A RESULT CONCERNING INTEGRAL BINARY QUADRATIC FORMS

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This paper contains an extension of an earlier work by Dickson ([1], p. 95), in which the following theorem was proven:

THEOREM 1. (Dickson's Theorem). *If a number is represented properly by a form $[a, b, c]$ of discriminant $D = 4ac - b^2$, then any divisor of that number is represented by some form of the same discriminant D.*

DEFINITION. ([1], p. 68). A positive form $[a, b, c]$ is called reduced if $-a < b \leq a, c \geq a$, with $b \geq 0$ if $c = a$.

As a consequence of the above definition it follows that $4a^2 \leq 4ac = D + b^2 \leq D + a^2$, $3a^2 \leq D$, and finally $a \leq \sqrt{(1/3)\, D}$

THEOREM 2. *Let M be properly represented by the integral positve definite quadratic form $a\alpha^2 + b\alpha\gamma + c\gamma^2$ of discriminant $D = 4ac - b^2$. If $M \leq 3D/16$ and $(D, M) = 1$, then in every factorization of M one of the factors is a_i, one of the minimal values of a primitive quadratic form of discriminant D. In other words, $M = M_1M_2$ where M_1 is a unit or a prime and M_2 is the product of no more than two a_i.*

Proof. According to the remark following the definition $a_i \leq \sqrt{D/3}$, where equality for a primitive reduced form is possible only if $a_i = b_i = c_i = 1$ and hence $D = 3$ so that the inequality $0 < M \leq 3D/16$ cannot be satisfied. Thus $a_i < \sqrt{D/3}$.

Now assume $M = r_1 r_2$. Then according to Theorm 1 it follows that

$$r_1 = a_i\alpha_i^2 + b_i\alpha_i\gamma_i + c_i\gamma_i^2 , \quad r_2 = a_j\alpha_j^2 + b_j\alpha_j\gamma_j + c_j\gamma_j^2$$

where the two quadratic forms are primitive reduced forms of discriminant D. Hence

$$(4a_i r_1)(4a_j r_2) = [(2a_i\alpha_i + b_i\gamma_i)^2 + D\gamma_i^2][(2a_j\alpha_j + b_j\gamma_j)^2 + D\gamma_j^2]$$
$$= (\beta_i^2 + D\gamma_i^2)(\beta_j^2 + D\gamma_j^2) = 16a_i a_j M$$
$$< 16(D/3)M \leq (16D/3)(3D/16) = D^2 ,$$

where $\beta_i = (2a_i\alpha_i + b_i\gamma_i)$ and $\beta_j = (2a_j\alpha_j + b_j\gamma_j)$. This implies that $\gamma_i\gamma_j = 0$, say $\gamma_i = 0$, and therefore $r_1 = a_i$.

To prove the final statement of the theorem, assume $M \neq a_i$ and

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let r_2 be a minimal factor of M so that $r_2 \neq a_j$. If M_1 is any prime factor of r_2, then $M = M_1 M_2$ where $M_2 = (M/r_2)(r_2/M_1) = a_i a_j$.

REFERENCE

1. L. E. Dickson, *Introduction to the Theory of Numbers*, Dover Publications, Inc., New York, 1929.

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