\mu_K)$  for some  $S(K) \subset K$ . If  $K' \supset K$ , compact, then

$$\begin{aligned}
C_{S(K)}L^1(d\mu) &= C_{S(K)}L^1(d\mu_K) = \mathcal{M}_K = C_K C_{K'} \mathcal{M} \\
&= C_K C_{S(K')} L^1(d\mu_{K'}) = C_{S(K') \cap K} L^1(d\mu_{K'}) \\
&= C_{S(K') \cap K} L^1(d\mu),
\end{aligned}$$

so that  $S(K) = S(K') \cap K$  (modulo null sets).

Let  $\mathcal{H}$  denote the set of all continuous functions with compact support and let  $\sigma$  be the linear functional on  $\mathcal{H}$  defined by

$$(2.4) \quad \sigma(\varphi) = \int_{S(K)} \varphi d\mu$$

for  $\varphi \in \mathcal{H}$  where  $K$  is any compact subset containing the support of  $\varphi$ . Then  $\sigma$  is well-defined and is continuous in the  $L^1$ -norm, so can be uniquely extended to a bounded linear functional on  $L^1(d\mu)$ , which we again denote by  $\sigma$ . Let  $\sigma$  be realized by the  $L^\infty$ -function  $g$  so that

$$(2.5) \quad \sigma(f) = \int fg d\mu$$

for all  $f \in L^1(d\mu)$ . From (2.4) and (2.5) it is easy to see that  $g|_K = C_{S(K)}$  a.e. for every compact subset  $K$ ; so we may assume  $g = C_S$  for some measurable  $S$  with  $S \cap K = S(K)$  (modulo null sets). Now

$$C_K C_S L^1(d\mu) = C_{S \cap K} L^1(d\mu) = C_{S(K)} L^1(d\mu) = \mathcal{M}_K = C_K \mathcal{M}$$

for all compact  $K$ . Since for any  $f \in L^1(d\mu)$ ,  $C_K f \rightarrow f$  in  $L^1(d\mu)$ , it follows from the above that  $C_S L^1(d\mu) = \mathcal{M}$ .

**REMARK.** The assumption that  $\mathcal{A}$  is an algebra is crucial in both Theorems 1 and 2; the conclusion would be false if  $\mathcal{A}$  were merely a linear subspace satisfying the rest of the assumptions. The following example shows that, in the locally compact case for instance, a conjugate-closed linear subspace of  $L^\infty(d\mu)$  may be weakly\* dense on every compact subset but not on the whole space.

Let  $X$  be a locally compact space and  $\mu$  a non-finite Radon measure on  $X$ . Let  $f \in L^1(d\mu)$  be real and have a support of infinite  $\mu$ -measure. Then the support is non-compact. Let  $\mathcal{A} = \left\{ g \in L^\infty(d\mu) : \int gf d\mu = 0 \right\}$ . Then  $\mathcal{A}$  is clearly not weakly\* dense in  $L^\infty(d\mu)$ . But if  $g$  is any continuous function with compact support which is "orthogonal" to  $\mathcal{A}$ , then  $g$  must be in the linear span of  $f$  in  $L^1(d\mu)$ . It follows from our assumption on  $f$  that  $g$  is the zero function. Hence  $\mathcal{A}$  is weakly\* dense on every compact subset.

**3. Dunford-Pettis theorem.** Let  $X$  denote a locally compact Hausdorff space and  $\mu$  a positive Radon measure on  $X$ . Let  $E$  be a

separable Banach space and  $\mathcal{H}_E$  denote the space of continuous functions from  $X$  into  $E$  with compact support. For  $1 \leq p < \infty$ , let  $\mathcal{F}_E^p$  be the space of all functions  $f$  from  $X$  into  $E$  with

$$N_p(f) = \left( \int_X^* \|f(x)\|^p d\mu(x) \right)^{1/p} < \infty$$

where  $\int^*$  denotes the upper integral.  $\mathcal{F}_E^p$  is then a locally convex space with respect to the seminorm  $N_p$ . Let  $\mathcal{L}_E^p$  denote the closure of  $\mathcal{H}_E$  in  $\mathcal{F}_E^p$  and let  $L_E^p = \mathcal{L}_E^p / \mathcal{N}_E^p$  where  $\mathcal{N}_E^p$  is the set of all functions  $f \in \mathcal{L}_E^p$  with  $N_p(f) = 0$ . Then  $L_E^p$  is a Banach space with the norm induced by  $N_p$  in the obvious way.

Denote by  $\mathcal{L}_{E^*}^\infty$  the space of all weakly\* measurable functions  $f$  on  $X$  to the dual  $E^*$  of  $E$  such that  $\|f(x)\| \leq A < \infty$  l.a.e. ( $\|f(x)\| \leq A$  a.e. on every compact subset). For  $f \in \mathcal{L}_{E^*}^\infty$  let

$$N_\infty(f) = \sup_K (\text{ess. sup}_{x \in K} \|f(x)\|)$$

where  $K$  ranges over all compact subsets of  $X$ . Then  $N_\infty$  is a seminorm which makes  $\mathcal{L}_{E^*}^\infty$  a locally convex space. Let  $L_{E^*}^\infty$  be the quotient of  $\mathcal{L}_{E^*}^\infty$  by the space of all functions in  $\mathcal{L}_{E^*}^\infty$  which vanish l.a.e. Then  $L_{E^*}^\infty$  is a Banach space.

The following theorem is well-known (cf. for instance [1; p. 46, Corollaire 2]):

**THEOREM (Dunford-Pettis).** *Let  $F$  be a separable Banach space. For  $f \in L_{F^*}^\infty$  and  $g \in L^1(d\mu)$ , let*

$$w_f(g) = \int_X g f d\mu.$$

*Then  $w_f(g) \in F^*$  and the mapping  $f \rightarrow w_f$  induces an isometric isomorphism from  $L_{F^*}^\infty$  onto  $\mathcal{L}(L^1, F^*)$ , the space of bounded linear maps from  $L^1(d\mu)$  to  $F^*$ .*

We need the following variant of the Dunford-Pettis theorem:

**THEOREM 3.** *Let  $E, F$  be separable Banach spaces. For any bounded linear map  $u$  of  $L_E^1$  into  $F^*$  there exists a function  $\Phi$  from  $X$  into  $\mathcal{L}(E, F^*)$  such that*

- (i)  $\langle \Phi(x)s, t \rangle$  is measurable for every  $s \in E, t \in F$ ,
- (ii)  $N_\infty(\Phi) < \infty$ , and
- (iii)  $u(f) = \int_X \Phi(x)f(x)d\mu(x)$  for every  $f \in L_E^1$  with  $\|u\| = N_\infty(\Phi)$ .

*Conversely, any function  $\Phi$  satisfying (i) and (ii) defines a bounded linear map  $u$  satisfying (iii).*

*Proof.* Only the direct part needs a proof. First we note that  $\mathcal{L}(E, F^*)$  can be regarded as the strong dual of the projective tensor product  $E \hat{\otimes} F$ . Indeed, the strong dual of  $E \hat{\otimes} F$  is canonically identified with the space  $B(E, F)$  of bounded bilinear forms on  $E \times F$  and  $\mathcal{L}(E, F^*)$  is canonically isomorphic with  $B(E, F)$ . Since  $E, F$  are separable, so is  $E \hat{\otimes} F$  and therefore  $\mathcal{L}(E, F^*)$  can be regarded as the strong dual of a separable Banach space.

Let  $u$  be a bounded linear map of  $L^1_E$  into  $F^*$ . Then  $u$  induces a bounded bilinear form  $\tilde{u}$  on  $L^1 \times E$  into  $F^*$  by  $\tilde{u}(f, s) = u(f \otimes s)$  for  $f \in L^1, s \in E$ . For any fixed  $f \in L^1, s \rightarrow \tilde{u}(f, s)$  is a bounded linear map of  $E$  into  $F^*$  which we shall denote by  $u_f$ . Then  $u_1: f \rightarrow u_f$  is a bounded linear map from  $L^1$  into  $\mathcal{L}(E, F^*)$  with  $\|u_1\| = \|u\|$ . By the Dunford-Pettis theorem, there exists a function  $\Phi: X \rightarrow \mathcal{L}(E, F^*)$  such that

- (i)  $\langle \Phi(x)s, t \rangle$  is measurable for each  $s \in E, t \in F$
- (ii)  $N_\infty(\Phi) = \|u_1\|$ , and
- (iii)  $u_1(f) = u_f = \int_X f(x)\Phi(x)d\mu(x)$ .

Hence

$$\begin{aligned} u(f \otimes s) &= \tilde{u}(f, s) = u_f(s) = \int_X f\Phi s d\mu \\ &= \int_X \Phi(f \otimes s) d\mu. \end{aligned}$$

Because of the continuity of  $u$ , the theorem follows.

**4. Doubly invariant subspaces.** In this section we prove Wiener's theorem in the general setup. Let as usual  $X$  denote a locally compact Hausdorff space,  $\mu$  a positive Radon measure on  $X$ ,  $\mathcal{H}$  a separable Hilbert space and  $\mathcal{K}_{\mathcal{H}}$  the space of continuous functions from  $X$  into  $\mathcal{H}$  with compact support. Let  $A$  be a subalgebra of the algebra of bounded continuous functions on  $X$  and  $\mathcal{A}$  denote the algebra generated by  $A \cup \bar{A}$  and the constants. A subspace  $\mathcal{M}$  of  $L^p_{\mathcal{H}}$  is clearly invariant under multiplication by functions in  $A \cup \bar{A}$  if and only if it is  $\mathcal{A}$ -invariant. We recall that  $\mathcal{M}$  is *doubly invariant* if

- (i)  $\mathcal{M}$  is closed in  $L^p_{\mathcal{H}}$  if  $1 \leq p < \infty$  and weakly\* closed if  $p = \infty$ ,
- (ii)  $\mathcal{M}$  is  $\mathcal{A}$ -invariant.

Then we have

**THEOREM 4.** *If  $\mathcal{A} \upharpoonright K$  is dense in  $L^2(d\mu \upharpoonright K)$  for every compact subset  $K$ , then every doubly invariant subspace  $\mathcal{M}$  of  $L^p_{\mathcal{H}}$  ( $1 \leq p \leq \infty$ ) is of the form  $\hat{P}L^p_{\mathcal{H}}$  for some measurable range function  $P$ ;  $\mathcal{M}$  determines  $P$  uniquely.*

*Proof.* We divide the proof into three parts; in the first and the



second we assume  $\mu(X) < \infty$  and the proof is an imitation of that of the scalar case in [4]. In the last part we treat the case of arbitrary measure spaces and an indication of the proof in this case was given in the proof of Theorem 2.

(i)  $\mu(X) < \infty$ ,  $1 \leq p \leq 2$ . By Theorem 2,  $\mathcal{A}$  is weakly\* dense in  $L^\infty(d\mu)$  and in this case the theorem has been proved in [4] for  $p = 2$ . Let  $1 \leq p < 2$  and  $\mathcal{N} = \mathcal{M} \cap L^2_{\mathcal{H}}$ . Then  $\mathcal{N}$  is a doubly invariant subspace of  $L^2_{\mathcal{H}}$  and so  $\mathcal{N} = \hat{P}L^2_{\mathcal{H}}$  for some measurable range function  $P$ . We wish to show that  $\mathcal{M} = \hat{P}L^p_{\mathcal{H}}$ .

For any  $f \in \mathcal{M}$  let  $f_1(x) = \|f(x)\|^{1-(p/2)}$  and  $f_2(x) = f_1(x)^{-1}f(x)$  (of course  $f_2(x) = 0$  if  $f_1(x) = 0$ ). Then  $f_1 \in L^s(d\mu)$  where  $(1/s) + (1/2) = (1/p)$  and  $f_2 \in L^2_{\mathcal{H}}$ . Let  $\mathcal{N}_2$  be the doubly invariant subspace of  $L^2_{\mathcal{H}}$  generated by  $f_2$ . Then  $\mathcal{N}_2 = \hat{P}_2L^2_{\mathcal{H}}$  for a measurable range function  $P_2$ . Here we may assume that  $P_2(x) = 0$  for those  $x$  for which  $f_1(x) = 0$ . For any  $\varphi \in \mathcal{H}_{\mathcal{H}}$

$$f_1\hat{P}_2\varphi \in f_1\hat{P}_2L^2_{\mathcal{H}} = f_1\mathcal{N}_2 \subset \mathcal{M}.$$

On the other hand, since  $s > 2$ ,

$$f_1\hat{P}_2\varphi \in L^s_{\mathcal{H}} \subset L^2_{\mathcal{H}}$$

as  $f_1 \in L^s$ ,  $\hat{P}_2\varphi$  is bounded and  $\mu(X) < \infty$ . Hence

$$f_1\hat{P}_2\varphi \in \mathcal{M} \cap L^2_{\mathcal{H}} = \mathcal{N} = \hat{P}L^2_{\mathcal{H}}.$$

This means that  $\hat{P}\hat{P}_2f_1\varphi = \hat{P}_2f_1\varphi$  for all  $\varphi \in \mathcal{H}_{\mathcal{H}}$ . So,  $P_2(x) \leq P(x)$  l.a.e. Thus we have  $\mathcal{N}_2 = \hat{P}_2L^2_{\mathcal{H}} \subset \hat{P}L^2_{\mathcal{H}}$ . Hence

$$f = f_1f_2 \in f_1\mathcal{N}_2 \subset f_1\hat{P}L^2_{\mathcal{H}} \subset \hat{P}L^p_{\mathcal{H}};$$

the last inclusion resulting from the fact that  $f_1 \in L^s$  where  $(1/s) + (1/2) = (1/p)$ . This shows that  $\mathcal{M} \subset \hat{P}L^p_{\mathcal{H}}$ .

Since  $\mathcal{M} \supset \mathcal{N} = \hat{P}L^2_{\mathcal{H}}$ , we have  $\mathcal{M} \supset \hat{P}\mathcal{H}_{\mathcal{H}}$ . But  $\mathcal{H}_{\mathcal{H}}$  is dense in  $L^p_{\mathcal{H}}$  and  $\hat{P}$  is  $L^p$ -continuous. So  $\mathcal{M} \supset \hat{P}L^p_{\mathcal{H}}$  and we have  $\mathcal{M} = \hat{P}L^p_{\mathcal{H}}$ .

(ii)  $\mu(X) < \infty$ ,  $2 < p \leq \infty$ . Let  $\mathcal{M}' = \{f \in L^q_{\mathcal{H}} : f \perp \mathcal{M}\}$  where  $(1/q) + (1/p) = 1$ . Then  $1 \leq q < 2$  and  $\mathcal{M}'$  is doubly invariant in  $L^q_{\mathcal{H}}$ . Hence by (i)  $\mathcal{M}' = \hat{P}'L^q_{\mathcal{H}}$  for some measurable range function  $P'$ . Then it is easy to see that  $\mathcal{M} = \hat{P}L^p_{\mathcal{H}}$  where  $P(x) = I - P'(x)$ ,  $I$  denoting the identity operator on  $\mathcal{H}$ .

(iii)  $\mu(X)$  not necessarily finite,  $1 \leq p \leq \infty$ . Consider any compact subset  $K$  of  $X$ . Let  $\mathcal{M}_K = C_K\mathcal{M}$ ,  $\mathcal{A}_K = C_K\mathcal{A}$  and  $\mu_K = C_K\mu$ . We shall identify  $L^p_{\mathcal{H}}(d\mu|K)$ ,  $L^p_{\mathcal{H}}(d\mu_K)$  and  $C_KL^p_{\mathcal{H}}(d\mu)$  which are obviously mutually isometrically isomorphic and denote any of them by  $L^p_{\mathcal{H}}(K)$ . Now  $\mathcal{M}_K$  is a doubly invariant subspace of  $L^p_{\mathcal{H}}(d\mu_K)$  (with  $\mathcal{A}_K$  replacing  $\mathcal{A}$ ) and  $\mathcal{A}_K$  is dense in  $L^2(d\mu_K)$ . Hence by (i)

and (ii) above,  $\mathcal{M}_K = \hat{P}_K L_{\mathcal{H}}^p(K)$ . We extend  $P_K$  to the whole of  $X$  by defining  $P_K(x) = 0$  outside of  $K$ .

For any two compact subsets  $K_1, K_2$  with  $K_1 \supset K_2$  we have

$$\begin{aligned} \hat{P}_{K_2} L_{\mathcal{H}}^p &= \hat{P}_{K_2} L_{\mathcal{H}}^p(K_2) = \mathcal{M}_{K_2} = C_{K_2} C_{K_1} \mathcal{M} = C_{K_2} \hat{P}_{K_1} L_{\mathcal{H}}^p(K_1) \\ &= \hat{P}_{K_1} C_{K_2} L_{\mathcal{H}}^p(K_1) = \hat{P}_{K_1} C_{K_2} L_{\mathcal{H}}^p. \end{aligned}$$

Hence  $P_{K_2} = P_{K_1} C_{K_2}$  a.e. It follows from this that the map  $\sigma: \mathcal{K}_{\mathcal{H}} \rightarrow \mathcal{H}$  given by

$$\sigma(\varphi) = \int_X P_K(x) \varphi(x) d\mu(x),$$

where  $K$  is any compact subset containing the support of  $\varphi$ , is well-defined.  $\sigma$  is clearly continuous with respect to the  $L_{\mathcal{H}}^1$ -norm and so can be uniquely extended to the whole of  $L_{\mathcal{H}}^1$  to be continuous. We shall denote the extended map by  $\tilde{\sigma}$ . By Theorem 3 there exists a weakly measurable bounded operator-valued function  $\Phi: X \rightarrow \mathcal{L}(\mathcal{H}, \mathcal{H})$  such that

$$\tilde{\sigma}(f) = \int_X \Phi(x) f(x) d\mu(x)$$

for all  $f \in L^1$ . Then, since  $\tilde{\sigma}$  extends  $\sigma$ , it is obvious that

$$\Phi|_K = P_K \text{ a.e.}$$

for every compact set  $K$ ; so there exists a measurable range function  $P$  such that  $\Phi = P$  l.a.e.

We assert that  $\mathcal{M} = \hat{P} L_{\mathcal{H}}^p$ . This follows from the fact that  $C_K \mathcal{M} = C_K \hat{P} L_{\mathcal{H}}^p$  for every compact set  $K$  and every  $f \in \mathcal{M}$  is the  $L^p$ -limit (or the weak\* limit if  $p = \infty$ ) of  $C_K f$ . This completes the proof.

The uniqueness of  $P$  (for a given  $\mathcal{M}$ ) follows from the uniqueness established in [4] for finite measure spaces.

**5. Decomposable operators.** Let  $X, \mu, A$  and  $\mathcal{A}$  be as in §4 and let  $T$  be an operator in  $L_{\mathcal{H}}^p$  bounded if  $1 \leq p < \infty$  and in addition weakly\* continuous if  $p = \infty$ . Clearly  $T$  commutes with multiplication by functions in  $A \cup \bar{A}$  if and only if it commutes with functions in  $\mathcal{A}$ , and any operator  $T$  which operates pointwise (l.a.e.), meaning

$$(Tf)(x) = T(x)f(x) \text{ l.a.e.}$$

for an operator-valued function  $T(x)$ , clearly has this property. We wish to prove the following converse.

**THEOREM 5.** *If  $T$  is a bounded (and weakly\* continuous, if*

$p = \infty$ ) linear map from  $L^p_{\mathcal{H}}$  into  $L^p_{\mathcal{H}}$  ( $1 \leq p \leq \infty$ ) which commutes with multiplication by functions in  $\mathcal{A}$ , then there exists an operator-valued function  $T(x)$  defined a.e. with  $T(x) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$  which is weakly measurable and uniformly bounded such that

$$(Tf)(x) = T(x)f(x) \text{ a.e. } ((Tf)(x) = T(x)f(x) \text{ l.a.e. if } p = \infty)$$

This theorem is usually stated for  $L^2_{\mathcal{H}}$  [2; p. 162, Theoreme 1] and as far as we are aware, the existing proofs require  $L^2_{\mathcal{H}}$  to be separable. We use the variant of Dunford-Pettis theorem established by us in § 3 to get around the difficulties that may be caused by non-separability (we of course assume that the Hilbert space  $\mathcal{H}$  is separable).

*Proof of Theorem 5.* We first consider the case  $1 \leq p < \infty$ , for convenience we assume that  $T$  is bounded by 1. Let  $f \in L^p_{\mathcal{H}}$ . Then

$$\int_x \| (Tf)(x) \|^p d\mu(x) \leq \int_x \| f(x) \|^p d\mu(x).$$

Since  $T$  commutes with multiplication by functions in  $\mathcal{A}$ , this yields

$$\int_x |\alpha(x)|^p \| (Tf)(x) \|^p d\mu(x) \leq \int_x |\alpha(x)|^p \| f(x) \|^p d\mu(x)$$

for all  $\alpha \in \mathcal{H}$ . From the weak\* density of  $\mathcal{A}$  in  $L^\infty$ , it follows that

$$\| (Tf)(x) \| \leq \| f(x) \| \text{ a.e.}$$

If  $L^p_{\mathcal{H}}$  is separable, we can obtain  $T(x)$  by an explicit construction. In the general case we argue as follows:

Define a map  $u: \mathcal{K}_{\mathcal{H}} \rightarrow \mathcal{H}$  by setting

$$u(\varphi) = \int_x (T\varphi)(x) d\mu(x), \quad \varphi \in \mathcal{K}_{\mathcal{H}}.$$

Then  $u$  is continuous with respect to the  $L^1_{\mathcal{H}}$ -norm on  $\mathcal{K}_{\mathcal{H}}$  because

$$\begin{aligned} \left\| \int_x (T\varphi)(x) d\mu(x) \right\| &\leq \int_x \| (T\varphi)(x) \| d\mu(x) \\ &\leq \int_x \| \varphi(x) \| d\mu(x). \end{aligned}$$

Since  $\mathcal{K}_{\mathcal{H}}$  is dense in  $L^1_{\mathcal{H}}$ ,  $u$  can be extended by continuity to the whole  $L^1_{\mathcal{H}}$  without increasing its norm. We denote the extended map also by  $u$ . By Theorem 3 there exists a function  $\Phi(x)$  from  $X$  into  $\mathcal{L}(\mathcal{H}, \mathcal{H})$  such that  $\Phi$  is weakly measurable, uniformly bounded with  $\| \Phi(x) \| \leq \| u \| \leq 1$  and

$$u(f) = \int_x \Phi(x)f(x) d\mu(x)$$

for every  $f \in L^1_{\mathcal{H}}$ . Thus for any  $\varphi \in \mathcal{H}_{\mathcal{H}}$

$$\int_x (T\varphi)(x) d\mu(x) = u(\varphi) = \int_x \Phi(x)\varphi(x) d\mu(x).$$

Since  $T$  commutes with multiplication by functions in  $\mathcal{A}$  and every  $\alpha \in \mathcal{A}$  is continuous, we get

$$\begin{aligned} \int_x \alpha(x)\Phi(x)\varphi(x) d\mu(x) &= \int_x \Phi(x)\alpha(x)\varphi(x) d\mu(x) \\ &= \int_x (T\alpha\varphi)(x) d\mu(x) = \int_x \alpha(x)(T\varphi)(x) d\mu(x). \end{aligned}$$

By the weak\* density of  $\mathcal{A}$  in  $L^\infty$ , this implies

$$(T\varphi)(x) = \Phi(x)\varphi(x) \text{ a.e.}$$

for all  $\varphi \in \mathcal{H}_{\mathcal{H}}$ . If  $\hat{\Phi}$  denotes the operator in  $L^p_{\mathcal{H}}$  defined by

$$(\hat{\Phi}f)(x) = \hat{\Phi}(x)f(x) \text{ a.e.,}$$

then we have  $T\varphi = \hat{\Phi}\varphi$  for all  $\varphi \in \mathcal{H}_{\mathcal{H}}$ . Since both  $T$  and  $\hat{\Phi}$  are bounded in  $L^p_{\mathcal{H}}$  and  $\mathcal{H}_{\mathcal{H}}$  is dense in  $L^p_{\mathcal{H}}$ , it follows that  $T = \hat{\Phi}$ . Now we have only to put  $\Phi(x) = T(x)$  in order to get the theorem.

If  $p = \infty$  and  $T$  is bounded and weakly\* continuous, then the transposed map  $T^*$  of  $T$  maps  $L^1_{\mathcal{H}}$  into  $L^1_{\mathcal{H}}$ . Since  $T^*$  commutes with multiplication by functions in  $\mathcal{A}$ ,  $T^*$  is expressed by an operator-valued function which is weakly measurable and uniformly bounded. Therefore  $T$  is also a uniformly bounded and weakly measurable operator-valued function  $T(x)$ . In this case, we clearly have

$$(Tf)(x) = T(x)f(x) \text{ l.a.e.}$$

for all  $f \in L^\infty_{\mathcal{H}}$ .

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