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**BASIC SEQUENCES AND THE PALEY-WIENER CRITERION**

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**1. Introduction.** Throughout the paper  $X$  will denote a complete metric linear space (i.e., a complete topological linear space with topology derived from a metric  $d$  with the property that  $d(x, y) = d(x - y, 0)$ , for all  $x, y \in X$ ) or some specialization thereof over the real or complex field;  $\|x\|$  will denote  $d(x, 0)$ ; and if  $\{x_n\}$  is a sequence in  $X$ ,  $[x_n]$  will denote the closed linear span of the elements  $\{x_n\}_{n \in \omega}$ .

A sequence  $\{x_n\}$  is said to be a *basic sequence of vectors* if  $\{x_n\}$  is a basis of vectors of the space  $[x_n]$ , i.e., for each  $x \in [x_n]$  there corresponds a unique sequence of scalars  $\{a_i\}$  such that

$$(1.1) \quad x = \sum_{i=1}^{\infty} a_i x_i,$$

the convergence being in the topology of  $X$ . We say that the basis is unconditional if the convergence in (1.1) is unconditional. It is well known that if  $\{x_n\}$  is a basic sequence of vectors, then every  $x \in [x_n]$  can be represented in the form  $x = \sum_{i=1}^{\infty} f_i(x) x_i$  where  $\{f_i\}$  is the sequence of continuous coefficient functionals biorthogonal to  $\{x_i\}$  (Arsove [1, p. 368], Dunford and Schwartz [4, p. 71]).

Similarly, we say that a sequence  $\{M_i\}$  of nontrivial subspaces of a complete metric linear space  $X$  is a *basis of subspaces* of  $X$ , if for each  $x \in X$ , there corresponds a unique sequence  $\{x_i\}$ ,  $x_i \in M_i$  for each  $i$ , such that

$$(1.2) \quad x = \sum_{i=1}^{\infty} x_i.$$

This concept has been studied by Fage [5], Markus [9], and others in separable Hilbert space and by Grimblyum [6] and McArthur [10] in complete metric linear spaces. We say that the basis of subspaces is *unconditional* if the convergence in (1.2) is unconditional.

If  $\{M_i\}$  is a basis of subspaces for  $X$ , for each  $i \in \omega$  define  $E_i$  from  $X$  into  $X$  by  $E_i(x) = x_i$  where  $\sum_{i=1}^{\infty} x_i$  is the unique representation of  $x \in X$ .  $E_i$  is a projection (linear and idempotent);  $E_i E_j = 0$  if  $i \neq j$ ; the range of  $E_i$  is  $M_i$ ; for each  $x \in X$ ,  $x = \sum_{i=1}^{\infty} E_i(x)$  and if  $E_i(x) = 0$  for each  $i$ , then  $x = 0$ .  $\{M_i\}$  will be called a *Schauder basis of subspaces* if each  $E_i$  is continuous.

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A sequence  $\{M_i\}$  of non-trivial subspaces of  $X$  is a (*unconditional basic sequence of subspaces*) if  $\{M_i\}$  is a (unconditional) basis of subspaces of  $[M_i]$ , the closed linear span of  $\bigcup_{i \in \omega} M_i$ . If  $\{M_i\}$  is a basic sequence of subspaces and  $x \in [M_i]$  then  $x = \sum_{i=1}^{\infty} E_i(x)$ , where  $E_i$  is now defined on  $[M_i]$ .

The classical Paley-Wiener theorem can be formulated in  $X$  as follows.

**1.3. THEOREM.** *Let  $\{x_n\}$  and  $\{y_n\}$  be sequences in  $X$  and let  $\lambda$  be a real number ( $0 < \lambda < 1$ ) such that*

$$(1.3a) \quad \left\| \sum_{n=1}^m a_n(x_n - y_n) \right\| \leq \lambda \left\| \sum_{n=1}^m a_n x_n \right\|$$

*holds for arbitrary scalars  $a_1, \dots, a_m$ . Then (1) if  $\{x_n\}$  is a basis so is  $\{y_n\}$ ; (2) if  $\{x_n\}$  is fundamental (i.e.,  $[x_n] = X$ ) so is  $\{y_n\}$ .*

Recently Arsove [1] showed that Theorem 1.3 is valid in a complete metric linear space. It is the purpose of this paper to show that this result and results similar to those of Pollard [13], Hilding [7], and Nagy [11] (all of which generalize condition 1.3a) are valid for basic sequences of subspaces in  $X$ . As a corollary to Theorem 4.3 we obtain a new version of the Paley-Wiener theorem.

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**2. Basic sequences of subspaces.** Special cases of the following lemma have been used by Hilding [7, p. 93], Nagy [11, p. 76], and others to prove theorems similar to Theorems 2.3 and 2.4.

**2.1. LEMMA.** *Let  $\{M_i\}$  and  $\{N_i\}$  be sequences of nontrivial subspaces of the complete metric linear space  $X$ . Suppose that for each  $i \in \omega$  there exists a one-to-one linear transformation  $T_i$  of  $M_i$  onto  $N_i$  and suppose further that there are positive numbers  $m, M$  such that*

$$(2.1a) \quad m \left\| \sum_{i=1}^p x_i \right\| \leq \left\| \sum_{i=1}^p T_i(x_i) \right\| \leq M \left\| \sum_{i=1}^p x_i \right\|$$

*holds for arbitrary  $x_i \in M_i$ ,  $i = 1, \dots, p$ . Then*

*(i) there is a linear homeomorphism  $T$  of  $[M_i]$  onto  $[N_i]$  such that the restriction of  $T$  to  $M_i$  equals  $T_i$  for each  $i \in \omega$  and such that*

$$(2.1b) \quad m \|x\| \leq \|T(x)\| \leq M \|x\|, \text{ for all } x \in [M_i].$$

(ii)  $\{M_i\}$  is a (unconditional) basic sequence of subspaces if and only if  $\{N_i\}$  is a (unconditional) basic sequence of subspaces.

*Proof.* Let  $X_0$  denote the space of finite linear combinations of  $\bigcup_{i \in \omega} M_i$ . These, of course, are reducible to the form  $\sum_{i=1}^n x_i$ ,  $x_i \in M_i$ . If  $x_i, x'_i \in M_i, i = 1, \dots, p$  and  $\sum_{i=1}^p x_i = \sum_{i=1}^p x'_i$  then from 2.1a it follows that  $\sum_{i=1}^p T_i(x_i) = \sum_{i=1}^p T_i(x'_i)$ . Thus we may define a linear transformation  $S$  from  $X_0$  into  $[N_i]$  by  $S(\sum_{i=1}^p x_i) = \sum_{i=1}^p T_i(x_i)$  and have  $m \|x\| \leq \|S(x)\| \leq M \|x\|$ , for all  $x \in X_0$ . It is clear that  $S$  restricted to  $M_i$  is equal to  $T_i$  and that  $S$  is continuous. Thus defined on a dense subset of  $[M_i]$ ,  $S$  has a unique linear extension  $T$  to  $[M_i]$  satisfying 2.1b. From 2.1b it follows that  $T$  is one-to-one and  $T^{-1}$  is continuous. We show  $T$  is onto  $[N_i]$ .

Let  $y \in [N_i]$ . Then  $y = \lim_k g_k$  where  $g_k$  is of the form  $g_k = \sum_{i=1}^{n(k)} y_i^{(k)}$ ,  $y_i^{(k)} \in N_i, i = 1, \dots, n(k)$ . For each such  $y_i^{(k)}$  there is a unique  $x_i^{(k)} \in M_i$  such that  $T_i(x_i^{(k)}) = y_i^{(k)}$ . Let  $h_k = \sum_{i=1}^{n(k)} x_i^{(k)}$ . Then from 2.1b,  $\|h_p - h_q\| \leq (1/m) \|g_p - g_q\|$ , so  $\{h_k\}$  is Cauchy and there is an  $x_0 \in [M_i]$  such that  $\{h_k\} \rightarrow x_0$ . Clearly,  $T(x_0) = y$ .

To verify (ii) suppose  $\{M_i\}$  is basic, i.e., a basic sequence of subspaces. Let  $y \in [N_i]$ . Then  $y = T(x)$  for some  $x \in [M_i]$ .  $x$  has a unique expansion  $x = \sum_{i=1}^\infty x_i$ ,  $x_i \in M_i$  and  $y = \sum_{i=1}^\infty T(x_i)$ ,  $T(x_i) \in N_i$ . Now if  $y = \sum_{i=1}^\infty y_i, y_i \in N_i$ , then  $y_i = T(x'_i)$  for some unique  $x'_i \in M_i$ . Hence  $0 = T(\sum_{i=1}^\infty x_i - x'_i)$  which implies  $x_i = x'_i$ . Since the expansion for  $y$  is unique, it follows that  $\{N_i\}$  is basic. The converse follows from (i) in the same way. If in the preceding argument  $\{M_i\}$  had been assumed an unconditional basis of subspaces for  $[M_i]$  then the series  $\sum_{i=1}^\infty x_i$  would have been unconditionally convergent to  $x$  and since  $T$  is a linear homeomorphism it follows that  $\sum_{i=1}^\infty T(x_i)$  would be unconditionally convergent.

2.2. DEFINITION. Two sequences  $\{x_i\}$  and  $\{y_i\}$  (in the given order) in  $X$  are said to have the property:

(P-W) (for Paley-Wiener) if there is a real number  $\lambda$  ( $0 < \lambda < 1$ ) such that  $\|\sum_{i=1}^n a_i(x_i - y_i)\| \leq \lambda \|\sum_{i=1}^n a_i x_i\|$  holds for arbitrary scalars  $a_1, a_2, \dots, a_n$ ;

(P-H) (for Pollard-Hilding) if for each positive real number  $k$ , there are real numbers  $\lambda_1, \lambda_2$  ( $0 \leq \lambda_i < \min[1; 2^{1-1/k}], i = 1, 2$ ) such that

$$\left\| \sum_{i=1}^n a_i(x_i - y_i) \right\| \leq \left[ \lambda_1 \left\| \sum_{i=1}^n a_i x_i \right\|^k + \lambda_2 \left\| \sum_{i=1}^n a_i y_i \right\|^k \right]^{1/k}$$

holds for arbitrary scalars  $a_1, \dots, a_n$ ;

(N) (for Nagy) if there are real numbers  $\lambda', \mu, \nu$  ( $0 \leq \lambda' < 1, 0 \leq \nu < 1, 0 \leq \mu, \mu^2 \leq [1 - \lambda'][1 - \nu]$ ) such that

$$\left\| \sum_{i=1}^n a_i(x_i - y_i) \right\|^2 \leq \lambda' \left\| \sum_{i=1}^n a_i x_i \right\|^2 + \mu \left\| \sum_{i=1}^n a_i x_i \right\| \cdot \left\| \sum_{i=1}^n a_i y_i \right\| + \nu \left\| \sum_{i=1}^n a_i y_i \right\|^2$$

holds for arbitrary scalars  $a_1, \dots, a_n$ .

If  $k = 1$  and  $\lambda_1 = \lambda_2$  property P-H reduces to

$$(2.2a) \quad \left\| \sum_{i=1}^n a_i(x_i - y_i) \right\| \leq \lambda \left[ \left\| \sum_{i=1}^n a_i y_i \right\| + \left\| \sum_{i=1}^n a_i x_i \right\| \right]$$

where  $\lambda = \lambda_1 = \lambda_2$ .

**2.3. LEMMA.** *If  $\{x_n\}$  and  $\{y_n\}$  are sequences in  $X$  with property P-W, P-H or N then 2.2a holds, with  $\lambda$  ( $0 < \lambda < 1$ ) an appropriately chosen constant.*

*Proof.* That property P-W implies 2.2a is obvious. If  $\{x_n\}, \{y_n\}$  have property P-H, let  $\lambda = [\max(\lambda_1, \lambda_2)]^{1/k}$ ; if  $\{x_n\}, \{y_n\}$  have property N let  $\lambda = [\max(\lambda', \mu, \nu)]^{1/2}$ .

**2.4. THEOREM.** *Suppose  $\{M_i\}$  and  $\{N_i\}$  are sequences of nontrivial subspaces of  $X$  and suppose that for each  $i \in \omega$ ,  $T_i$  is a one-to-one linear transformation of  $M_i$  onto  $N_i$ . Suppose further that there is a  $\lambda$  ( $0 < \lambda < 1$ ) such that*

$$(2.4a) \quad \left\| \sum_{i=1}^n (x_i - T_i(x_i)) \right\| \leq \lambda \left( \left\| \sum_{i=1}^n x_i \right\| + \left\| \sum_{i=1}^n T_i(x_i) \right\| \right)$$

holds for arbitrary  $x_i \in M_i, i = 1, \dots, n$ . Then

(i) *there is a linear homeomorphism  $T$  of  $[M_i]$  onto  $[N_i]$  such that  $T$  restricted to  $M_i$  equals  $T_i$  for each  $i$  and such that*

$$(2.4b) \quad [(1 - \lambda)/(1 + \lambda)] \|x\| \leq \|T(x)\| \leq [(1 + \lambda)/(1 - \lambda)] \|x\|$$

for each  $x \in [M_i]$ ;

(ii)  *$\{M_i\}$  is a (unconditional) basic sequence of subspaces if and only if  $\{N_i\}$  is a (unconditional) basic sequence of subspaces.*

*Proof.*

$$\begin{aligned} \left\| \sum_{i=1}^n T_i(x_i) \right\| &\leq \left\| \sum_{i=1}^n (T_i(x_i) - x_i) \right\| + \left\| \sum_{i=1}^n x_i \right\| \\ &\leq (1 + \lambda) \left\| \sum_{i=1}^n x_i \right\| + \lambda \left\| \sum_{i=1}^n T_i(x_i) \right\|, \end{aligned}$$

i.e.,

$$\left\| \sum_{i=1}^n T_i(x_i) \right\| \leq [(1 + \lambda)/(1 - \lambda)] \left\| \sum_{i=1}^n x_i \right\|.$$

Similarly,

$$\left\| \sum_{i=1}^n x_i \right\| \leq [(1 + \lambda)/(1 - \lambda)] \left\| \sum_{i=1}^n T_i(x_i) \right\| .$$

Thus

$$[(1 - \lambda)/(1 + \lambda)] \left\| \sum_{i=1}^n x_i \right\| \leq \left\| \sum_{i=1}^n T_i(x_i) \right\| \leq [(1 + \lambda)/(1 - \lambda)] \left\| \sum_{i=1}^n x_i \right\| .$$

The conclusions follow from Lemma 2.1.

**2.5. COROLLARY.** *Suppose  $\{M_i\}$  and  $\{N_i\}$  are sequences of non-trivial subspaces of  $X$  and suppose that for each  $i \in \omega$ ,  $T_i$  is a one-to-one linear transformation of  $M_i$  onto  $N_i$ . Suppose further that  $\{x_i\}$  and  $\{T_i(x_i)\}$  have property P-W, P-H or N, for arbitrary  $x_i \in M_i$  (observe that since  $x_i \in M_i$  is arbitrary,  $x_i$  and  $T_i(x_i)$  include the scalar  $a_i$  for each  $i$ ) then the conclusions of Theorem 2.4 hold. In particular, if Property P-W holds and  $\{M_i\}$  is a basis of subspaces for  $X$ , so is  $\{N_i\}$ .*

*Proof.* The first part of the corollary follows from Lemma 2.3. Arsove [1, p. 367] has shown how to prove the other assertion of the corollary. We repeat the proof for completeness.

Since Property P-W holds there exists a linear operator  $T$  from  $X$  into  $X$  satisfying  $\|x - T(x)\| \leq \lambda \|x\|, x \in X$  and such that  $T$  restricted to  $M_i$  equals  $T_i$ . Let  $A = T - I$ , where  $I$  is the identity operator.  $A$  is continuous at each  $x \in X$  and furthermore  $\|A^n(x)\| \leq \lambda^n \|x\|$  for each  $x \in X$  and positive integer  $n$ . Thus a linear operator  $U$  of  $X$  onto  $X$  may be defined by  $U(x) = \sum_{n=0}^{\infty} (-A^n(x)), x \in X$ . It follows that  $\|U(x)\| \leq (1 - \lambda)^{-1} \|x\|$ , so  $U$  is continuous. Given  $y \in X$ , let  $x = U(y)$ . Then  $y = (I + A)x = T(x)$  so  $T$  is onto  $X$ . Thus  $\{N_i\}$  is a basis of subspaces for  $X$ .

**3. Basic sequences of vectors.** If  $X$  has a basis of vectors  $\{x_n\}$ , then  $\{x_n\}$  induces in a natural way a basis of subspaces  $\{M_i\}$  for  $X$ . We have only to define  $M_i$  to be the span of the single element  $x_i$  (denoted by  $sp(x_i)$ ). From the remarks in the introduction we have  $x = \sum_{i=1}^{\infty} f_i(x)x_i$  for each  $x \in X$ , so  $E_i(x) = f_i(x)x_i$ . Since  $h(a) = ax_i$  is a linear homeomorphism of the scalar field into  $X$  and  $f_i(x)$  is a continuous linear functional it follows that  $E_i$  is continuous for each  $i \in \omega$  and so  $\{M_i\}$  is a Schauder basis of subspaces for  $X$ . Thus, for Schauder bases of vectors, we obtain the following theorems as corollaries to the theorems of § 2.

**3.1. THEOREM.** *Suppose  $\{x_i\}$  and  $\{y_i\}$  are nontrivial (i.e.,  $x_i \neq 0$ ,  $y_i \neq 0$ , for each  $i \in \omega$ ) sequences in  $X$  and suppose there is a  $\lambda (0 < \lambda < 1)$  such that*

$$(3.1a) \quad \left\| \sum_{i=1}^n a_i(x_i - y_i) \right\| \leq \lambda \left( \left\| \sum_{i=1}^n a_i x_i \right\| + \left\| \sum_{i=1}^n a_i y_i \right\| \right)$$

*holds for arbitrary scalars  $a_1, \dots, a_n$ . Then,*

(i) *there exists a linear homeomorphism  $T$  of  $[x_i]$  onto  $[y_i]$  such that  $T(x_i) = y_i$  for each  $i \in \omega$ , and*

(ii)  *$\{x_i\}$  is a (unconditional) basic sequence of vectors if and only if  $\{y_i\}$  is a (unconditional) basic sequence of vectors.*

*Proof.* Let  $M_i = sp(x_i)$  and  $N_i = sp(y_i)$ . Define a linear operator  $T_i$  from  $M_i$  onto  $N_i$  by  $T_i(ax_i) = ay_i$  where  $a$  is an arbitrary scalar. Clearly,  $T_i$  is one-to-one and continuous. 3.1a can be rewritten

$$(3.1b) \quad \left\| \sum_{i=1}^n (x'_i - T_i(x'_i)) \right\| \leq \lambda \left( \left\| \sum_{i=1}^n x'_i \right\| + \left\| \sum_{i=1}^n T_i(x'_i) \right\| \right)$$

for arbitrary  $x'_i \in M_i$ ,  $i = 1, \dots, n$ . The conclusions follow from Theorem 2.4.

Thus in particular, if  $\{x_n\}$  and  $\{y_n\}$  are nontrivial sequences in  $X$  with property P-W, P-H or N, the conclusions of 3.1 are valid.

We have remarked that if  $\{x_n\}$  and  $\{y_n\}$  have property P-W and  $\{x_n\}$  is a basis of vectors for  $X$ , then  $\{y_n\}$  is a basis of vectors for  $X$ . From 3.1 it follows that if  $\{x_n\}$  is an unconditional basis of vectors for  $X$ , then  $\{y_n\}$  is an unconditional basis of vectors for  $X$ .

**4. Basic sequences in Banach spaces.** From Grinblyum [6] the following can be derived (a proof is given in [10]).

**4.1. LEMMA.** *Let  $\{M_i\}$  be sequence  $a$  of nontrivial closed subspaces in a Banach space  $X$ .  $\{M_i\}$  is a Schauder basis of subspace for  $[M_i]$  if and only if there is a  $K \geq 1$  such that for arbitrary  $p, q \in \omega$ ,  $p \leq q$  we have  $\| \sum_{i=1}^p x_i \| \leq K \| \sum_{i=1}^q x_i \|$ , for arbitrary  $x_i \in M_i$ ,  $i = 1, \dots, q$ .*

**4.2. LEMMA.** *Let  $\{M_i\}$  be a sequence of nontrivial closed subspaces of a Banach space  $X$ .  $\{M_i\}$  is an unconditional Schauder basis of subspaces of  $[M_i]$  if and only if there is a  $K \geq 1$  such that for arbitrary finite sets of positive integers  $F, F'$  with  $F \subset F'$  we have  $\| \sum_{i \in F} x_i \| \leq K \| \sum_{i \in F'} x_i \|$ , for arbitrary  $x_i \in M_i$ .*

**4.3. THEOREM.** *Suppose  $\{M_i\}$  and  $\{N_i\}$  are sequences of closed nontrivial subspaces of a Banach space  $X$ .*

(1) *If there is a  $\lambda(0 < \lambda < 1)$  such that for an arbitrary finite set of integers  $F'$  and arbitrary  $y_i \in N_i, i \in F'$ , there exists  $x_i \in M_i, i \in F'$  such that*

$$(4.3a) \quad \left\| \sum_{i \in F'} (y_i - x_i) \right\| \leq \lambda \left[ \left\| \sum_{i \in F'} x_i \right\| + \left\| \sum_{i \in F'} y_i \right\| \right]$$

*holds for arbitrary  $F' \subset F$  then  $\{N_i\}$  is an unconditional (Schauder) basic sequence of subspaces if  $\{M_i\}$  is an unconditional (Schauder) basic sequence of subspaces;*

(2) *if there is a  $\lambda(0 < \lambda < 1)$  such that for arbitrary  $q \in \omega$  and arbitrary  $y_1, \dots, y_q, y_i \in N_i, i = 1, \dots, q$  there exist  $x_1, \dots, x_q, x_i \in M_i, i = 1, \dots, q$  such that*

$$(4.3b) \quad \left\| \sum_{i=1}^p (y_i - x_i) \right\| \leq \lambda \left[ \left\| \sum_{i=1}^p x_i \right\| + \left\| \sum_{i=1}^p y_i \right\| \right]$$

*holds for all  $p \leq q$  then  $\{N_i\}$  is a (Schauder) basic sequence of subspaces if  $\{M_i\}$  is a (Schauder) basic sequence of subspaces.*

*Proof.* We prove (2). The proof of (1) is analogous using Lemma 4.2 instead of 4.1.

Suppose  $\{M_i\}$  be a basis of subspaces for  $[M_i]$ . By Lemma 4.1 there is a  $K \geq 1$  such that

$$\left\| \sum_{i=1}^p x_i \right\| \leq K \left\| \sum_{i=1}^q x_i \right\|, x_i \in M_i, p \leq q.$$

We have

$$\left\| \sum_{i=1}^p y_i \right\| \leq \left\| \sum_{i=1}^p (y_i - x_i) \right\| + \left\| \sum_{i=1}^p x_i \right\|$$

and from (4.4b) it follows that

$$\left\| \sum_{i=1}^p y_i \right\| \leq \frac{1 + \lambda}{1 - \lambda} \left\| \sum_{i=1}^p x_i \right\|.$$

Also

$$\left\| \sum_{i=1}^q x_i \right\| \leq \frac{1 + \lambda}{1 - \lambda} \left\| \sum_{j=1}^q y_j \right\|.$$

Thus we have

$$\left\| \sum_{i=1}^p y_i \right\| \leq \left[ \frac{1 + \lambda}{1 - \lambda} \right]^2 K \left\| \sum_{i=1}^q y_i \right\|.$$

Thus by Lemma 4.1,  $\{N_i\}$  is a basis of subspaces for  $[N_i]$ .



4.4. COROLLARY. Let  $\{x_i\}$  and  $\{y_i\}$  be non-trivial sequences in a Banach space  $X$ .

(1) If there is a  $\lambda (0 < \lambda < 1)$  such that for an arbitrary finite set of indices  $F'$  and arbitrary scalars  $\{a_i\}$ ,  $i \in F'$ , there exist scalars  $\{b_i\}$ ,  $i \in F'$ , such that

$$(4.4a) \quad \left\| \sum_{i \in F'} (a_i y_i - b_i x_i) \right\| \leq \lambda \left[ \left\| \sum_{i \in F'} a_i y_i \right\| + \left\| \sum_{i \in F'} b_i x_i \right\| \right]$$

holds for arbitrary  $F' \subset F$  then  $\{y_i\}$  is an unconditional (Schauder) basic sequence of vectors if  $\{x_i\}$  is an unconditional (Schauder) basic sequence of vectors;

(2) if there is a  $\lambda (0 < \lambda < 1)$  such that for arbitrary  $q \in \omega$  and arbitrary scalars  $a_1, \dots, a_q$  there are scalars  $b_1, \dots, b_q$  such that

$$(4.4b) \quad \left\| \sum_{i=1}^q (a_i y_i - b_i x_i) \right\| \leq \lambda \left[ \left\| \sum_{i=1}^q b_i x_i \right\| + \left\| \sum_{i=1}^q a_i y_i \right\| \right]$$

holds for all  $p \leq q$  then  $\{y_i\}$  is a (Schauder) basic sequence of vectors if  $\{x_i\}$  is a (Schauder) basic sequence of vectors.

*Proof.* Let  $M_i = sp(x_i)$ ,  $N_i = sp(y_i)$  and apply the preceding theorem.

4.4 is a new form of the Paley-Wiener theorem for we no longer require the coefficients of  $x_i$  and  $y_i$  to be the same. We could now define properties similar to properties P-W, P-H and N by merely asserting the existence of a scalar  $b_i$  to replace the coefficient of  $x_i$  in each of the properties defined in 2.2. It is easy to see that these new forms of properties P-W, P-H and N imply the hypotheses of corollary 4.5.

It is unknown<sup>to</sup> the author whether  $[x_n]$  is linearly homeomorphic to  $[y_n]$  or not.

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# Pacific Journal of Mathematics

Vol. 14, No. 3

July, 1964

Erik Balslev and Theodore William Gamelin, <i>The essential spectrum of a class of ordinary differential operators</i> .....	755
James Henry Bramble and Lawrence Edward Payne, <i>Bounds for derivatives in elliptic boundary value problems</i> .....	777
Hugh D. Brunk, <i>Integral inequalities for functions with nondecreasing increments</i> .....	783
William Edward Christilles, <i>A result concerning integral binary quadratic forms</i> .....	795
Peter Crawley and Bjarni Jónsson, <i>Refinements for infinite direct decompositions of algebraic systems</i> .....	797
Don Deckard and Carl Mark Percy, <i>On continuous matrix-valued functions on a Stonian space</i> .....	857
Raymond Frank Dickman, Leonard Rubin and P. M. Swingle, <i>Another characterization of the <math>n</math>-sphere and related results</i> .....	871
Edgar Earle Enochs, <i>A note on reflexive modules</i> .....	879
Vladimir Filippenko, <i>On the reflection of harmonic functions and of solutions of the wave equation</i> .....	883
Derek Joseph Haggard Fuller, <i>Mappings of bounded characteristic into arbitrary Riemann surfaces</i> .....	895
Curtis M. Fulton, <i>Clifford vectors</i> .....	917
Irving Leonard Glicksberg, <i>Maximal algebras and a theorem of Radó</i> .....	919
Kyong Taik Hahn, <i>Minimum problems of Plateau type in the Bergman metric space</i> .....	943
A. Hayes, <i>A representation theory for a class of partially ordered rings</i> .....	957
J. M. C. Joshi, <i>On a generalized Stieltjes transform</i> .....	969
J. M. C. Joshi, <i>Inversion and representation theorems for a generalized Laplace transform</i> .....	977
Eugene Kay McLachlan, <i>Extremal elements of the convex cone <math>B_n</math> of functions</i> .....	987
Robert Alan Melter, <i>Contributions to Boolean geometry of <math>p</math>-rings</i> .....	995
James Ronald Retherford, <i>Basic sequences and the Paley-Wiener criterion</i> .....	1019
Dallas W. Sasser, <i>Quasi-positive operators</i> .....	1029
Oved Shisha, <i>On the structure of infrapolynomials with prescribed coefficients</i> .....	1039
Oved Shisha and Gerald Thomas Cargo, <i>On comparable means</i> .....	1053
Maurice Sion, <i>A characterization of weak* convergence</i> .....	1059
Morton Lincoln Slater and Robert James Thompson, <i>A permanent inequality for positive functions on the unit square</i> .....	1069
David A. Smith, <i>On fixed points of automorphisms of classical Lie algebras</i> .....	1079
Sherman K. Stein, <i>Homogeneous quasigroups</i> .....	1091
J. L. Walsh and Oved Shisha, <i>On the location of the zeros of some infrapolynomials with prescribed coefficients</i> .....	1103
Ronson Joseph Warne, <i>Homomorphisms of <math>d</math>-simple inverse semigroups with identity</i> .....	1111
Roy Westwick, <i>Linear transformations on Grassman spaces</i> .....	1123