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**ON THE STRUCTURE OF INFRAPOLYNOMIALS WITH
PRESCRIBED COEFFICIENTS**

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Introduction. The main result of this paper is Theorem 5 which deals with the structure of infrapolynomials with prescribed coefficients. This theorem was quoted (without proof) in a previous paper [Shisha and Walsh, 1961]¹, and was used there to prove a few results concerning the geometrical location of the zeros of some infrapolynomials with prescribed coefficients [loc. cit., Theorems 11, 12, 16, 17]. Two similar results are given here in Theorem 6.

We refer the reader to the Introduction of the last mentioned paper for a review of the development of the concept of infrapolynomial. Here we shall just mention two of the underlying definitions.

A. Let n and q be natural numbers ($q \leq n$), n_1, n_2, \dots, n_q integers such that $0 \leq n_1 < n_2 < \dots < n_q \leq n$, and S a set in the complex plane². An n th *infrapolynomial on S with respect to (n_1, n_2, \dots, n_q)* is a polynomial $A(z) \equiv \sum_{\nu=0}^n a_\nu z^\nu$ such that no $B(z) \equiv \sum_{\nu=0}^n b_\nu z^\nu$ exists, satisfying the following properties.

- (1) $B(z) \not\equiv A(z)$,
- (2) $b_{n_\nu} = a_{n_\nu}$ ($\nu = 1, 2, \dots, q$),
- (3) $|B(z)| < |A(z)|$ whenever $z \in S$ and $A(z) \neq 0$, and
- (4) $B(z) = 0$ whenever $z \in S$ and $A(z) = 0$.

B. Let n be a natural number. A *simple n -sequence* is a sequence having one of the forms

$$(0, 1, \dots, k, n-l, n-l+1, \dots, n) [k \geq 0, l \geq 0, k+l+2 \leq n],$$

$$(0, 1, \dots, k) [0 \leq k < n], (n-l, n-l+1, \dots, n) [0 \leq l < n].$$

Theorem 5 may yield information on the location of the zeros of an n th infrapolynomial $A(z)$ on a set S with respect to a simple n -sequence σ . For it allows (under quite general conditions) to set $A(z) \equiv B(z) D(z)$ where $D(z)$ is a polynomial all of whose zeros lie in S , whereas $B(z)$ is a divisor of a polynomial $Q(z)$ whose structure is given by the theorem. By studying the location of the zeros of $Q(z)$, one may get information on the location of the zeros of $A(z)$. By this method, Theorems 11, 12, 16, 17 [loc. cit.] were proved. (Compare

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¹ Dates in square brackets refer to the bibliography.

² We deal throughout this paper with the *open* plane of complex numbers.

also the proof of Theorem 6 below.)

Theorem 5 is a generalization of Fekete's structure theorem [1951], and we use his method of proof [cf. also Fekete 1955]. The concept of a "juxtafunction" (Definition 1) is a generalization of Fekete's "nearest polynomial" [1955], later termed "juxtapolynomial" [Walsh and Motzkin 1957]. Theorems 1-4 and Lemmas 1-4 are contained in the author's Ph. D. thesis [1958]; they are needed for the proof of Theorem 5, and they generalize previous results of Fekete [1951, 1955]. The principal results of the present paper were published by the author (without proof) in abstracts (1958a, 1959, 1961].

1. DEFINITION 1. Let S be a set in the complex plane and let Π be a set of complex functions defined on³ S such that whenever $f_1 \in \Pi$, $f_2 \in \Pi$ and c_1, c_2 are complex numbers, then⁴ $c_1 f_1 + c_2 f_2 \in \Pi$. Let f be a complex function defined on S . A juxtafunction to f on S with respect to Π is an element p of Π having the property: there does not exist a $q \in \Pi$ satisfying

- (a) $q(z) \neq f(z)$ for at least one $z \in S$,
- (b) $|f(z) - q(z)| < |f(z) - p(z)|$ whenever $z \in S$ and $p(z) \neq f(z)$,
- (c) $q(z) = f(z)$ whenever $z \in S$ and $p(z) = f(z)$.

EXAMPLES A. Let $S (\neq \emptyset)$ be⁵ a closed and bounded set in the complex plane. Let $f, p_1, p_2, \dots, p_n, \mu$ be complex functions with domain S which are continuous on S , and assume, furthermore, that $\mu(z) \neq 0$ throughout S . For every complex function ψ with domain S which is continuous on S , let $\|\psi\| = \max\{|\mu(z)\psi(z)|, z \text{ on } S\}$. It is known that there exist complex numbers $\lambda_1^*, \lambda_2^*, \dots, \lambda_n^*$ such that for every complex $\lambda_1, \lambda_2, \dots, \lambda_n$,

$$(1) \quad \left\| f - \sum_{\nu=1}^n \lambda_{\nu}^* f_{\nu} \right\| \leq \left\| f - \sum_{\nu=1}^n \lambda_{\nu} f_{\nu} \right\|.$$

Consider the linear space Π of all linear combinations (with complex coefficients) of p_1, p_2, \dots, p_n . Then $p = \sum_{\nu=1}^n \lambda_{\nu}^* p_{\nu}$ is a juxtafunction to f on S with respect to Π . Indeed, suppose that some $q = \sum_{\nu=1}^n \lambda'_{\nu} p_{\nu}$ satisfies (a), (b) and (c) of Definition 1. Let ζ be a point of S such that

$$\|f - q\| = |\mu(\zeta)(f(\zeta) - q(\zeta))|.$$

Then by (a) $q(\zeta) \neq f(\zeta)$, and therefore, by (c), $p(\zeta) \neq f(\zeta)$. From (b) we get $\|f - \sum_{\nu=1}^n \lambda'_{\nu} f_{\nu}\| = \|f - q\| = |\mu(\zeta)(f(\zeta) - q(\zeta))| < |\mu(\zeta)(f(\zeta) - p(\zeta))|$.

³ i.e. their domains include S .

⁴ The domain of $c_1 f_1 + c_2 f_2$ is the intersection of those of f_1 and f_2 .

⁵ \emptyset denotes the empty set.

– $p(\zeta) \mid \leq \|f - q\| = \|f - \sum_{\nu=1}^n \lambda_{\nu}^* f_{\nu}\|$, contradicting (1).

B. Let f, p_1, p_2, \dots, p_n be real functions with domain $S = [0, 1]$, continuous there, and assume furthermore that p_1, p_2, \dots, p_n are orthonormal on $[0, 1]$. Let Π be again the set of all linear combinations (with complex coefficients) of p_1, p_2, \dots, p_n . Let $\lambda_{\nu}^* = \int_0^1 f(x)p_{\nu}(x)dx$ ($\nu = 1, 2, \dots, n$). Then $p = \sum_{\nu=1}^n \lambda_{\nu}^* p_{\nu}$ is a juxtafunction to f on S with respect to Π . Indeed, if $p = f$, then the last assertion follows from Lemma 1 below. We thus assume that $p(x_0) \neq f(x_0)$ for some $x_0 \in [0, 1]$. Suppose there exists a $q = \sum_{\nu=1}^n \lambda_{\nu} p_{\nu}$ satisfying (a), (b) and (c) of Definition 1. Then $|f(x) - q(x)| \leq |f(x) - p(x)|$ throughout $[0, 1]$, and $|f(x_0) - q(x_0)| < |f(x_0) - p(x_0)|$. Thus

$$\int_0^1 [f(x) - \sum_{\nu=1}^n \operatorname{Re}(\lambda_{\nu}) p_{\nu}(x)]^2 dx < \int_0^1 [f(x) - \sum_{\nu=1}^n \lambda_{\nu}^* p_{\nu}(x)]^2 dx,$$

contradicting the least squares property of the Fourier coefficients λ_{ν}^* .

LEMMA 1. *Let S and Π be as in Definition 1 and let f be an element of Π with domain S . Then f is the unique function with domain S which is a juxtafunction to f on S with respect to Π .*

Proof. f is such a juxtafunction, since (a) and (c) of Definition 1 are mutually contradictory when p is f . If p (with domain S) belongs to Π and $p \neq f$, then $q = \frac{1}{2}(p + f)$ belongs to Π and satisfies (a), (b) and (c), so that p is not a juxtafunction to f on S with respect to Π .

THEOREM 1.

Hypotheses.

1. $S (\neq \emptyset)$ is a closed and bounded set in the complex plane, f, p_1, p_2, \dots, p_n are complex functions defined and continuous on⁶ S .
2. Π is the set of all complex functions defined on S which can be represented throughout S as linear combinations (with complex coefficients) of the p 's.
3. p is a juxtafunction to f on S with respect to Π , and $p(z) \neq f(z)$ throughout S .

⁶ As the domain of f may properly include S , its continuity on S means that if $a \in S$, and if $(a_j)_{j=1}^{\infty}$ is a sequence of points of S converging to a , then $\lim_{j \rightarrow \infty} f(a_j) = f(a)$. Similarly for p_1, p_2, \dots, p_n and in Lemma 2.

Conclusion. There exist distinct points z_1, z_2, \dots, z_m of S ($1 \leq m \leq 2n + 1$) and positive $\lambda_1, \lambda_2, \dots, \lambda_m$ such that:

(I). $p(z)$ is a juxtafunction to f on $s = \{z_1, z_2, \dots, z_m\}$ with respect to Π ,

(II). No complex b_1, b_2, \dots, b_n exist such that $|f(z) - \sum_{\nu=1}^n b_\nu p_\nu(z)| < |f(z) - p(z)|$ throughout s ,

(III). $\sum_{\mu=1}^m \lambda_\mu p_\nu(z_\mu) / \{f(z_\mu) - p(z_\mu)\} = 0, \nu = 1, 2, \dots, n$.

REMARK 1. Observe that (I) is implied by (II).

For the proof of Theorem 1 we shall need two lemmas.

LEMMA 2. Let $S (\neq \emptyset)$ be a closed and bounded set in the complex plane, and Π a set of complex functions, defined and continuous on S such that whenever $f_1 \in \Pi, f_2 \in \Pi$, and c_1 and c_2 are complex numbers, then $c_1 f_1 + c_2 f_2 \in \Pi$. Let f be a complex function defined and continuous on S , and let p be an element of Π such that $p(z) \neq f(z)$ throughout S . A necessary and sufficient condition for the existence of a $q \in \Pi$ satisfying throughout S

$$(2) \quad |f(z) - q(z)| < |f(z) - p(z)|$$

is the existence of an $r \in \Pi$, satisfying throughout S

$$(3) \quad |f(z) - p(z) - r(z)| < |f(z) - p(z) + r(z)|.$$

Proof of Lemma 2.

Necessity. Let $r = q - p$. Then throughout S

$$|f(z) - p(z) - r(z)| < |f(z) - p(z)| < |f(z) - p(z)| \{2 - |f(z) - q(z)| \times |f(z) - p(z)|^{-1}\} \leq |2\{f(z) - p(z)\} - \{f(z) - q(z)\}| = |f(z) - p(z) + r(z)|.$$

Sufficiency. We use the fact that if a, b are arbitrary complex numbers, the inequalities $|a - b| < |a + b|, \operatorname{Re}(b \bar{a}) > 0$, are equivalent. Since throughout S

$$\operatorname{Re}[r(z) / \{f(z) - p(z)\}] = |f(z) - p(z)|^{-2} \operatorname{Re}[r(z) \overline{\{f(z) - p(z)\}}] > 0,$$

we have there $\alpha |r(z) / \{f(z) - p(z)\}|^2 < 2 \operatorname{Re}[r(z) / \{f(z) - p(z)\}]$ where $\alpha = \min [|\{f(z) - p(z)\} / r(z)|^2 \operatorname{Re}(r(z) / \{f(z) - p(z)\}), z \text{ on } S]$. Let $q = p + \alpha r$. Then throughout S ,

$$|f(z) - q(z)| = |f(z) - p(z)| |1 - \alpha r(z) \{f(z) - p(z)\}^{-1}| = |f(z) - p(z)| \times [1 + \alpha^2 |r(z) / (f(z) - p(z))|^2 - 2\alpha \operatorname{Re}\{r(z) (f(z) - p(z))^{-1}\}]^{1/2} < |f(z) - p(z)|.$$

LEMMA 3. Let the Hypotheses 1, 2 of Theorem 1 hold, and let p be an element of Π such that $p(z) \neq f(z)$ throughout S . For every $z \in S$, let $F(z)$ denote the point $(x_1(z), y_1(z), x_2(z), y_2(z), \dots, x_n(z), y_n(z))$ of the (real) Euclidean $2n$ -space E_{2n} , where $x_\nu(z)$ is the real part and $y_\nu(z)$ the imaginary part of $p_\nu(z) \{f(z) - p(z)\}$. A necessary and sufficient condition for the existence of a $q \in \Pi$ satisfying (2) throughout S , is that the point $\Omega_{2n} = (0, 0, \dots, 0)$ of E_{2n} does not belong to the convex hull H of⁷ $F(s)$.

Proof of Lemma 3.

Necessity. By Lemma 2 there exists an $r \in \Pi$ such that (3), i.e. the inequality

$$(3a) \quad \operatorname{Re}[r(z)\overline{\{f(z) - p(z)\}}] > 0$$

holds throughout S . Let $s_1, t_1, s_2, t_2, \dots, s_n, t_n$ be reals such that throughout S , $r(z) = \sum_{\nu=1}^n (s_\nu - it_\nu)p_\nu(z)$. Then throughout S we have

$$(4) \quad \sum_{\nu=1}^n s_\nu x_\nu(z) + t_\nu y_\nu(z) > 0$$

and thus $F(s)$ is a subset of the half-space

$$(5) \quad s_1 x_1 + t_1 x_2 + \dots + s_n x_{2n-1} + t_n x_{2n} > 0.$$

Therefore H is also a subset of this half-space, and consequently $\Omega_{2n} \notin H$.

Sufficiency. Since H is compact and $\Omega_{2n} \notin H$, we can find a half-space (5) containing $F(S)$. Thus (4) holds for every $z \in S$. Setting $r = \sum_{\nu=1}^n (s_\nu - it_\nu)p_\nu$, we have throughout S , (3a), and therefore (3). Thus, by Lemma 2, there exists a $q \in \Pi$ satisfying (2) throughout S .

Proof of Theorem 1. f cannot belong to Π , for otherwise, by Lemma 1, the restrictions of f and of p to S would coincide, contradicting Hypothesis 3. By Definition 1, there does not exist a $q \in \Pi$ satisfying (2) throughout S . Using notations of the last lemma, it follows that $\Omega_{2n} \in H$. By a well known theorem of Carathéodory there exist in $F(S)$ distinct points A_1, A_2, \dots, A_m ($m \leq 2n + 1$) and there exist positive $\lambda_1, \lambda_2, \dots, \lambda_m$ such that

$$(6) \quad \Omega_{2n} = \sum_{\mu=1}^m \lambda_\mu A_\mu.$$

Let

⁷ $F(s)$ is, as usual, the set of all $F(z)$, $z \in S$.

$$(7) \quad A_\mu = F(z_\mu), \quad z_\mu \in S \quad (\mu = 1, 2, \dots, m).$$

Then the z_μ are distinct, and from (6) we get by taking components,

$$(8) \quad \sum_{\mu=1}^m A_\mu p_\nu(z_\mu) \overline{\{f(z_\mu) - p(z_\mu)\}} = 0 \quad (\nu = 1, 2, \dots, n).$$

Thus

$$\sum_{\mu=1}^m \lambda_\mu p_\nu(z_\mu) / \{f(z_\mu) - p(z_\mu)\} = 0 \quad (\nu = 1, 2, \dots, n)$$

where $\lambda_\mu = A_\mu |f(z_\mu) - p(z_\mu)|^2 > 0$ ($\mu = 1, 2, \dots, m$). Let $s = \{z_1, z_2, \dots, z_m\}$, and let π be the set of all functions defined on s which can be represented throughout s as linear combinations (with complex coefficients) of the p_ν . Obviously $p \in \pi$, since $p \in \Pi$. From (6) and (7) it follows that Ω_{2n} belongs to the convex hull of $F(s)$ and therefore, by Lemma 3 (taking there s in place of S and π in place of Π) there does not exist a $q \in \pi$ satisfying (2) throughout s . This concludes the proof.

REMARK 2. Suppose that one of the p_ν in Theorem 1 equals throughout S a constant $c (\neq 0)$. Then from (8) we obtain $\sum_{\mu=1}^m A_\mu \{f(z_\mu) - p(z_\mu)\} = 0$. Thus 0 belongs to the convex hull of the image of s (and a fortiori of S) under $f - p$. [Compare Motzkin and Walsh 1953, § 2, and Fekete 1955, § 18].

REMARK 3. Let $s' = \{z_1, z_2, \dots, z_M\}$ be a finite set in the complex plane and suppose that f, p_1, p_2, \dots, p_n are complex functions defined on s' . Let π' be the set of all complex functions representable throughout s' as a linear combination with complex coefficients of p_1, p_2, \dots, p_n . Let p be an element of π' such that $p(z) \neq f(z)$ throughout s' , and suppose there exist nonnegative reals $\lambda'_1, \dots, \lambda'_M$ (not all zero) such that

$$\sum_{\mu=1}^M \lambda'_\mu p_\nu(z_\mu) / \{f(z_\mu) - p(z_\mu)\} = 0 \quad (\nu = 1, 2, \dots, n).$$

Then there does not exist a $q \in \pi'$ such that (2) holds throughout s' . Indeed, we have

$$\sum_{\mu=1}^M A'_\mu p_\nu(z_\mu) \overline{\{f(z_\mu) - p(z_\mu)\}} = 0 \quad (\nu = 1, 2, \dots, n)$$

where A'_μ are nonnegative reals, not all zero. Therefore (using notations of Lemma 3) Ω_{2n} belongs to the convex hull of $F(s')$. By Lemma 3, there does not exist a $q \in \pi'$ satisfying (2) throughout s' . Consequently, p is a juxtafunction to f on s' with respect to π' .

THEOREM 2. *Let the hypotheses of Theorem 1 hold and suppose furthermore that $f - p, p_1, p_2, \dots, p_n$ are real valued throughout S . Then the inequality $1 \leq m \leq 2n + 1$ in the conclusion of Theorem 1 can be replaced by $1 \leq m \leq n + 1$.*

Theorem 2 is proved with the aid of the following lemma, in the same way that Theorem 1 was proved with the aid of Lemma 3.

LEMMA 4. *Let the hypotheses 1, 2 of Theorem 1 hold, let p be an element of Π such that $f(z) \neq p(z)$ throughout S , and suppose that $f - p, p_1, p_2, \dots, p_n$ are real throughout S . For every $z \in S$, let $F_1(z)$ denote the point $(p_1(z)\{f(z) - p(z)\}, p_2(z)\{f(z) - p(z)\}, \dots, p_n(z)\{f(z) - p(z)\})$ of the (real) Euclidean n -space E_n . A necessary and sufficient condition for the existence of a $q \in \Pi$ satisfying (2) throughout S , is that the point $\Omega_n = (0, 0, \dots, 0)$ of E_n does not belong to the convex hull of $F_1(S)$.*

The proof of the last lemma is analogous to that of Lemma 3.

We shall make frequent use of the concept of unisolvence. We mention therefore the following

DEFINITION 2. *Let S be a set in the complex plane, and $(p_\nu(z))_{\nu=1}^n$ a finite sequence of complex functions defined on S . The sequence will be called unisolvent on S if and only if for every complex c_1, c_2, \dots, c_n (not all zero) the set of all $z \in S$ for which $\sum_{\nu=1}^n c_\nu p_\nu(z) = 0$, contains less than n points.*

REMARK 4. Thus $(p_\nu(z))_{\nu=1}^n$ is unisolvent on S if and only if this sequence is linearly independent on every n -point subset of S . A simple example is the sequence $(z^{\nu-1})_{\nu=1}^n$, which is unisolvent on every subset of the complex plane. A unisolvent sequence has been termed also (for an important particular case) a "Tchebycheff system". Other terms used in this connection are "Haar system" and "interpolational system".

THEOREM 3. *Let the hypotheses of Theorem 1 hold and suppose that each of the sequences $(p_\nu(z))_{\nu=1}^j$ ($j = 1, 2, \dots, n$) is unisolvent on S . Then the inequalities*

$$(9) \quad 1 \leq m \leq 2n + 1$$

in Theorem 1, can be replaced by the sharper estimate $n + 1 \leq m \leq 2n + 1$. Furthermore, if the additional hypothesis of Theorem 2 is

made too, (9) can be replaced by $m = n + 1$.

Proof. Choose distinct points z_1, z_2, \dots, z_m of S and positive $\lambda_1, \lambda_2, \dots, \lambda_m$ such that (I), (II) and (III) of Theorem 1 hold, where $1 \leq m \leq 2n + 1$ and where, furthermore, $1 \leq m \leq n + 1$ in case the additional hypothesis of Theorem 2 holds. We shall prove that $n + 1 \leq m$. Indeed: suppose $m \leq n$. Then since $(p_\nu(z))_{\nu=1}^m$ is unisolvent on S , the determinant whose j th row is $p_1(z_j) p_2(z_j) \dots p_m(z_j)$ is different from zero. Therefore there exist constants c_1, \dots, c_m such that $f(z) = \sum_{\nu=1}^m c_\nu p_\nu(z)$ throughout s . Let π have the same meaning as in the proof of Theorem 1; then $f \in \pi$. By Theorem 1, (II), p is a juxtafunction to f on s with respect to π . By Lemma 1 (with S replaced by s , Π by π , and f by the restriction of our f to s) we have $f(z) = p(z)$ throughout s , contradicting hypothesis 3 of Theorem 1.

2. We apply now Theorems 1, 2 and 3 to n th infrapolynomials (cf. the Introduction).

THEOREM 4. *Let n and q be natural numbers ($q \leq n$), n_1, n_2, \dots, n_q integers such that $0 \leq n_1 < n_2 < \dots < n_q \leq n$, and S a closed and bounded set in the complex plane. Let $A(z)$ ($\neq 0$ throughout S) be an n th infrapolynomial on S with respect to (n_1, \dots, n_q) . Then⁸ there exist distinct points z_1, z_2, \dots, z_m of S ,*

$$(10) \quad 1 \leq m \leq 2(n - q) + 3$$

and positive $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $A(z)$ is an n th infrapolynomial on $s = \{z_1, z_2, \dots, z_m\}$ with respect to (n_1, n_2, \dots, n_q) and such that

$$(11) \quad \sum_{\mu=1}^m \lambda_\mu z_\mu^{\nu} / A(z_\mu) = 0 \quad (\nu = 1, 2, \dots, n + 1 - q)$$

where $l_1, l_2, \dots, l_{n+1-q}$ ($l_1 < l_2 < \dots < l_{n+1-q}$) are the elements of $\{0, 1, \dots, n\} - \{n_1, n_2, \dots, n_q\}$. If the polynomials $A(z), z^{l_1}, \dots, z^{l_{n+1-q}}$ are real valued throughout S , then (10) can be replaced by $1 \leq m \leq n + 2 - q$. If each of the sequences $(z^{l_j})_{j=1}^n$ ($j = 1, 2, \dots, n + 1 - q$) is unisolvent on S , then (10) can be replaced by

$$(12) \quad n - q + 2 \leq m \leq 2(n - q) + 3.$$

If the polynomials $A(z), z^{l_1}, \dots, z^{l_{n+1-q}}$ are real valued throughout S and each of the sequences $(z^{l_j})_{j=1}^n$ ($j = 1, 2, \dots, n + 1 - q$) is unisolvent on S , then (10) can be replaced by $m = n - q + 2$.

REMARK 5. If (n_1, n_2, \dots, n_q) of Theorem 4 is a simple n -sequ-

⁸ As is easily seen, S cannot be empty. [Cf. Shisha and Walsh, 1961, footnote 7 on p. 117].

ence (cf. the Introduction) and if, in case $n_1 = 0$, $0 \notin S$, then as is easily seen, the sequences $(z^{l_j})_{j=1}^j$ ($j = 1, 2, \dots, n + 1 - q$) are unisolvent on S .

Proof of Theorem 4. Let Π be the set of all complex functions defined on S which are expressible throughout S as linear combinations of $z^{l_1}, z^{l_2}, \dots, z^{l_{n+1-q}}$ with complex coefficients, and let $f(z) \equiv \sum_{\nu=1}^q a_{n_\nu} z^{n_\nu}$, $p(z) \equiv - \sum_{\nu=1}^{n+1-q} a_{l_\nu} z^{l_\nu}$. It is easily seen that $p(z)$ is a juxtafunction to f on S with respect to Π . Therefore, by Theorem 1 there exist distinct points z_1, \dots, z_m ($m \leq 2(n + 1 - q) + 1 = 2(n - q) + 3$) of S and positive $\lambda_1, \lambda_2, \dots, \lambda_m$ such that (11) holds, and such that no complex $b_1, b_2, \dots, b_{n+1-q}$ exist satisfying

$$\left| \sum_{\nu=1}^q a_{n_\nu} z^{n_\nu} - \sum_{\nu=1}^{n+1-q} b_\nu z^{l_\nu} \right| < |A(z)|$$

throughout $s = \{z_1, z_2, \dots, z_m\}$. Thus $A(z)$ is an n th infrapolynomial on s with respect to (n_1, n_2, \dots, n_q) . The rest of Theorem 4 follows from Theorems 2 and 3.

REMARK 6. Let n, n_1, n_2, \dots, n_q be integers ($q \leq n, 0 \leq n_1 < n_2 < \dots < n_q \leq n$), $A(z) \equiv \sum_{\nu=0}^n a_\nu z^\nu$ a polynomial, z_1, z_2, \dots, z_M points of the complex plane, and $\lambda'_1, \lambda'_2, \dots, \lambda'_M$ ($\sum_{\mu=1}^M \lambda'_\mu > 0$) nonnegative reals such that $A(z_\mu) \neq 0$ ($\mu = 1, 2, \dots, M$), and such that $\sum_{\mu=1}^M \lambda'_\mu z_\mu^{l_\nu} / A(z_\mu) = 0$ ($\nu = 1, 2, \dots, n + 1 - q$), where the l_ν have the same meaning as in Theorem 4. Then $A(z)$ is an n th infrapolynomial on $s' = \{z_1, z_2, \dots, z_M\}$ with respect to (n_1, n_2, \dots, n_q) . Indeed: let f and p be as in the last proof, and let π' be the set of all complex functions representable throughout s' as a linear combination (with complex coefficients) of $z^{l_1}, z^{l_2}, \dots, z^{l_{n+1-q}}$. The asserted conclusion follows from Remark 3.

We give now the following structure theorem which is the main result of this paper.

THEOREM 5. Let n and q ($1 \leq q \leq n$) be integers, and σ a simple n -sequence of q elements. Let S be a closed and bounded set in the complex plane, and in case $0 \in \sigma$, assume that $0 \notin S$. Let $A(z)$ ($\neq 0$) be an n th infrapolynomial on S with respect to σ , and let $B(z)$ ($\neq 0$ throughout S) be a divisor of $A(z)$. Assume also that the degree⁹ r of $B(z)$ is $\geq q$. Then $B(z)$ is a divisor of some

$$(13) \quad Q(z) \equiv P(z)g(z) + z^K \sum_{\mu=1}^{M-q+2} \lambda_\mu g(z)/(z - z_\mu).$$

⁹ By degree of a polynomial ($\neq 0$) we mean its exact degree. The polynomial 0 is assigned the degree-1.

Here M is an integer satisfying $r \leq M \leq 2r - q + 1$, the z_ν are distinct points of S , $g(z) \equiv \prod_{\mu=1}^{M-q+2} (z - z_\mu)$, the λ_μ are positive reals with $\sum_{\mu=1}^{M-q+2} \lambda_\mu = 1$, $P(z)$ is a polynomial of degree $\leq q - 1$ such that $P(z)g(z) + z^{\kappa+M-q+1}$ is of degree $\leq M$, and K is $\min[\nu, \nu \notin \sigma, \nu = 0, 1, 2, \dots]$.

REMARK 7. As will be seen from the proof of Theorem 5, if S and the coefficients of $B(z)$ are real, the inequality $r \leq M \leq 2r - q + 1$ of the theorem can be replaced by the equality $M = r$.

In the proof of Theorem 5 use will be made of the following

LEMMA 5. Let n, q, σ and K be as in the last theorem, let S be a set in the complex plane, and let $A(z)$ ($\neq 0$) be an n th infrapolynomial on S with respect to σ . Let $B(z)$ be a polynomial of degree $r(\geq q)$ dividing $A(z)$. Then $B(z)$ is an r th infrapolynomial on S with respect to σ_0 , where σ_0 is that simple r -sequence of q elements for which $K = \min[\nu, \nu \notin \sigma_0, \nu = 0, 1, 2, \dots]$.

The proof of Lemma 5 is straightforward and may be omitted.

Proof of Theorem 5. By Lemma 5, $B(z)$ is an r th infrapolynomial on S with respect to the sequence σ_0 defined there. We choose (cf. Theorem 4 and Remark 5) distinct points z_1, z_2, \dots, z_m of S and positive $\lambda_1, \lambda_2, \dots, \lambda_m$ such that $\sum_{\mu=1}^m \lambda_\mu = 1$ and

$$(14) \quad \sum_{\mu=1}^m \lambda_\mu z_\mu^\rho / B(z_\mu) = 0$$

for every integer ρ satisfying $0 \leq \rho \leq r, \rho \notin \sigma_0$. Here m is an integer satisfying $r - q + 2 \leq 2(r - q) + 3$, and in case S and the coefficients of $B(z)$ are real we may take $m = r - q + 2$. Set

$$(15) \quad g(z) \equiv \prod_{\mu=1}^m (z - z_\mu), \quad N(z) \equiv \sum_{\mu=1}^m \lambda_\mu z_\mu^{r-q+\kappa+1} g(z) / \{B(z_\mu)(z - z_\mu)\}.$$

If μ and ν are integers, $1 \leq \mu \leq m, 0 \leq \nu \leq r - q + K$, then

$$[\lambda_\mu z_\mu^{r-q+\kappa+1} g(z) / \{B(z_\mu)(z - z_\mu)\}]_{z=0}^{(\nu)} = - \sum_{j=0}^{\nu} \lambda_\mu z_\mu^{r-q+\kappa-j} \binom{\nu}{j} j! g^{(\nu-j)}(0) / B(z_\mu)$$

(the equality is obvious if $z_\mu = 0$, and otherwise it is obtained by Leibnitz's rule for differentiating a product). Therefore, from (15) we get

$$(16) \quad N^{(\nu)}(0) = - \sum_{j=0}^{\nu} \binom{\nu}{j} j! g^{(\nu-j)}(0) \sum_{\mu=1}^m \lambda_\mu z_\mu^{r-q+\kappa-j} / B(z_\mu) \\ (\nu = 0, 1, \dots, r - q + K).$$

Since $\{0, 1, \dots, r\} - \{\sigma_0\} = \{r - q + K - j\}_{j=0}^{r-q}$, therefore (16) and (14) yield $N^{(\nu)}(0) = 0, \nu = 0, 1, \dots, r - q$. Hence we can write $N(z) \equiv z^{r-q+1}M_1(z)$ where $M_1(z)$ is a polynomial (of degree $\leq m - 2$). Let

$$M_2(z) \equiv \sum_{\mu=1}^m \lambda_\mu z_\mu^K g(z) / \{B(z_\mu)(z - z_\mu)\} .$$

By (14),

$$\sum_{\mu=1}^m \lambda_\mu z_\mu^K / B(z_\mu) = 0$$

and therefore the degree of $M_2(z)$ is $\leq m - 2$. For every z_j different from zero we have by (15), $M_1(z_j) = z_j^{-r+q-1}N(z_j) = \lambda_j z_j^K g'(z_j) / B(z_j) = M_2(z_j)$. Since there are at least $m - 1$ such z_j , we have $M_1(z) \equiv M_2(z)$. Consider now the polynomial

$$R(z) \equiv B(z)M_2(z) - \sum_{\mu=1}^m \lambda_\mu z_\mu^K g(z) / (z - z_\mu) .$$

For $j = 1, 2, \dots, m$ we have $R(z_j) = B(z_j)M_2(z_j) - \lambda_j z_j^K g'(z_j) = 0$. Therefore we can write $R(z) \equiv g(z)U(z)$, where $U(z)$ is some polynomial. Also, the relation $N(z) \equiv z^{r-q+1}M_2(z)$ and the definition of $R(z)$ imply that the degree of the latter is $\leq m + q - 2$. Therefore the degree of $U(z)$ is at most $q - 2$. If $K \geq 1$, then the relation

$$B(z)M_2(z) \equiv g(z)U(z) + \sum_{\mu=1}^m \lambda_\mu z_\mu^K g(z) / (z - z_\mu)$$

yields, upon putting $z_\mu^K = [z + (z_\mu - z)]^K$ and developing the last right member,

$$B(z)M_2(z) \equiv g(z)[U(z) + A_{K-1}(z)] + z^K \sum_{\mu=1}^m \lambda_\mu g(z) / (z - z_\mu) ,$$

where $A_{K-1}(z)$ is a polynomial of degree $K - 1$. The last relation (with $A_{K-1}(z) \equiv 0$) holds also when $K = 0$. We set now $P(z) \equiv U(z) + A_{K-1}(z)$, and get that $B(z)$ is a divisor of

$$Q(z) \equiv P(z)g(z) + z^K \sum_{\mu=1}^m \lambda_\mu g(z) / (z - z_\mu) .$$

The degree of $Q(z)$, i. e. of $B(z)M_2(z)$, is $\leq m + q - 2$. Thus the degree of $P(z)$ is $\leq q - 1$, and that of $P(z)g(z) + z^{K+m-1}$ is $\leq m + q - 2$. We set now $M = m + q - 2$, and observe that the conclusions of the theorem are all satisfied.

REMARK 8. The polynomial $Q(z)$ of (13) is an M th infrapolynomial on $\{z_1, z_2, \dots, z_{M-q+2}\}$ with respect to σ_1 , where σ_1 is that simple

M -sequence of q elements for which $\min [\nu, \nu \notin \sigma_1, \nu = 0, 1, 2, \dots] = K$. This follows from Theorem 1 of Shisha and Walsh [1961].

THEOREM 6. *Let S be a closed and bounded set in the complex plane, $A(z) \equiv \sum_{\nu=0}^n a_\nu z^\nu$ ($n \geq 1$, $a_n \neq 0$) an n th infrapolynomial on S with respect to $(n-1)$, and suppose that $A(z) \neq 0$ throughout S . Then:*

(a) *Every zero ζ of $A(z)$ is of the form*

$$(17) \quad c(\zeta) - \lambda(\zeta)[a_{n-1}/a_n]$$

where $c(\zeta)$ belongs to the convex hull of S and where $0 \leq \lambda(\zeta) \leq 1$.¹⁰

(b) *Suppose that S lies in a closed disc $C: |z - a| \leq r$ ($r \geq 0$). Then all zeros of $A(z)$ belong to $C \cup C_1$, where C_1 is the closed disc $|z - [a - (a_{n-1}/a_n)]| \leq r$. If C and C_1 are disjoint then $A(z)$ has at least $n-1$ zeros belonging to C . [Multiplicities are always being counted].*

Proof. We choose distinct points z_1, z_2, \dots, z_m of S and positive $\lambda_1, \lambda_2, \dots, \lambda_m$ ($m \leq 2n+1$) such that $\sum_{\mu=1}^m \lambda_\mu = 1$ and $\sum_{\mu=1}^m \lambda_\mu z_\mu^\rho / A(z_\mu) = 0$ for all integers ρ with $0 \leq \rho \leq n$, $\rho \neq n-1$. Then $1 = \sum_{\mu=1}^m \lambda_\mu A(z_\mu) / A(z_\mu) = \sum_{\mu=1}^m \lambda_\mu a_{n-1} z_\mu^{n-1} / A(z_\mu)$, and so

$$\sum_{\mu=1}^m \lambda_\mu z_\mu^{n-1} / A(z_\mu) = 1/a_{n-1}.$$

We set $g(z) \equiv \prod_{\mu=1}^m (z - z_\mu)$, $N(z) \equiv \sum_{\mu=1}^m \lambda_\mu z_\mu^{n-1} g(z) / \{A(z_\mu)(z - z_\mu)\} \equiv a_{n-1}^{-1} z^{m-1} + \dots$. We follow the proof of Theorem 5 from the sentence following (15). Again we have $N^{(\nu)}(0) = 0$ for every ν satisfying $0 \leq \nu \leq n-2$. Thus we may set $N(z) \equiv z^{n-1} M_1(z)$, where $M_1(z) \equiv a_{n-1}^{-1} z^{m-n} + \dots$ is some polynomial. Let $M_2(z) \equiv \sum_{\mu=1}^m \lambda_\mu g(z) / \{A(z_\mu)(z - z_\mu)\}$. If $n=1$, then $M_2(z) \equiv N(z) \equiv M_1(z)$. If $n > 1$ then for each z_j different from zero, $M_1(z_j) = \lambda_\mu g'(z_j) / A(z_j) = M_2(z_j)$, and since there are at least $m-1$ such z_j and $M_1(z)$ and $M_2(z)$ are of degrees $\leq m-2$, we have again $M_2(z) \equiv M_1(z)$. Consider now the polynomial $R(z) \equiv A(z)M_2(z) - \sum_{\mu=1}^m \lambda_\mu g(z) / (z - z_\mu) \equiv (a_n/a_{n-1})z^m + \dots$. For $j=1, 2, \dots, m$, $R(z_j) = 0$, and therefore $R(z) \equiv (a_n/a_{n-1})g(z)$. Thus, $A(z)$ is a divisor of $Q(z) \equiv (a_n/a_{n-1})g(z) + \sum_{\mu=1}^m \lambda_\mu g(z) / (z - z_\mu)$. Let ζ be a zero of $A(z)$. Then $g(\zeta) \neq 0$, and thus $a_n/a_{n-1} + \sum_{\mu=1}^m \lambda_\mu / (\zeta - z_\mu) = 0$. Since $\sum_{\mu=1}^m \lambda_\mu / (\zeta - z_\mu)$ can be written [Shisha and Walsh 1961, Lemma on p. 127] as $\lambda(\zeta) / (\zeta - c(\zeta))$ where $c(\zeta)$ and $\lambda(\zeta)$ are as required in (a) of our theorem, ζ is of the form (17). Suppose now that S lies in a closed disc $C: |z - a| \leq r$ ($r \geq 0$). Then by a theorem due to J. L. Walsh [cf. 1922, Theorem VI; see also Shisha and Walsh 1961, p. 147] all zeros of $Q(z)$ lie in

¹⁰ Thus ζ belongs to the set swept by the convex hull of S while being displaced, the displacement being given by the vector $-a_{n-1}/a_n$.

$C \cup C_1$, and if C and C_1 are disjoint, the number of zeros of $Q(z)$ in them is, respectively, $m - 1$ and 1. From this follow the conclusions of part (b) of our theorem.

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