Pacific Journal of Mathematics

INVERSION AND REPRESENTATION THEOREMS FOR A GENERALIZED LAPLACE TRANSFORM

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Vol. 14, No. 3 July 1964

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1. Introduction. In a series of recent papers I have discussed various properties and inversion theorems etc. for the transform

$$F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1}\int_0^\infty (xy)^{\beta}{}_1F_1(\beta+\eta+1; \alpha+\beta+\eta+1; -xy)f(y)dy \,. \tag{1.1}$$

where $f(y) \in L0, \infty)$, $\beta \geqq 0$, $\eta > 0$.

$$= A\int_0^\infty (xy)^{\beta}\psi(x, y)f(y)dy$$

where for convenience we denote $\Gamma(\beta+\eta+1)/\Gamma(\alpha+\beta+\eta+1)$ by A and ${}_1F_1(a; b; -xy)$ by $\psi(xy)$; a and b standing respectively for $\beta+\eta+1$ and $a+\alpha$. For $\alpha=\beta=0$ (1.1) reduces to the wellknown Laplace transform

$$F(x) = \int_0^\infty e^{-xy}f(y)dy \,. \tag{1.2}$$

The transform (1.1), which may be called a generalization of the Laplace transform, arises if we apply Kober's operators of fractional integration [2] to the function $x^{\beta}e^{-x}$[1].

The object of the present paper is to obtain an inversion and a representation theorem for the transform (1.1) by using properties of Kober's operators defined below.

2. Definition of operations. The operators given by Kober are defined as follows.

$$I^+_{\eta,\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)}x^{-\eta-\alpha}\int_0^x (x-u)^{\alpha-1}u^{\eta}f(u)du$$

$$K^-_{\zeta\,\alpha}[f(x)] = \frac{1}{\Gamma(\alpha)}x^{\zeta}\int_n^\infty (u-x)^{\alpha-1}u^{-\zeta-\alpha}f(u)du$$

where $f(x) \in L_p(0, \infty)$, $1/p + 1/q = 1$, if $1 < p < \infty$ and $1/p$ or $1/q$ 0 if p or $q=1$, $\alpha > 0$, $\zeta > -(1/p)$, $\eta > -(1/q)$.

The Mellin transform $\bar{M}f(x)$ of a function $f(x) \in L_p(0, \infty)$ is defined as

Received November 4, 1963.

$$\bar{M}f(x) = \int_0^\infty f(x)x^{it}du \qquad (p=1)$$

and

$$= \underset{x\to\infty}{\text{index}V \lim} \int_{1/x}^{x} f(x)^{it-1/q}dn \qquad (p>1)\,.$$

The inverse Mellin transform $M^{-1}\phi(t)$ of a function $\phi(t) \in L_q(-\infty, \infty)$ is defined by

$$M^{-1}\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(t)x^{-it}dt \qquad (q=1) \tag{2.1}$$

and

$$= \frac{1}{2\pi} \underset{T\to\infty}{\text{index}p \lim} \int_{-T}^{T} \phi(t)x^{-it-1/p}dt \qquad (q>1)\,.$$

If Mellin transform is applied to Kober's operators and the orders of integrations are interchanged we obtain, under certain conditions

$$\bar{M}\{I_{\eta}^{+}{}_{\alpha}f(x)\} = \frac{\Gamma\left(\eta + \frac{1}{q} - it\right)}{\Gamma\left(\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right]}\bar{M}f(x)$$

and

$$\bar{M}\{K_{\zeta}^{-}{}_{\alpha}f(x)\} = \frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]}\bar{M}f(x)\,.$$

But

$$\bar{M}(e^{-x}\cdot x^{\beta}) = \int_0^\infty e^{-x}x^{\beta+it-1/q}dx = \Gamma\left(\beta + it + \frac{1}{p}\right), \quad \text{if } Re\left(\beta + \frac{1}{p}\right) > 0\,.$$

Therefore

$$\bar{M}\{I_{\eta,\alpha}^{+}(x^{\beta}e^{-x})\} = \frac{\Gamma\left[\left(\eta + \frac{1}{q} - it\right)\right]\Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left\{\eta + \frac{1}{q} - it\right\}\right]}$$

and

$$\bar{M}\{K^{-}_{\zeta,\alpha}(x^{\beta}e^{-x})\} = \frac{\Gamma\left(\beta + it + \frac{1}{p}\right)P\left(\zeta + it + \frac{1}{p}\right)}{\Gamma\left[\alpha + \left\{\zeta + \frac{1}{p} + it\right\}\right]} .$$

By (2.1) we then have

$$I^{+}_{\eta,\alpha}(x^{\beta}e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma\left(\eta + \frac{1}{q} - it\right)\Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\eta + \frac{1}{q} - it\right)\right]}x^{-it-1/p}dt \tag{2.2}$$

and

$$K^{-}_{\zeta,\alpha}(x^{\beta}e^{-x}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Gamma\left(\zeta + \frac{1}{p} + it\right)\Gamma\left(\beta + \frac{1}{p} + it\right)}{\Gamma\left[\alpha + \left(\zeta + \frac{1}{p} + it\right)\right]}x^{-it-1/p}dt ,$$

provided that $1/p > 0$, $\eta + 1/q > 0$ and $\zeta + 1/p > 0$.

3. **Inversion theorem.** We now define an inversion operator which will serve to invert (1.1).

An operator is defined for integral values of n by the relations

$$W_0[G(x)] = G(x) ,$$

$$W_n[G(x)] = (-)^n n^{\beta+n+1}\left(\frac{d}{dx}\right)^n[x^{-\beta}G(x)],\ (n = 1, 2, \cdots)$$

$$Q_{n,t}[G(x)] = \frac{1}{\Gamma(n + 1 + \beta - \alpha)}[W_n[G(x)]]_{n=n/t}(n = 1, 2, \cdots) .$$

THEOREM 3.1. *If $f(t)$ is bounded in $(0 < t < \infty)$ then, provided that the integral* (1.1) *converges, $\eta > 0$, $\beta \geqq 0$*

$$f(t) = \lim_{n\to\infty} Q_{n,t}[F(x)]$$

for almost all positive t.

Proof. Let x be any number greater than zero. Then, since the integral (1.1) converges, we can differentiate under the integral sign. Also (2.2) gives

$$\left(\frac{d}{dx}\right)[x^{-\beta}I_{n,\alpha}(x^{\beta}e^{-x})] = -x^{-\beta}I_{\eta+1,\alpha}[x^{\beta}e^{-x}] . \tag{3.1}$$

Using this relation we get

$$W_n[F(n)] = (-)^n n^{\beta+n+1}\int_0^\infty x^{-\beta} y^n I_{\eta+n,\alpha}\{(xy)^\beta e^{-xy}\} f(y)dy$$
$$= \frac{\Gamma(\beta+\eta+n+1)}{\Gamma(\alpha+\beta+\eta+n+1)}\int_0^\infty y^{\beta+n}{}_1F_1(\beta+\eta+n+1;$$
$$\alpha+\beta+\eta+n+1 - xy) f(y)dy\ .$$

Therefore

$$Q_{n,t}\{F(x)\}$$
$$= \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\left(\frac{n}{t}\right)^{\beta+n+1} \frac{1}{\Gamma(n+\beta+1-\alpha)}$$
$$\times \int_0^\infty y^{\beta+n}{}_1F_1(n+\beta+\eta+1; \alpha+\beta+\eta+1+n; -xy) f(y)dy$$
$$= \frac{1}{\Gamma(n+\beta+1-\alpha)}\left(\frac{n}{t}\right)^{\beta+n+1} \frac{\Gamma(a)}{\Gamma(b)}$$
$$\times \int_0^\infty y^{\beta+n}{}_1F_1(a+n; b+n; -xy) f(y)dy$$

in the notation of § 1.

$$= \frac{\Gamma(a+n)}{\Gamma(b+n)\Gamma(n+\beta+1-\alpha)}\left(\frac{n}{t}\right)^{n+\beta+1}$$
$$\times \int_0^\infty (tv)^{n+\beta}{}_1F_1(a+n; b+n; -nv) f(tv)dt$$
$$= \frac{\Gamma(a+n)}{\Gamma(b+n)\Gamma(n+\beta+1-\alpha)}\left(\frac{n}{t}\right)^{n+\beta+1}$$
$$\times \int_0^\infty v^{n+\beta}{}_1F_1(\beta+\eta+n+1; \alpha+\beta+\eta+n+1; -nv) f(tv)dt$$

by a simple change of variable. Now by using a result of Slater [4] we have

$$\frac{\Gamma(a+n)}{\Gamma(b+n)}{}_1F_1(a+n; b+n; -v) \sim (nv)^{a-b} e^{-nv} \qquad (n \longrightarrow \infty).$$

Therefore

$$\lim_{n\to\infty} Q_{n,t}\{F(n)\} = \lim_{n\to\infty} \frac{n^{\beta+n+1-\alpha}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty v^{n+\beta-\alpha} e^{-nv} f(tv)dv\ .$$

But [3] we have for almost all positive t

$$\lim_{n\to\infty} \frac{n^{\beta+n+1-\alpha}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty y^{n+\beta-\alpha} e^{-ny}\{f(ty) - f(t)\}dy = 0$$

and so we have our theorem.

5. **Representation theorem.** In this section we propose to give a set of necessary and sufficient conditions for the representation of a function as an integral of the form (1.1). We shall need a lemma which we now prove.

LEMMA 4.1. *If n is a positive integer and x and t are positive variables then*

$$\left(\frac{\partial}{\partial t}\right)^n\left[t^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] = \frac{n^n}{t^{n+1-\beta}} I_{\eta+n,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}.$$

Proof. It is plain that

$$\left(\frac{t}{x}\right)^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}$$

is a homogeneous function of zero order. Therefore applying Euler's theorem we get

$$t\left(\frac{\partial}{\partial t}\right)\left[\left(\frac{t}{x}\right)^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] + n\left(\frac{\partial}{\partial x}\right)\left[\left(\frac{t}{x}\right)^{\beta+n-1}I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] = 0$$

or

$$\left(\frac{\partial}{\partial t}\right)\left[\frac{t^{\beta+n-1}}{x^{\beta+n}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] = -\left(\frac{\partial}{\partial x}\right)\left[\frac{t^{\beta+n-2}}{x^{\beta+n-1}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right]$$

or

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\left[\frac{t^{\beta+n-1}}{x^{\beta+n}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right] &= -\frac{\partial^2}{\partial t\partial x}\left[\frac{t^{\beta+n-2}}{x^{\beta+n-1}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right]\\ &= -\left(\frac{\partial}{\partial x}\right)\left[\frac{\partial}{\partial t}\left\{\frac{t^{\beta+n-2}}{x^{\beta+n-1}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right\}\right]\\ &= (-)^2\frac{\partial^2}{\partial x^2}\left[\frac{t^{\beta+n-3}}{x^{\beta+n-2}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right].\end{aligned}$$

Proceeding in the same manner we have

$$\frac{\partial^n}{\partial t^n}\left[\frac{t^{\beta+n-1}}{x^{\beta+n}} I_{\eta,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-n/t}\right\}\right] = \frac{t^{\beta-n-1}}{x^\beta} I_{\eta+n,\alpha}\left\{\left(\frac{x}{t}\right)^\beta e^{-x/t}\right\}\right]$$

using (3.1).

THEOREM 4.1. *The necessary and sufficient conditions that a given function $F(x)$ may have the representation* (1.1) *with $f(y)$ bounded and $Re\,\eta > 0$ $Re\,\beta \geqq 0$ are that*

(i) *$F(x)$ has derivatives of all orders in $0 < x < \infty$.*

(ii) *$F(x)$ tends to zero as x tends to infinity and*
(iii) $|Q_{n,t}\{F(x)\}| < M$ *for all integral* n $(0 < t < \infty)$.

Proof. First let us suppose that $F(x)$ has the representation (1.1). Under the conditions of the theorem it is obvious that all the derivatives of $F(x)$ exist. Also

$$\begin{aligned} F(x) &\leqq M' \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \\ &\quad \times \int_0^\infty (xy)^\beta {}_1F_1(\beta+\eta+1;\ \alpha+\beta+\eta+1;\ -xy)dy \\ &= \frac{M'\Gamma(\eta)\Gamma(\beta+1)}{x\Gamma(\alpha+\eta)} \end{aligned}$$

since $f(y)$ is bounded. So $F(x)$ tends to zero as x tends to infinity. To prove the necessity of (iii) we see, as in Theorem 3.1, that

$$|Q_{n,t}\{F(x)\}| \leqq \left\{\frac{n^{\beta+n+1-\alpha}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty v^{n+\beta-\alpha}e^{-nv}dv\right\}\left\{\operatorname*{lub}_{0\leqq t<\infty} |f(tv)|\right\} = M .$$

To prove the sufficiency let us suppose that the conditions are satisfied. If we now set

$$J_n = \int_0^\infty I_{\eta,\alpha}\{(xy)^\beta e^{-xy}\}Q_{n,y}\{F(x)\}dy$$

we have

$$\begin{aligned} J_n &= \frac{1}{\Gamma(n+1+\beta-\alpha)}\int_0^\infty \frac{n}{t^2} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\} W_n\{F(x)\}dn \\ &= (-)^n \int_0^\infty nt^{n+\beta-1} I_{\eta,\alpha}\left\{\left(\frac{nx}{t}\right)^\beta e^{-nx/t}\right\}\left(\frac{d}{dt}\right)^n \{t^{-\beta}F(t)\}dt . \end{aligned}$$

It will be seen in the course of the arguement that this integral exists. Integrating by parts we have

$$\begin{aligned} J_n &= \frac{(-)^n n}{\Gamma(n+\beta+1-\alpha)}\left[t^{n+\beta-1}I_{\eta,\alpha}\left\{\left(\frac{nn}{t}\right)^\beta e^{-nn/t}\right\}\left(\frac{d}{dt}\right)^{n-1}\{t^{-\beta}F(t)\}\right]_0^\infty \\ &\quad + \frac{(-)^{n-1}n}{\Gamma(n+1+\beta-\alpha)}\int_0^\infty \left(\frac{d}{dt}\right)^{n-1}\{t^{-\beta}F(t)\}\left(\frac{\partial}{\partial t}\right)\{t^{n+\beta-1}I_{\eta,\alpha}\phi\}dt \end{aligned}$$

where

$$\phi \equiv \left(\frac{nx}{t}\right)^\beta e^{-nx/t} .$$

Now

$$\begin{aligned} I_{\eta\alpha}\phi &= 0(t^{\eta+1}) \qquad (t \to 0) \\ &= 0(1) \quad \beta = 0(t \to \infty) \\ &= 0(1) \quad \beta > 0(t \to \infty) \end{aligned}$$

for [1]

$$I_{\eta,\alpha}(\phi) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)}\left(\frac{nx}{t}\right)^{\beta} {}_1F_1\left(\beta+\eta+1; \alpha+\beta+\eta+1; -\frac{nx}{t}\right).$$

Also the hypotheses of the theorem by implications mean that

$$F(x) = 0(x^{-1})$$

and in general

$$F^{(n)}(x) = 0(x^{-n-1})$$

and

$$\left(\frac{d}{dt}\right)^{n-1}[t^{-\beta}F(t)]$$
$$= \{(-)^{n-1}\beta(\beta+1)\cdots(\beta+n-2)t^{-\beta-n+1}F(t) + \cdots t^{-\beta}F^{(n-1)}(t)\}.$$

Therefore the integrated part

$$= 0[t^{\eta+1}\{A_1F(t) + \cdots t^{n-1}F^{(n-)}(t)\}] \to 0 \quad \text{as} \quad t \to 0.$$

Also it is

$$= 0[A_1F(t) + \cdots tF^{(n-1)}(t)] \to 0 \qquad \text{as} \quad t \to \infty.$$

Therefore the integrated part is zero and integrating by parts again

$$J_n = \frac{(-)^{n-1}n}{\Gamma(n+\beta+1-\alpha)}\left[\frac{\partial}{\partial t}(t^{n+\beta-1}I_{\eta\alpha}\phi)\left(\frac{d}{dt}\right)^{n-2}\{t^{-\beta}F(t)\}\right]_0^\infty$$
$$+ \frac{(-)^{n-2}n}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty \left(\frac{d}{dt}\right)^{n-2}\{t^{-\beta}F(t)\}\frac{\partial^2}{\partial t^2}(t^{n+\beta-1}I_{\eta,\alpha}\phi)dt.$$

Now

$$\left(\frac{\partial}{\partial t}\right)\{t^{\beta+n-1}I_{\eta,\alpha}\phi\} = [(n-1)t^{\beta+n-2}I_{\eta,\alpha}\phi + \cdots + nnt^{\beta+n-3}I_{\eta+1,\alpha(\varphi)}]$$

and

$$\left(\frac{d}{dt}\right)^{n-2}\{t^{-\beta}F(t)\}$$
$$= \{(-)^{n-2}\beta(\beta+1)\cdots(\beta+n-3)t^{-\beta-n+2}F(t) + \cdots t^{-\beta}F^{(n-2)}(t)\}.$$

Therefore as before the integrated part again approaches zero when t tends to zero and t tends to infinity. Proceeding in the same manner we obtain

$$J_n = \frac{n}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty t^{-\beta}F(t)\frac{\partial^n}{\partial t^n}\{t^{\beta+n-1}I_{\eta,\alpha}\phi\}dt$$
$$= \frac{n}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty t^{-\beta}F(t)\frac{(nx)^n}{t^{n+1}}t^{\beta}I_{\eta+n,\alpha}(\phi)dt$$

by the Lemma 4.1. Hence

$$J_n = \frac{n^{n+\beta+1}n^{n+\beta}\Gamma(a)}{\Gamma(n+\beta+1-\alpha)\Gamma(b)}\int_0^\infty t^{-\beta-n-1}{}_1F_1\left(a;b;-\frac{nx}{t}\right)F(t)dt\ .$$

It is clear that this integral exists under the hypotheses of the theorem and therefore all the previous integrals exist. By a simple substitution this gives on using the asymptotic expansion of ${}_1F_1(a;b;x)$ [4]

$$J_n \sim \frac{n^{\beta+n+1}n^{n+\beta}}{\Gamma(n+\beta+1-\alpha)}\int_0^\infty u^{\beta+n-1}e^{-nxu}F\left(\frac{1}{u}\right)du\ .$$

Let

$$(1/u)F\left(\frac{1}{u}\right) \equiv \psi(u)\ .$$

Now

$$(1/u)F(1/u) = 0(1) \quad (u \to \infty) \quad \text{and} \quad F\left(\frac{1}{u}\right) = 0(1) \quad (u \to 0)\ .$$

Hence it is easily seen
(i) $\psi(u) \in L$ $(1/R \leqq t < R)$ for every $R > 1$.
(ii) $\int_1^\infty \psi(u)e^{-cu}du$ converges for any fixed $c > 0$, and
(iii) $\int_0^1 u\psi(u)du$ also converges. Therefore [3]

$$\lim_{n\to\infty} J_n = \frac{1}{u}\psi\left(\frac{1}{u}\right) = F(u)\ .$$

Now if

$$\chi(x,y) = \frac{\Gamma(a)}{\Gamma(b)}(xy)^{\beta}{}_1F_1(a;b;-xy)\ .$$

Then $\chi(xy) \in L$ in $0 \leqq y < \infty$ under the conditions assumed for the convergence of (1.1). Therefore by a theorem on weak compactness of a set of functions [5] the inequalities in the hypothesis (iii) of the theorem imply the existence of a subset $\{n_i\}$ of the positive integers and a bounded function $f(y)$ such that

$$\lim_{i\to\infty}\int_0^\infty [Q_{n_i,y}\{F(x)\}]\chi(x, y) = \int_0^\infty \chi(x, y)f(y)dy .$$

Hence

$$F(x) = \int_0^\infty \chi(x, y)f(y)dy$$

and the theorem is established.

I am indebted to Dr. K. M. Saksena for guidance and help in the preparation of the paper.

References

1. A. Erdelyi, *On some functional transformations,* Rend. del Semin. Matematico. **10** (1950-51), 217-234.
2. H. Kober, *On fractional integrals and derivatives,* Quart. Jour. Math. **11** (1940), 193-211.
3. J. M. C. Joshi, *A singular integral and a real inversion theorem for a generalized Laplace transform,* to appear.
4. L. J. Slater, *Confluent Hypergeometric Functions,* Cambridge University Press, 1960.
5. D. V. Widder, *The Laplace Transform,* Princeton University Press 1946.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

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