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QUASI-POSITIVE OPERATORS

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1. **Introduction.** The classical results of Perron and Frobenius ([6], [7], [12]) assert that a finite dimensional, nonnegative, non-nilpotent matrix has a positive eigenvalue which is not exceeded in absolute value by any other eigenvalue and the matrix has a nonnegative eigenvector corresponding to this positive eigenvalue. If the matrix has strictly positive entries, then there is a positive eigenvalue which exceeds every other eigenvalue in absolute value, and the corresponding space of eigenvectors is one-dimensional and is spanned by a vector with strictly positive coordinates. Numerous generalizations of these results to order-preserving linear operators acting in ordered linear spaces have appeared in recent years; a short bibliography is included at the end of this paper. In this paper a generalization in a different direction is obtained which reduces, in the finite dimensional case, to the assertion that the Perron-Frobenius theorems hold if it is only required that all but a finite number of the powers of the matrix satisfy the given conditions. The principal results are theorems of the Perron-Frobenius type which are applicable to any compact linear operator (the compactness condition is weakened somewhat), acting in an ordered real Banach space B , which satisfies a condition weaker than order-preserving. In addition, the results apply to the case when the "cone" of positive elements in B has no interior.

2. **Preliminaries.** Throughout the sequel, B will denote a real Banach space with norm $\|\cdot\|$. The complex extension of B , \tilde{B} , is the complex Banach space $\tilde{B} = \{x + iy \mid x, y \in B\}$ with the obvious definitions of addition and complex scalar multiplication and the norm in \tilde{B} is $\|x + iy\| = \sup_{\theta} \|\cos \theta \cdot x + \sin \theta \cdot y\|$. If T is a (real) linear operator on B into B , the (complex) linear operator \tilde{T} on \tilde{B} into \tilde{B} is defined by $\tilde{T}(x + iy) = Tx + iTy$. T is bounded if and only if \tilde{T} is bounded, in which case $\|T\| = \|\tilde{T}\|$. The spectrum, $\sigma(T)$, and the resolvent, $\rho(T)$, are defined to be the corresponding sets associated with the operator \tilde{T} . We denote the spectral radius of T by r_T , $r_T = \lim_{n \rightarrow \infty} \|T^n\|^{1/n} = \sup_{\lambda \in \sigma(T)} |\lambda|$ (provided $\|T\| < \infty$).

In all of our results there will be a basic assumption that the linear operator under consideration is quasi-compact, a notion which we will now define. A bounded linear operator T is compact (also called completely continuous) if each sequence Tx_1, Tx_2, \dots , with

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$\|x_i\| \leq 1, i = 1, 2, \dots$, has a convergent subsequence. T is quasi-compact if there exists a positive integer n and a bounded linear operator V such that $T^n - V$ is compact and $r_V < r_T$.¹ There are a number of properties possessed by quasi-compact operators some of which we state now without proof.² If $\lambda_0 \in \sigma(T)$ and $|\lambda_0| = r_T$, then λ_0 is an isolated point in $\sigma(T)$ and is in the point spectrum, i.e., $(\lambda_0 I - \tilde{T})$ is not one-to-one. The resolvent operator, $R(\lambda, T) \equiv (\lambda I - \tilde{T})^{-1}$, exists in a neighborhood of λ_0 (excluding λ_0) and, in this neighborhood, $R(\lambda, T)$ has a Laurent series expansion of the form

$$R(\lambda, T) = \sum_{k=1}^{n(\lambda_0)} \frac{(\lambda_0 I - \tilde{T})^{k-1}}{(\lambda - \lambda_0)^k} P(\lambda_0, T) + \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k A_k(\lambda_0, T)$$

where $A_k(\lambda_0, T)$ is a bounded linear operator and the series on the right is convergent in the uniform operator topology. The integer $n(\lambda_0)$ is the index of λ_0 , i.e., $n(\lambda_0)$ is the smallest integer n such that $\{x | (\lambda_0 I - \tilde{T})^{n+1}x = 0\} = \{x | (\lambda_0 I - \tilde{T})^n x = 0\}$. $P(\lambda_0, T)$ is a projection onto the finite dimensional space $\{x | (\lambda_0 I - \tilde{T})^{n(\lambda_0)}x = 0\}$. The minimal property of $n(\lambda_0)$ implies that $(\lambda_0 I - \tilde{T})^{n(\lambda_0)-1}P(\lambda_0, T) \neq 0$.

We recall that for an arbitrary bounded linear operator, the resolvent $R(\lambda, T) = (\lambda I - \tilde{T})^{-1}$ is an analytic function of λ for $\lambda \in \rho(T)$ and the expansion $R(\lambda, T) = \sum_{k=0}^{\infty} (1/\lambda)^{k+1} \tilde{T}^k$ is valid for $|\lambda| > r_T$.

3. Quasi-positive operators. A cone in B is a convex set K which contains λx for all $\lambda \geq 0$ if it contains x . K is a proper cone if $x \in K$ and $-x \in K$ imply $x = 0$. A cone K induces an ordering \geq in B with $x \geq y$ if and only if $x - y \in K$. This transitive ordering satisfies

- (1) if $x \geq y, u \geq v$, then $x + u \geq y + v$,
- (2) if $x \geq y$ and $\lambda \geq 0$, then $\lambda x \geq \lambda y$, and
- (3) $x \geq y$ if and only if $-y \geq -x$.

If the cone is proper, then the ordering satisfies, in addition,

- (4) if $x \geq y$ and $y \geq x$, then $x = y$.

We will use the notation $x > y$ to denote $x \geq y, x \neq y$. Associated with a cone K is a closed cone K^+ in the conjugate space B^* of continuous, real-valued, linear functions on B , consisting of those $x^* \in B^*$ with the property that $x^*(x) \geq 0$ for all $x \in K$. K^+ is a proper cone if and only if the linear space spanned by K is dense in B (a set with this property is called *fundamental*). This is an easy consequence of the Hahn-Banach theorem on the extension of linear functionals. We will use the notations $x^* \geq y^*$ and $x^* > y^*$ to denote $x^* - y^* \in K^+$

¹ Note that a compact operator is quasi-compact if and only if it has a positive spectral radius.

² For details, see Yu. L. Smvl'yan, *Completely continuous perturbations of operators*, Amer. Math. Soc. Translations **10**, 341-344.

and $x^* - y^* \in K^+$, $x^* \neq y^*$, respectively. An element $x > 0$ ($x^* > 0$) will be called *strictly positive* if $x^*(x) > 0$ for all $x^* > 0$ ($x^*(x) > 0$ for all $x > 0$).

The following theorem is a characterization of a closed cone and its interior (when the latter is nonvoid) in terms of K^+ . The proof may be found, for example, in [11] (Theorem 1.3 and its corollaries, pg. 16).

THEOREM 1. *Let K be a closed cone in B . Then $x \in K$ if and only if $x^*(x) \geq 0$ for all $x^* \geq 0$. If K has a nonvoid interior, then*

(1) *x is in the interior of K if and only if x is strictly positive and*

(2) *for each x on the boundary of K there exists an $x^* > 0$ such that $x^*(x) = 0$.*

COROLLARY. *If K is a closed proper cone, K^+ is a total set of functionals, i.e., for each $x \neq 0$, $x \in B$, there exists $x^* > 0$ such that $x^*(x) \neq 0$.*

Proof. Since either $x \notin K$ or $-x \notin K$ if $x \neq 0$, this follows immediately from Theorem 1.

A linear operator T on B into B will be called *positive with respect to a cone K* if $TK \subseteq K$. In the absence of ambiguity we will simply say T is positive. In our applications K will be a closed cone and in this case, in view of Theorem 1, T is positive if and only if $x^*(Tx) \geq 0$ for all $x \geq 0$, $x^* \geq 0$. Since $Tx \geq 0$ if $x \geq 0$, we have $x^*(T^2x) \geq 0$ and, in general, $x^*(T^n x) \geq 0$ for all n and all $x \geq 0$, $x^* \geq 0$. We define T to be *quasi-positive* if for each pair $x \geq 0$, $x^* \geq 0$, there exists an integer $n(x, x^*) \geq 1$ such that $x^*(T^n x) \geq 0$ if $n \geq n(x, x^*)$. We define T to be *strictly quasi-positive* if for each pair $x > 0$, $x^* > 0$, there exists an integer $n(x, x^*) \geq 1$ such that $x^*(T^n x) > 0$ if $n \geq n(x, x^*)$. Finally we define T to be *strongly quasi-positive* if it is not nilpotent³ and for each pair $x > 0$, $x^* > 0$, $\liminf_{n \rightarrow \infty} x^*(T^n x) / \|T^n\| > 0$.

4. **Spectral properties.** Throughout this section, K will denote a closed proper cone in B and K will be assumed to be fundamental. T will denote a quasi-compact bounded linear operator with spectral radius 1. This restriction on the spectral radius is for convenience only and the results given may be interpreted for a general (quasi-compact) bounded linear operator S with spectral radius $r_s > 0$ by considering the operator $T = (1/r_s) S$ which has spectral radius 1.

³ An operator T is nilpotent if $T^n = 0$ for some n .

THEOREM 2. *If T is quasi-positive and quasi-compact with spectral radius 1, then $1 \in \sigma(T)$ and the index of 1 is not exceeded by the index of any other point $\lambda \in \sigma(T)$, $|\lambda| = 1$.*

Proof. Assume that $1 \in \rho(T)$. Since $\rho(T)$ is open and $R(\lambda, T)$ is analytic in λ for $\lambda \in \rho(T)$, it follows that the function $g(\lambda) = x^*(R(1/\lambda, T)x)$, $x > 0$, $x^* > 0$, is analytic for $1/\lambda \in \rho(T)$, in particular for λ in some neighborhood of 1. Moreover, $R(\lambda, T) = \sum_{k=0}^{\infty} (1/\lambda)^{k+1} \tilde{T}^k$ if $|\lambda| > 1$, hence $g(\lambda) = \sum_{k=0}^{\infty} \lambda^{k+1} x^*(T^k x)$ if $|\lambda| < 1$. A theorem of Pringsheim states that if a power series has nonnegative coefficients and converges in the open unit disk, either 1 is a singularity of the series or the series has radius of convergence greater than 1.⁴ Clearly it is sufficient to assume that all but a finite number of the coefficients are nonnegative. Since $x^*(T^n x) \geq 0$ if $n \geq n(x, x^*)$, and $g(\lambda)$ is analytic in a neighborhood of 1, we conclude that the series $\sum_{k=0}^{\infty} \lambda^{k+1} x^*(T^k x)$ converges in $|\lambda| < 1 + \delta$ for some $\delta > 0$. By assumption $r_T = 1$, hence $R(\lambda, T)$ has a singularity somewhere on $|\lambda| = 1$, say at λ_0 . Since T is quasi-compact, the expansion

$$R(\lambda, T) = \sum_{k=1}^n \frac{(\lambda_0 I - \tilde{T})^{k-1}}{(\lambda - \lambda_0)^k} P(\lambda_0, T) + \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k A_k(\lambda_0, T)$$

is valid for $0 < |\lambda - \lambda_0| < \delta'$, where $n = n(\lambda_0)$ is the index of λ_0 and $(\lambda_0 I - \tilde{T})^{n-1} P(\lambda_0, T) \neq 0$. We may choose $x > 0$ such that $(\lambda_0 I - \tilde{T})^{n-1} P(\lambda_0, T)x = y \neq 0$ since K is fundamental and by Theorem 1 we may choose $x^* > 0$ such that $x^*(y) \neq 0$. It follows easily that

$$g(\lambda) = (\lambda/\lambda_0)^n (1/\lambda_0 - \lambda)^{-n} h(\lambda), \quad |1/\lambda - \lambda_0| < \delta,$$

where $h(\lambda)$ is analytic and $h(1/\lambda_0) = x^*(y) \neq 0$. Thus g has a pole at $1/\lambda_0$ which contradicts the fact that g has a Taylor's series about the origin with radius of convergence greater than 1. Our assumption that $1 \in \rho(T)$ leads to a contradiction, hence $1 \in \sigma(T)$.

Now let the index of 1 be n . It is easy to see that $\lim_{\lambda \rightarrow 1} (\lambda - 1)^k R(\lambda, T) = 0$ if $k > n$. It follows that for $|\lambda| > 1$, $\lim_{\lambda \rightarrow 1} (\lambda - 1)^k \sum_{m=0}^{\infty} (1/\lambda)^{m+1} x^*(T^m x) = 0$ for every pair $x > 0$, $x^* > 0$ and clearly this implies $\lim_{\lambda \rightarrow 1} (\lambda - 1)^k \sum_{m=j}^{\infty} (1/\lambda)^{m+1} x^*(T^m x) = 0$ if $k > n$ and $j \geq 0$. If $\lambda_0 \in \sigma(T)$, $|\lambda_0| = 1$ and λ_0 has index l , then $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^l R(\lambda, T) \neq 0$. We may choose $x > 0$ and $x^* > 0$ such that $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^l x^*(R(\lambda, T)x) \neq 0$ and it follows that for $|\lambda| > 1$, $\lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)^l \sum_{m=j}^{\infty} (1/\lambda)^{m+1} x^*(T^m x) \neq 0$. Let $\lambda_0 = e^{i\varphi}$, $\lambda = \rho e^{i\varphi}$, $\rho > 1$. If $j \geq n(x, x^*)$, $|(\lambda - \lambda_0)^l \sum_{m=j}^{\infty} (1/\lambda)^{m+1} x^*(T^m x)| \leq (\rho - 1)^l \sum_{m=j}^{\infty} (1/\rho)^{m+1} x^*(T^m x)$. The expression on the right in this last inequality tends to zero as

⁴ See Titchmarsh, *Theory of Functions*, pg. 214. Acknowledgement is due here to S. Karlin for the essence of the proof in Theorem 2 (see [10], Theorem 4).

ρ tends to 1 if $l > n$, hence $l \leq n$. This completes the proof.

THEOREM 3. *If T is quasi-positive and quasi-compact with spectral radius 1, there exist elements $u > 0$ and $u^* > 0$ such that $Tu = u$, $T^*u^* = u^*$.⁵*

Proof. By Theorem 2, $1 \in \sigma(T)$. We have

$$R(\lambda, T) = \sum_{k=1}^n \frac{(I - \tilde{T})^{k-1}}{(\lambda - 1)^k} P(1, T) + \sum_{k=0}^{\infty} (\lambda - 1)^k A_k(1, T)$$

where $P(1, T)$ is a projection onto the finite-dimensional space $\{x \mid (I - \tilde{T})^n x = 0\}$ and $(I - \tilde{T})^{n-1} P(1, T) \neq 0$. Let $\Gamma = (I - \tilde{T})^{n-1} P(1, T)$. It is easy to see that $R(\lambda, T)B \subseteq B$ for λ real. Since $\Gamma = \lim_{\lambda \rightarrow 1} (\lambda - 1)^n R(\lambda, T)$, it follows that $\Gamma B \subseteq B$. Also $\tilde{T}\Gamma = \Gamma\tilde{T} = \Gamma$. Let $x \geq 0$, $x^* \geq 0$ be arbitrary and let $N = n(x, x^*)$. If $\lambda > 1$, we have $x^*(T^N R(\lambda, T)x) = \sum_{m=0}^{\infty} (1/\lambda)^{m+1} x^*(T^{N+m}x) \geq 0$. It follows that for $\lambda > 1$, $x^*(T^N \Gamma x) = \lim_{\lambda \rightarrow 1} (\lambda - 1)^n \sum_{m=0}^{\infty} (1/\lambda)^{m+1} x^*(T^{N+m}x) \geq 0$. Since $T^N \Gamma = \Gamma$, Γ is a positive operator. We choose $v > 0$ such that $\Gamma v = u \neq 0$. Then $u > 0$ and $Tu = T\Gamma v = \Gamma v = u$. We choose $v^* > 0$ such that $v^*(u) > 0$. Letting $u^* = \Gamma^* v^*$, we see that for $x \geq 0$, $u^*(x) = (\Gamma^* v^*)(x) = v^*(\Gamma x) \geq 0$ since $v^* > 0$ and Γ is a positive operator. Hence $u^* \geq 0$, and since $u^*(v) = (\Gamma^* v^*)(v) = v^*(\Gamma v) = v^*(u) > 0$, $u^* > 0$. Finally, we have $\Gamma T = \Gamma$ which implies $T^* \Gamma^* = \Gamma^*$, hence $T^* u^* = T^*(\Gamma^* v^*) = \Gamma^* v^* = u^*$ which completes the proof.

For strictly quasi-positive operators we obtain stronger results in the next two theorems.

THEOREM 4. *If T is strictly quasi-positive and quasi-compact with spectral radius 1, then $1 \in \sigma(T)$, 1 has index one and \tilde{T} has a representation of the form $\tilde{T} = \sum_{j=1}^m \lambda_j P_j + S$ where $\lambda_1 = 1$, $|\lambda_j| = 1$, $P_j^2 = P_j$, $SP_j = P_j S = 0$, $j = 1, 2, \dots, m$, $P_i P_j = 0$ if $i \neq j$, and $r_s < 1$.*

Proof. By Theorem 2, $1 \in \sigma(T)$. By Theorem 3, there exists $u^* > 0$ such that $T^* u^* = u^*$ and for $x > 0$, $u^*(x) = u^*(T^n x) > 0$ if $n \geq n(x, u^*)$, hence u^* is strictly positive. Let the index of 1 be n . Then $\Gamma = \lim_{\lambda \rightarrow 1} (\lambda - 1)^n R(\lambda, T) \neq 0$. For $\lambda > 1$ and arbitrary x we have

$$\begin{aligned} u^*(\Gamma x) &= \lim_{\lambda \rightarrow 1} (\lambda - 1)^n \sum_{k=0}^{\infty} (1/\lambda)^{k+1} u^*(T^k x) = \lim_{\lambda \rightarrow 1} u^*(x) (\lambda - 1)^n \sum_{k=0}^{\infty} (1/\lambda)^{k+1} \\ &= u^*(x) \lim_{\lambda \rightarrow 1} (\lambda - 1)^{n-1} = 0 \end{aligned}$$

⁵ T^* is the adjoint of T , defined on B^* by $(T^* x^*)(x) = x^*(Tx)$.

unless $n = 1$. In proving Theorem 3 we showed that Γ is a positive operator, hence there exists $x > 0$ such that $\Gamma x > 0$ and therefore $u^*(\Gamma x) > 0$. It follows that $n = 1$. By Theorem 2, every $\lambda_0 \in \sigma(T)$, $|\lambda_0| = 1$, has index 1 and hence $P(\lambda_0, T) = \lim_{\lambda \rightarrow \lambda_0} (\lambda - \lambda_0)R(\lambda, T)$ exists and is a projection onto the finite dimensional space $\{x \mid (\lambda_0 I - \tilde{T})x = 0\}$. Let $\lambda_1 = 1, \lambda_2, \dots, \lambda_m$ be an enumeration of the points in $\sigma(T)$ with absolute value 1 and let $P_j = P(\lambda_j, T)$. Since \tilde{T} commutes with $R(\lambda, T)$ and $P_j = \lim_{\lambda \rightarrow \lambda_j} (\lambda - \lambda_j)R(\lambda, T)$, it follows that \tilde{T} commutes with P_j . For $i \neq j$ we have $\lambda_i P_i P_j = \tilde{T} P_i P_j = P_i \tilde{T} P_j = \lambda_j P_i P_j$, hence $P_i P_j = 0$. Define the bounded linear operator S by the equation $\tilde{T} = \sum_{j=1}^m \lambda_j P_j + S$. Since $\tilde{T} P_j = P_j \tilde{T} = \lambda_j P_j$, $P_j^2 = P_j$ and $P_i P_j = 0$ if $i \neq j$, it follows that $P_j S = S P_j = 0$. This implies $\tilde{T}^n = \sum_{j=1}^m \lambda_j^n P_j + S^n$. Suppose $r_s \geq 1$. T is quasi-compact, hence $\tilde{T}^n = U + V$ for some n where U is compact and $r_V < 1$. The operator U' defined by $U'x = Ux - \sum_{j=1}^m \lambda_j^n P_j x$ is compact⁶ and $S^n = U' + V$. Therefore S is quasi-compact. Let $\lambda \in \sigma(S)$, $|\lambda| = r_s \geq 1$. Then $Sx = \lambda x$ for some $x \in \tilde{B}$, $x \neq 0$. Since $P_j S = S P_j = 0$, it follows that $\tilde{T}x = \lambda x$ and therefore for some j , $\lambda = \lambda_j$ and $P_j x = x$. This implies $Sx = S P_j x = 0$, a contradiction. Therefore $r_s < 1$ and the proof is complete.

Before stating our next result, we state the following lemma which is easily proved.

LEMMA 1. *If E is a finite dimensional real Banach space, K is a cone in E and K is fundamental, then K contains an open set.*

THEOREM 5. *If T is strictly quasi-positive and quasi-compact with spectral radius 1, the eigenspace for T corresponding to the eigenvalue 1 is one-dimensional.*

Proof. By Theorem 4 we have $\tilde{T} = \sum_{j=1}^m \lambda_j P_j + S$ where P_j is a projection onto the eigenspace corresponding to λ_j , $\lambda_1 = 1$, $|\lambda_j| = 1$, $P_j S = S P_j = 0$, $j = 1, 2, \dots, m$ and $P_i P_j = 0$ if $i \neq j$. By a theorem of Kronecker, there exists a sequence n_1, n_2, \dots of positive integers such that $\lim_{k \rightarrow \infty} \lambda_j^{n_k} = 1$, $j = 1, 2, \dots, m$.⁷ Since $r_s < 1$, it follows that $\lim_{n \rightarrow \infty} \|S^n\| = 0$. This implies $\lim_{k \rightarrow \infty} \tilde{T}^{n_k} = \sum_{j=1}^m P_j$. Let $P = \sum_{j=1}^m P_j$. For $x \in B$ we have $Px = \lim_{k \rightarrow \infty} \tilde{T}^{n_k} x$, hence $PB \subseteq B$. For $x \geq 0$ and $x^* \geq 0$, $x^*(Px) = \lim_{k \rightarrow \infty} x^*(\tilde{T}^{n_k} x) \geq 0$, hence P is a positive operator. Consider the finite dimensional real Banach space PB with closed proper cone PK . Since K is fundamental in B , it is clear that PK is fundamental in PB . Therefore, by Lemma 1, PK contains an open set (open relative to PB). Since T is strictly quasi-positive, every

⁶ The compact operators from an ideal in the algebra of bounded linear operators and any bounded operator with a finite dimensional range is compact.

⁷ See, for example, Hardy & Wright, *The Theory of Numbers*, Oxford Univ. Press.

non-trivial fixed vector of T in K is strictly positive. By Theorem 3, there exists $u > 0$ such that $Tu = u$. Let $Tx = x, x \neq 0$. We wish to show u and x are linearly dependent and for this purpose we may assume $x \notin K$ (otherwise replace x by $-x$). It is clear that $u \in PK$ and $x \in PB$. Let $t_0 = \sup \{t \mid u + tx \in PK\}$. Since u is in the interior of PK and $x \notin PK$, it is easy to see that $0 < t_0 < \infty$ and that $u + t_0x$ is on the boundary of PK . Hence, by Theorem 1, there exists $x^* \in (PK)^+$ such that $x^*(u + t_0x) = 0$. We extend x^* to $y^* \in B^*$ by defining $y^*(y) = x^*(Py)$. Since $PK \subseteq K$, it follows that $y^* \in K^+$. We have $P(u + t_0x) = u + t_0x$, hence $y^*(u + t_0x) = x^*(u_0 + t_0x) = 0$. Now $u + t_0x$ is a fixed vector of T which is not strictly positive, hence $u + t_0x = 0$, which completes the proof.

Our next result is a characterization of strongly quasi-positive operators.

THEOREM 6. *If T is quasi-compact with spectral radius 1, then T is strongly quasi-positive if and only if the following conditions are satisfied:*

- (1) $1 \in \sigma(T)$ and 1 is the only point in $\sigma(T)$ with absolute value one,
- (2) the eigenspace for T corresponding to the eigenvalue 1 is one-dimensional and is spanned by a strictly positive element u ,
- (3) there exists a strictly positive element u^* such that $T^*u^* = u^*$.

Proof. In Theorems 3, 4, 5 we have seen that if T is strictly quasi-positive (in particular, if it is strongly quasi-positive), then $1 \in \sigma(T)$ and (2) and (3) hold. There remains to show 1 is the only point in $\sigma(T)$ with absolute value one. We define the operator $P = \sum_{j=1}^m P_j$ as in Theorem 5 and recall that PB is a finite dimensional real Banach space with closed proper cone PK containing interior elements. Let $\lambda = e^{i\theta}$ be a point in $\sigma(T)$ and let $\tilde{T}(x + iy) = e^{i\theta}(x + iy)$ for some x, y in B , not both zero. It is easy to see that $Px = x$ and $Py = y$, hence $x \in PB$ and $y \in PB$. At least one of the four elements $x + y, x - y, y - x, -x - y$ must be not in PK since otherwise $x + y = 0, x - y = 0$, hence $x = y = 0$. Therefore $ax + by \notin PK$ for some choice of $a = \pm 1$ and $b = \pm 1$. Now choose $t > 0$ such that $u + t(ax + by) = v$ is on the boundary of PK . By Theorem 1, there exists $x^* \in (PK)^+, x^* \neq 0$, such that $x^*(v) = 0$. We extend x^* to $y^* \in K^+ : y^*(y) = x^*(Py)$. Now choose a sequence of positive integers n_1, n_2, \dots such that $\lim_{k \rightarrow \infty} e^{in_k\theta} = 1$. It follows that $\lim_{k \rightarrow \infty} T^{n_k}v = v$. Since $r_T = 1$, we have $\|T^n\| \geq 1$ for all n and hence if $v > 0$,

$$\liminf_{n \rightarrow \infty} y^*(T^n v) \geq \liminf_{n \rightarrow \infty} y^*(T^n v) / \|T^n\| > 0 .$$

This is impossible since $\lim_{k \rightarrow \infty} y^*(T^{nk}v) = y^*(v) = 0$. Therefore $v = 0$, i.e., $ax + by = -(1/t)u$. Since $\tilde{T}(x + iy) = e^{i\theta}(x + iy)$, it follows that $u^*(x) + iu^*(y) = e^{i\theta}(u^*(x) + iu^*(y))$. This implies either $e^{i\theta} = 1$ or $u^*(x) = u^*(y) = 0$. The second alternative is incompatible with $ax + by = -(1/t)u$ since $u^*(u) > 0$. Therefore $e^{i\theta} = 1$ and the necessity of (1), (2), (3) is proved.

Now let T satisfy conditions (1), (2), (3). We assume without loss of generality that u^* is normalized so that $u^*(u) = 1$. Define the bounded linear operator S by $Tx = u^*(x)u + Sx$. As in Theorem 4, it can be shown that $r_s < 1$. We have $Su = Tu - u^*(u)u = u - u = 0$ and it follows that $T^n x = u^*(x)u + S^n x$. Since $r_s < 1$, $\|S^n\| \leq M$ for all n and hence $\|T^n\| \leq \|u^*\| \|u\| + \|S^n\| \leq M'$ for all n . Moreover, $S^n x \rightarrow 0$ as $n \rightarrow \infty$ for all x . Hence if $x > 0$ and $x^* > 0$,

$$\begin{aligned} \liminf_{n \rightarrow \infty} x^*(T^n x) / \|T^n\| &\geq \liminf_{n \rightarrow \infty} (u^*(x)x^*(u) + x^*(S^n x)) / M' \\ &\geq u^*(x)x^*(u) / M' > 0. \end{aligned}$$

Therefore T is strongly quasi-positive and the theorem is proved.

THEOREM 7. *Assume that B is a lattice⁸ with respect to the ordering given by K . Then Theorem 6 is true if “strongly quasi-positive” is replaced by “strictly quasi-positive.”*

Proof. Conditions (1), (2) and (3) in Theorem 6 imply T is strongly quasi-positive, hence, a fortiori, T is strictly quasi-positive. Now suppose T is strictly quasi-positive. Then $1 \in \sigma(T)$ and (2), (3) hold. It is easy to see from the representation of Theorem 4, $\tilde{T} = \sum_{j=1}^m \lambda_j P_j + S$, that $\|T^n\|$ is bounded independently of n . Hence, by a theorem of Krein-Rutman ([11], Theorem 8.1 and corollary), every $\lambda \in \sigma(T)$, $|\lambda| = 1$, is a root of unity. It is easily verified that every power of T is quasi-compact and strictly quasi-positive, hence the eigenspace for T^n corresponding to the eigenvalue 1 is one-dimensional for all n . If $\tilde{T}x = \lambda x$, $|\lambda| = 1$, $\lambda^n = 1$, then $\tilde{T}^n x = \lambda^n x = x$ and it follows that $\lambda = 1$ which completes the proof.

An immediate consequence is the following corollary.

COROLLARY. *If B is a lattice, every strictly quasi-positive and quasi-compact operator is strongly quasi-positive.*

The conclusion of this corollary is not true in general as we will illustrate by an example. Let B be three-dimensional (real) Euclidean

⁸ I.e., each pair of elements in B has a greatest lower bound and a least upper bound.

space, $B = \{(x_1, x_2, x_3)\}$, and let $K = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq x_3^2, x_3 \geq 0\}$. If we interpret "to the right" to mean any direction in which the x_3 coordinate is increasing, each non-trivial element $x^* \in K^+$ is represented by a plane through the origin whose unit normal at the origin directed to the right lies in K . Let T be a rotation about the x_3 axis through θ radians where θ and 2π are incommensurable. It is clear that $\|T^n\| = 1$ for all n and that $TK \subseteq K$. To show that T is strictly quasi-positive it suffices to consider $x^* \in K^+$ which is represented by a plane tangent to K . If p is in the interior of K , $T^n p$ is in the interior for all n , hence $x^*(T^n p) > 0$. Now let p be on the boundary of K . There exists exactly one point q which has the same x_3 coordinate as p and such that $x^*(q) = 0$. Since θ and 2π are incommensurable, there is at most one value of n such that $T^n p = q$. Therefore, $x^*(T^m p) > 0$ for all m sufficiently large and, hence, T is strictly quasi-positive. If p is on the boundary of K , so is $T^n p$ for all n . We can pick a sequence n_1, n_2, \dots such that $T^{n_i} p$ converges to a point q on the boundary of K and there exists $x^* \in K^+$ such that $x^*(q) = 0$, $x^* \neq 0$. This shows T is not strongly quasi-positive.

BIBLIOGRAPHY

1. T. Ando, *Positive linear operators in semi-ordered linear spaces*, J. Fac. Science, Hokkaido University, Ser. I, XIII, (1957), 214-228.
2. G. Birkhoff, *Extensions of Jentzsch's theorem*, Trans. Amer. Math. Soc., **85** (1957), 219-228.
3. F. F. Bonsall, *Endomorphisms of partially ordered vector spaces*, J. London Math. Soc., **30** (1955), 133-144.
4. ———, *Endomorphism of a partially ordered vector space without order unit*, J. London Math. Soc., **30** (1955), 144-153.
5. ———, *Linear operators in complete positive cones*, Proc. London Math. Soc., **3**, VIII (29), (1958), 53-75.
6. G. Frobenius, *Über Matrizen aus positiven Elementen*, Sitz. Ber. Preuss. Akademie der Wiss. Berlin (1908), 471-476, (1909), 514-518.
7. ———, *Über Matrizen aus nicht negativen Elementen*, Sitz. Ber. Preuss. Akademie der Wiss. Berlin (1912), 456-477.
8. E. Hopf, *Über lineare Integralgleichungen mit positiven Kern*, Sitz. Ber. Preuss. Akademie der Wiss. Berlin, XVIII (1928), 233-245.
9. R. Jentzsch, *Über Integralgleichungen mit positiven Kern*, Crelles J., **141** (1912), 235-244.
10. S. Karlin, *Positive Operators*, J. Math., **8** (1959), 907-938.
11. M. Krein and M. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Matem. Navk (1948), 3-95, Amer. Math. Soc. Translation No. 26.
12. O. Perron, *Zur Theorie der Matrizen*, Math. Ann., **64** (1907), 248-263.
13. H. Schaefer, *Positive Transformationen in lokal konvexen halbgeordneten Vektorräumen*, Math. Ann., **129** (1955), 323-329.
14. ———, *Some spectral properties of positive linear operators*, Pacific J. Math., **10** (1960), 1009-1019.

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