

# Pacific Journal of Mathematics

**WEAKLY COMPACT OPERATORS ON OPERATOR ALGEBRAS**

SHOICHIRO SAKAI

# WEAKLY COMPACT OPERATORS ON OPERATOR ALGEBRAS

SHÔICHIRO SAKAI

Let  $K$  be a compact space and  $C(K)$  be the commutative  $B^*$ -algebra of all complex valued continuous functions on  $K$ , then Grothendieck [3] (also we can see other proofs in [2]) proved the following remarkable properties:

(I) An arbitrary bounded operator of  $C(K)$  into a weakly sequentially complete Banach space is weakly compact.

(II) If  $T$  is a weakly compact operator of  $C(K)$  into a Banach space, then  $T$  maps weakly fundamental sequences into strongly convergent sequences.

On the other hand, let  $M$  be a  $W^*$ -algebra and  $M_*$  be the associated space of  $M$  (namely, the dual of  $M_*$  is  $M$  (cf. [8])) then the author [7] noticed that the Banach space  $M_*$  is weakly sequentially complete. Therefore, the above Grothendieck's theorems are applicable in the theory of operator algebras.

In this note, we shall show some applications, and state some related problems.

**PROPOSITION 1.** Let  $A$  be a  $B^*$ -algebra,  $E$  an abstract  $L$ -space,  $T$  be a bounded operator of  $A$  into  $E$ , then  $T$  is weakly compact.

*Proof.* Let  $T^*$  be the dual of  $T$ , then  $T^*$  is a bounded operator on the dual  $E^*$  of  $E$  to the dual  $A^*$  of  $A$ ;  $E^*$  is a Banach space of type  $C(K)$  (cf. [5]) and the second dual  $A^{**}$  of  $A$  is a  $W^*$ -algebra (cf. [9]), so that  $A^*$  is the associated space of a  $W^*$ -algebra; hence  $A^*$  is weakly sequentially complete; therefore  $T^*$  is weakly compact, so that by the well-known theorem,  $T$  is weakly compact. This completes the proof.

Now we shall show some applications.

1. Let  $G$  be a locally compact group,  $L^1(G)$  be the Banach space of all complex valued integrable functions on  $G$  with respect to a left, invariant Haar measure  $\mu$  and  $L^2(G)$  be the Banach space of all complex valued square integrable functions on  $G$  with respect to  $\mu$ . Under the convolutions (denoted by “\*”),  $L^1(G)$  is a Banach algebra.

On the other hand, for  $f \in L^1(G)$  and  $g \in L^2(G)$ , put  $L_f g = f * g$ ,

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then  $L_f$  is a bounded operator on  $L^2(G)$ ; we shall denote the uniform norm of  $L_f$  by  $\|L_f\|$ .

Now, let  $T$  be an operator on  $L^1(G)$ .  $T$  is said to be spectrally continuous, if it satisfies  $\|Th\|_1 \leq r\|L_h\|$  for all  $h \in L^1(G)$ , where  $\|\cdot\|_1$  is the  $L^1$ -norm and  $r$  is a fixed number.

Using the generalized Planchrel's theorem and the structure theorem of connected locally compact groups, Helgason [4] proved the following: Let  $G$  be a separable unimodular locally compact, non-compact, connected group, then a spectrally continuous operator on  $L^1(G)$  commuting with all right translations is identically 0.

In his review for the Helgason's paper, Mautner [6] asked whether these restrictions on the group  $G$  can be dropped.

Now we shall show

**THEOREM 1.** *Let  $G$  be a locally compact, non-compact group, then a spectrally continuous operator  $T$  on  $L^1(G)$  commuting with all right translations is identically zero.*

*Proof.* Let  $R(G)$  be the uniform closure of the set  $\{L_f | f \in L^1(G)\}$  in the  $B^*$ -algebra  $B$  of all bounded operators on  $L^2(G)$ , then  $R(G)$  is a  $B^*$ -algebra; since  $T$  is spectrally continuous, it can be uniquely extended to a bounded operator  $\tilde{T}$  of  $R(G)$  into  $L^1(G)$ ; by Proposition 1,  $\tilde{T}$  is weakly compact; let  $S$  be the unit sphere of  $R(G)$ ; since  $\|L_f\| \leq \|f\|_1$  for  $f \in L^1(G)$ ,  $L^1(G) \cap S$  contains the unit sphere of  $L^1(G)$ ; therefore the set  $\{Th | h \in L^1(G), \|h\|_1 \leq 1\}$  is relatively weakly compact in  $L^1(G)$ ; this implies that  $T$  is weakly compact as an operator on  $L^1(G)$ .

Since  $T$  commutes with all right translations, by the theorem of Wendel [10], there is a bounded Radon measure  $\nu$  such that  $Th = \nu * h$  for  $h \in L^1(G)$ ; let  $f$  be an element of  $L^1(G)$ , then the mapping  $h \rightarrow (f * \nu)^{**} (f * \nu) * h$  on  $L^1(G)$  is weakly compact, where  $(f * \nu)^*(x) = \rho(x) \overline{f * \nu(x^{-1})}$ , and  $d\mu(x^{-1}) = \rho(x) d\mu(x)$  for  $x \in G$ ; hence the mapping  $h \rightarrow \{(f * \nu)^{**} (f * \nu)\} * \{(f * \nu)^{**} (f * \nu)\} * h$  is strongly compact (cf. Cor 3.7 in [2]).

Put  $g = \{(f * \nu)^{**} (f * \nu)\} * \{(f * \nu)^{**} (f * \nu)\}$ , then  $g$  belongs to  $L^1(G)$ . Let  $S_1$  be the unit sphere of  $L^1(G)$ , then  $g * S_1$  is relatively strongly compact in  $L^1(G)$ , so that the set  $\{(g * f)^* = f^* * g^* | f \in S_1\}$  is also so; hence  $S_1 * g^*$  is relatively strongly compact; let  $\{v_\alpha\}_{\alpha \in \Pi}$  be a fundamental family of compact neighborhoods at a point  $s$  of  $G$  and let  $\{f_\alpha\}_{\alpha \in \Pi}$  be a family of continuous positive functions on  $G$  such that the support of  $f_\alpha$  is contained in  $v_\alpha$  and  $\int_G f_\alpha(x) dx = 1$ , then the directed set  $\{f_\alpha * g^*\}$  converges to  $sg^*$  in the  $L^1$ -norm, where  $sg^*(x) = g^*(s^{-1}x)$ ; therefore the set  $\{sg^* | s \in G\}$  is relatively strongly compact.

Now suppose that  $\|g^*\|_1 \neq 0$ , then it is enough to assume that

$\|g^*\|_1 = 1$ . There is a finite set  $\{s_1g^*, s_2g^*, \dots, s_n g^*\}$  where  $s_i \in G$  ( $i = 1, 2, \dots, n$ ) such that  $\inf_{1 \leq i \leq n} \|sg^* - s_i g^*\|_1 < 1/2$  for all  $s \in G$ .

On the other hand, let  $C$  be a compact subset of  $G$  such that

$$\int_{\sigma^{-\sigma}} |g^*(x)| d\mu(x) < \frac{1}{10} \quad \text{and} \quad \int_{\sigma^{-\sigma}} |g^*(s_i^{-1}x)| d\mu(x) < \frac{1}{10}$$

for  $i = 1, 2, \dots, n$ , and  $s$  be an element of  $G$  such that  $s \notin CC^{-1}$ , then  $s^{-1}C \cap C = (\phi)$ ; therefore

$$\begin{aligned} & \|sg^* - s_i g^*\|_1 \\ &= \int_{\sigma} |(sg^* - s_i g)(x)| d\mu(x) + \int_{\sigma^{-\sigma}} |(sg^* - s_i g^*)(x)| d\mu(x) \\ &\geq \int_{\sigma} |(s_i g^*)(x)| d\mu(x) - \int_{\sigma} |(sg^*)(x)| d\mu(x) \\ &\quad + \int_{\sigma^{-\sigma}} |(sg^*)(x)| d\mu(x) - \int_{\sigma^{-\sigma}} |(s_i g^*)(x)| d\mu(x) \\ &\geq \left(1 - \frac{1}{10}\right) - \int_{s^{-1}\sigma} |g^*(x)| d\mu(x) + \left(1 - \frac{1}{10}\right) - \frac{1}{10} \\ &\geq \left(1 - \frac{1}{10}\right) - \frac{1}{10} + \left(1 - \frac{1}{10}\right) - \frac{1}{10} = \frac{8}{5} \quad \text{for all } i. \end{aligned}$$

This is a contradiction; hence  $g^* = 0$ , so that  $g = f*\nu = 0$ ; since  $f$  is an arbitrary element of  $L^1(G)$ ,  $\nu = 0$ , so that  $T = 0$ . This completes the proof.

2. At first we shall show

**PROPOSITION 2.** Let  $A$  be a weakly sequentially complete  $B^*$ -algebra, then  $A$  is finite dimensional.

*Proof.* It is enough to assume that  $A$  has unit. Let  $C$  be a maximal abelian  $*$ -subalgebra of  $A$ , then  $C$  is a Banach space of type  $C(K)$  and weakly sequentially complete; by the Grothendieck's theorem, the identity mapping  $T$  on  $C$  is weakly compact, so that  $T^2 = T$  is strongly compact on  $C$  (cf. Cor 3.7 in [2]); hence  $C$  is finite-dimensional. Therefore there is a finite family of mutually orthogonal projections  $(e_1, e_2, \dots, e_n)$  by which  $C$  is linearly spanned; by the maximality of  $C$ ,  $e_i A e_i$  ( $i = 1, 2, \dots, n$ ) is one-dimensional.

For any  $x, y \in A$ , there is a complex number  $\lambda_i(x, y)$  such that  $e_i y^* x e_i = \lambda_i(x, y) e_i$ ; clearly  $\lambda_i(x, x) \geq 0$ , and if  $\lambda_i(x, x) = 0$ ,  $x e_i = 0$ ; moreover  $\|x e_i\| = \|e_i x^* x e_i\|^{1/2} = \lambda_i(x, x)^{1/2}$ ; therefore a Banach subspace  $A e_i$  of  $A$  is a hilbert space; since  $A = \sum_{i=1}^n A e_i$ ,  $A$  is reflexive, so that  $A$  is a reflexive  $W^*$ -algebra; since all irreducible  $*$ -representations of

$A$  are  $\sigma$ -continuous,  $A$  is of type  $I$ ; since the center of  $A$  is finite-dimensional,  $A$  is a direct sum of a finite family of type  $I$ -factors  $(A_1, A_2, \dots, A_m)$ ; since  $A_j$  can be considered the algebra of all bounded operators on a hilbert space  $\mathfrak{h}_j$  for  $j = 1, 2, \dots, m$ , the reflexivity of  $A_j$  implies the finite-dimensionality of  $\mathfrak{h}_j$  and so the finite-dimensionality of  $A_j$ ; hence  $A$  is finite-dimensional. This completes the proof.

**COROLLARY 1.** *Let  $A$  be an infinite dimensional  $B^*$ -algebra and  $E$  be a Banach space of type  $L^p$  ( $1 \leq p < +\infty$ ) or the associated space of a  $W^*$ -algebra, then the Banach space  $A$  is not topologically isomorphic to  $E$ .*

Since  $B^*$ -algebras are Banach spaces which have many analogous properties with  $C(K)$ ; therefore it is very natural to ask whether the theorems of Grothendieck are positive in  $B^*$ -algebras.

We have no solution for the property (I); here we shall show that the property (II) is negative, and show an application.

A negative example. Let  $B(\mathfrak{h})$  be the  $B^*$ -algebra of all bounded operators on an infinite dimensional hilbert space  $\mathfrak{h}$ , and  $e$  be an one-dimensional projection on  $\mathfrak{h}$ , then the Banach subspace  $B(\mathfrak{h})e$  of  $B(\mathfrak{h})$  is isometric to  $\mathfrak{h}$  [cf. [7]]; therefore the mapping  $x \xrightarrow{T} xe$  of  $B(\mathfrak{h})$  into  $B(\mathfrak{h})e$  is weakly compact; the unit sphere  $S$  of  $B(\mathfrak{h})e$  is weakly compact in  $B(\mathfrak{h})$ ; therefore if  $B(\mathfrak{h})$  satisfies the property (II),  $TS = S$  is strongly compact, so that  $B(\mathfrak{h})e$  is finite-dimensional, a contradiction.

Concerning the property (I), we can notice that many operators satisfy the property (I).

For instance, let  $A$  be a  $B^*$ -algebra,  $A^*$  the dual of  $A$ . For  $x, a \in A$  and  $f \in A^*$ , put  $(Laf)(x) = f(ax)$  and  $(Raf)(x) = f(xa)$ ; we can consider bounded operators  $a \xrightarrow{T} Raf$ ,  $a \xrightarrow{S} Laf$  of  $A$  into  $A^*$ , then  $T$  and  $S$  are weakly compact (cf. [8]).

Finally we shall show an application.

**THEOREM 2.** *Let  $A$  be a  $B^*$ -algebra having an infinite dimensional irreducible  $*$ -representation, and  $E$  be a Banach space of type  $L^p$  ( $1 \leq p \leq +\infty$ ) or type  $C(\Omega)$ , where  $\Omega$  is a locally compact space and  $C(\Omega)$  is the Banach space of all continuous functions vanishing at infinity, or the associated space of a  $W^*$ -algebra, then the Banach space  $A$  is not topologically isomorphic to  $E$ .*

*Proof.* It is enough to show that  $A$  is not topologically isomorphic to  $C(\Omega)$ . Suppose that  $A$  is topologically isomorphic to  $C(\Omega)$ , then there is an isomorphism  $T$  of  $A$  onto  $C(\Omega)$ . Take the second dual  $T^{**}$  of  $T$ , the  $T^{**}$  gives an isomorphism of  $A^{**}$  onto  $C(\Omega)^{**}$ ;  $A^{**}$  is

a  $W^*$ -algebra and  $C(\Omega)^{**}$  is a Banach space of type  $C(K)$ ; since a  $*$ -representation of  $A$  can be uniquely extended to a  $\sigma$ -continuous  $*$ -representation of  $A^{**}$  (cf. [8]),  $A^{**}$  has an infinite dimensional irreducible  $W^*$ -representation; hence there is a central projection  $z$  of  $A^{**}$  such that  $A^{**}z$  is a factor of type  $I_\infty$ ; from the above negative example,  $A^{**}z$  has not the property (II); on the other hand, since  $A^{**} = A^{**}z \oplus A^{**}(1-z)$ , where  $1$  is the unit of  $A^{**}$ ,  $C(\Omega)^{**} = T^{**}(A^{**}z) + T^{**}(A^{**}(1-z))$ ; since  $T^{**}(A^{**}z)$  has the closed complement subspace in  $C(\Omega)^{**}$ ,  $T^{**}(A^{**}z)$  has the property (II), so that  $A^{**}z$  has the property (II). This is a contradiction, and completes the proof.

**COROLLARY 2.** *Let  $F$  be the associated space of a  $W^*$ -algebra without a type  $I_n$  part ( $n < +\infty$ ), and  $E$  be a Banach space of type  $L^p$  or  $C(\Omega)$  then  $F$  is not topologically isomorphic to  $E$ .*

*Proof.* Suppose that  $F$  is topologically isomorphic to  $E$ , then  $F^*$  is topologically isomorphic to  $E^*$ . This is a contradiction.

**REMARK.** Theorem 2 and Corollary 2 imply that the above mentioned  $B^*$ -algebras or associated spaces (for instance, the  $B^*$ -algebra  $\mathcal{C}$  of all compact operators on an infinite dimensional hilbert space, the  $B^*$ -algebra  $B(\mathfrak{h})$  of all bounded operators on an infinite dimensional hilbert space  $\mathfrak{h}$ , the  $B^*$ -algebra  $R(G)$  corresponding to all non-almost periodic locally compact groups, and all  $W^*$ -factor with an exception of type  $I_n$  ( $n < +\infty$ ) and their associated spaces) are not topologically contained in the classes of the so-called classical Banach spaces  $((M), (m), (C), (c), (C^{(p)})_{p \geq 1}, (L^p)_{1 \leq p \leq +\infty}, (l^p)_{1 \leq p \leq +\infty})$  mentioned by Banach (cf. [1]); therefore it is very meaningful to examine whether many unsolved problems concerning Banach spaces are positive in these examples.

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