Pacific Journal of Mathematics

A RESULT CONCERNING INTEGRAL BINARY QUADRATIC FORMS

WILLIAM EDWARD CHRISTILLES

Vol. 14, No. 3 July 1964

A RESULT CONCERNING INTEGRAL BINARY QUADRATIC FORMS

WILLIAM EDWARD CHRISTILLES

This paper contains an extension of an earlier work by Dickson ([1], p. 95), in which the following theorem was proven:

THEOREM 1. (Dickson's Theorem). *If a number is represented properly by a form* $[a, b, c]$ *of discriminant* $D = 4ac - b^2$, *then any divisor of that number is represented by some form of the same discriminant* D.

DEFINITION. ([1], p. 68). A positive form $[a, b, c]$ is called reduced if $-a < b \leqq a, c \geqq a$, with $b \geqq 0$ if $c = a$.

As a consequence of the above definition it follows that $4a^2 \leqq 4ac = D + b^2 \leqq D + a^2, 3a^2 \leqq D$, and finally $a \leqq \sqrt{(1/3)\,D}$

THEOREM 2. *Let* M *be properly represented by the integral positve definite quadratic form* $a\alpha^2 + b\alpha\gamma + c\gamma^2$ *of discriminant* $D = 4ac - b^2$. *If* $M \leqq 3D/16$ *and* $(D, M) = 1$, *then in every factorization of* M *one of the factors is* a_i, *one of the minimal values of a primitive quadratic form of discriminant* D. *In other words,* $M = M_1M_2$ *where* M_1 *is a unit or a prime and* M_2 *is the product of no more than two* a_i.

Proof. According to the remark following the definition $a_i \leqq \sqrt{D/3}$, where equality for a primitive reduced form is possible only if $a_i = b_i = c_i = 1$ and hence $D = 3$ so that the inequality $0 < M \leqq 3D/16$ cannot be satisfied. Thus $a_i < \sqrt{D/3}$.

Now assume $M = r_1r_2$. Then according to Theorm 1 it follows that

$$r_1 = a_i\alpha_i^2 + b_i\alpha_i\gamma_i + c_i\gamma_i^2\ , \quad r_2 = a_j\alpha_j^2 + b_j\alpha_j\gamma_j + c_j\gamma_j^2$$

where the two quadratic forms are primitive reduced forms of discriminant D. Hence

$$\begin{aligned}(4a_ir_1)(4a_jr_2) &= [(2a_i\alpha_i + b_i\gamma_i)^2 + D\gamma_i^2]\,[(2a_j\alpha_j + b_j\gamma_j)^2 + D\gamma_j^2] \\ &= (\beta_i^2 + D\gamma_i^2)(\beta_j^2 + D\gamma_j^2) = 16a_ia_jM \\ &< 16(D/3)M \leqq (16D/3)(3D/16) = D^2\ ,\end{aligned}$$

where $\beta_i = (2a_i\alpha_i + b_i\gamma_i)$ and $\beta_j = (2a_j\alpha_j + b_j\gamma_j)$. This implies that $\gamma_i\gamma_j = 0$, say $\gamma_i = 0$, and therefore $r_1 = a_i$.

To prove the final statement of the theorem, assume $M \neq a_i$ and

Received November 21, 1963. The author is indebted to the referee for the suggested revision of both the statement and the proof of Theorem 2.

let r_2 be a minimal factor of M so that $r_2 \neq a_j$. If M_1 is any prime factor of r_2, then $M = M_1M_2$ where $M_2 = (M/r_2)(r_2/M_1) = a_ia_j$.

Reference

1. L. E. Dickson, *Introduction to the Theory of Numbers*, Dover Publications, Inc., New York, 1929.

St. Mary's University, San Antonio, Texas

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should by typewritten (double spaced), and on submission, must be accompanied by a separate author's résumé. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, L. J. Paige at the University of California, Los Angeles 24, California.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published quarterly, in March, June, September, and December. Effective with Volume 13 the price per volume (4 numbers) is $18.00; single issues, $5.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $8.00 per volume; single issues $2.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 14, No. 3 July, 1964