Pacific Journal of Mathematics

ON COMPARABLE MEANS

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Vol. 14, No. 3 July 1964

ON COMPARABLE MEANS

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1. Let $-\infty < a < b < \infty$, and let Φ denote the set of all functions, continuous and strictly monotone in $[a, b]$. For every $\varphi \in \Phi$, every positive integer n, every $x_1, x_2, \cdots, x_n$ of $[a, b]$, and every positive $q_1, q_2, \cdots, q_n$ with $\sum_{\nu=1}^n q_\nu = 1$, we consider the mean

$$M_\varphi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n) = \varphi^{-1}(\textstyle\sum_{\nu=1}^n q_\nu \varphi(x_\nu)) .$$

Let ψ and χ be elements of Φ. We write

(1) $$M_\psi \leqq M_\chi$$

if and only if the inequality $M_\psi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n) \leqq M_\chi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n)$ holds for every $n \geqq 1$, every $x_1, x_2, \cdots, x_n$ of $[a, b]$, and every positive $q_1, q_2, \cdots, q_n$ with $\sum_{\nu=1}^n q_\nu = 1$.

A well-known necessary and sufficient condition for (1) to hold is that $\chi(\psi^{-1}(x))$ be convex in $[\psi(a), \psi(b)]$ (or $[\psi(b), \psi(a)]$) if χ is increasing, and that $\chi(\psi^{-1}(x))$ be concave there if χ is decreasing.

It is not difficult to see that (1) holds if and only if $M_\psi(x_1, x_2 \mid q_1, q_2) \leqq M_\chi(x_1, x_2 \mid q_1, q_2)$ for every x_1, x_2 of $[a, b]$ and every positive q_1, q_2 with $q_1 + q_2 = 1$, which in turn holds if and only if $M_\psi(x_1, x_2 \mid 1/2, 1/2) \leqq M_\chi(x_1, x_2 \mid 1/2, 1/2)$ for every x_1, x_2 of $[a, b]$.

Similarly, we write

(2) $$M_\psi < M_\chi$$

if and only if the inequality

$$M_\psi(x_1, x_2, \cdots, x_n \mid q_1, q_2 \cdots, q_n) < M_\chi(x_1, x_2, \cdots, x_n \mid q_1, q_2 \cdots, q_n)$$

holds for every $n \geqq 2$, every $x_1, x_2, \cdots, x_n$ (not all equal) of $[a, b]$, and every positive $q_1, q_2, \cdots, q_n$ with $\sum_{\nu=1}^n q_\nu = 1$. A necessary and sufficient condition for (2) to hold is that $\chi(\psi^{-1}(x))$ be strictly convex in $[\psi(a), \psi(b)]$ (or $[\psi(b), \psi(a)]$) if χ is increasing, and that $\chi(\psi^{-1}(x))$ be strictly concave there if χ is decreasing. Also, (2) holds if and only if $M_\psi(x_1, x_2 \mid q_1, q_2) < M_\chi(x_1, x_2 \mid q_1, q_2)$ for every x_1, x_2 $(\neq x_1)$ of $[a, b]$ and every positive q_1, q_2 with $q_1 + q_2 = 1$, which in turn holds if and only if $M_\psi(x_1, x_2 \mid 1/2, 1/2) < M_\chi(x_1, x_2 \mid 1/2, 1/2)$ for every x_1 and x_2 $(\neq x_1)$ of $[a, b]$.

2. In this paper we give simple criteria for the validity of (1)

Received December 30, 1963. The work of the second author was supported by the National Science Foundation through grant NSF-GP 1086.

and of (2), and then we give a few applications.

THEOREM 1. *Let* ψ *and* χ *be elements of* Φ *differentiable in* (a, b), *and let* $\psi' \neq 0$ *there. A necessary and sufficient condition for* (1) *to hold is that* χ'/ψ' *be nondecreasing in* (a, b) *if* ψ *and* χ *are monotone in the same sense, and that* χ'/ψ' *be nonincreasing there if* ψ *and* χ *are monotone in opposite senses.*

Proof. Consider the function $u(x) \equiv \chi(\psi^{-1}(x))$. Let J denote the open interval joining $\psi(a)$ to $\psi(b)$, and let $\bar{J}$ be the closure of J. For every $\xi \in J$, we have

$$u'(\xi) = \chi'(\psi^{-1}(\xi))/\psi'(\psi^{-1}(\xi)) \,. \tag{3}$$

Suppose that ψ and χ are monotone in the same sense. Then (1) holds if and only if $u(x)$ is convex in $\bar{J}$ in case χ increases, and if and only if $u(x)$ is concave there in case χ decreases. So (1) holds if and only if $u'(x)$ is nondecreasing in J in case ψ increases, and if and only if $u'(x)$ is nonincreasing there in case ψ decreases. From this, with the aid of (3), one easily infers that (1) is equivalent to χ'/ψ' being nondecreasing in (a, b). Similariy one shows that (1) is equivalent to χ'/ψ' being nonincreasing in (a, b), if ψ and χ are monotone in opposite senses.

One can modify Theorem 1 by replacing in it (1) by (2), "nondecreasing" by "strictly increasing," and "nonincreasing" by "strictly decreasing."

3. Given a function ψ, one may construct by means of Riemann-Stieltjes integrals functions χ such that $M_\psi \leqq M_\chi$. In fact, we have the following

THEOREM 2. *Let* ψ *be a real function, continuous in* $[a, b]$ *and differentiable in* (a, b). *Let* $m(x)$ *be nondecreasing or nonincreasing in* $[a, b]$, *continuous in* (a, b), *and suppose* $m(x)\psi'(x) \neq 0$ *throughout* (a, b). *Let* C *be a real constant, and for every* $x \in [a, b]$ *let*

$$\chi(x) = C + \int_a^x m(t) d\psi(t) \,.$$

Then ψ *and* χ *belong to* Φ. *If* $m(x)$ *is positive in* (a, b) *and nondecreasing in* $[a, b]$, *or negative in* (a, b) *and nonincreasing in* $[a, b]$, *then* $M_\psi \leqq M_\chi$. *Otherwise,* $M_\chi \leqq M_\psi$.

Proof. Since $\psi' \neq 0$ in (a, b), by a well known property of the derivative, ψ' is either positive throughout (a, b), or negative throughout (a, b). Thus ψ is strictly monotone in $[a, b]$. Also, by well-known properties of the Riemann-Stieltjes integral, χ is continuous in $[a, b]$, and $\chi'(x) = m(x)\psi'(x)$ throughout (a, b) (and so χ is strictly monotone in $[a, b]$). If $m(x)$ is positive in (a, b) and nondecreasing in $[a, b]$, then ψ and χ are monotone in the same sense in $[a, b]$, χ'/ψ' is nondecreasing in (a, b), and hence by Theorem 1, $M_\psi \leqq M_\chi$. Similarly the rest of Theorem 2 follows.

Theorem 2 can be modified by replacing in it "nondecreasing" by "strictly increasing," "nonincreasing" by "strictly decreasing," "$M_\psi \leqq M_\chi$" by "$M_\psi < M_\chi$," and "$M_\chi \leqq M_\psi$" by "$M_\chi < M_\psi$."

4. A converse of Theorem 2 is the following

THEOREM 3. *Let ψ and χ be elements of Φ differentiable in (a, b), and suppose $\psi' \neq 0$ there. Suppose, furthermore, that $M_\psi \leqq M_\chi$. Then there exists a function $m(x)$, nondecreasing in (a, b) if ψ and χ are monotone in the same sense, and nonincreasing there if ψ and χ are monotone in opposite senses, such that throughout $[a, b]$*

$$\chi(x) = \chi(a) + \int_a^x m(t)\psi'(t)dt \qquad \text{(a Lebesgue integral)}\,. \tag{4}$$

Proof. For every $x \in (a, b)$, let $m(x) = \chi'(x)/\psi'(x)$. By Theorem 1, $m(x)$ has the monotonicity property steated in Theorem 3. Now for every $x \in [a, b]$

$$\chi(x) - \chi(a) = \int_a^x \chi'(t)dt = \int_a^x m(t)\psi'(t)dt$$

(cf. [5], Theorems 269 (p. 188) and 264 (p. 183)).

REMARK. Observe that the integral in (4) can be written, under appropriate conditions, as a Riemman-Stieltjes integral: $\int_a^x m(t)\,d\psi(t)$. [Cf. loc. cit, Theorem 322.1 (p. 254), and 322 (p. 253)].

Theorem 3 remains valid if we replace in it "$M_\psi \leqq M_\chi$" by "$M_\psi < M_\chi$," "nondecreasing" by "strictly increasing," and "nonincreasing" by "strictly decreasing."

5. It is known that if the end-point a is positive and $r < s$, $rs \neq 0$, then $M_{x^r} < M_{x^s}$, and $M_{x^{-|r|}} < M_{\log x} < M_{x^{|r|}}$. Consequently, if $a > 0$ then for every real r ($\neq 0, 1$), $M_{(x^r)'} < M_{x^r}$, and $M_{(\log x)'} < M_{\log x}$. The question thus arises: Under what conditions on a function φ does one have $M_{\varphi'} < M_\varphi$ (or $M_{\varphi'} \leqq M_\varphi$)?

THEOREM 4. *A necessary and sufficient condition for a real* *function* φ *to fulfill the conditions* (α)–(γ) *below is that* $\varphi(x)$ *should be (throughout* $[a, b]$*) of one of the forms* $A + \int_a^x \exp C(t)dt$, $A - \int_a^x \exp C(t)dt$, $A + \int_a^x \exp\{-C(t)\}dt$, $A - \int_a^x \exp\{-C(t)\}dt$, *where* A *is a real number, and* $C(t)$ *is a function, continuous and convex in* $[a, b]$, *differentiable in* (a, b), *and satisfying there* $C'(x) < 0$.

(α) φ *is twice differentiable in* (a, b), $\varphi'(a)$ *and* $\varphi'(b)$ *exist as right and left hand derivatives, respectively,* $\varphi'(a)\varphi'(b) \neq 0$, *and* φ' *is continuous in* $[a, b]$.

(β) $\varphi'\varphi'' \neq 0$ *throughout* (a, b) *(and hence* φ *and* φ' *are strictly monotone in* $[a, b]$*).*

(γ) $M_{\varphi'} \leqq M_{\varphi}$.

Proof.

Necessity. By Theorem 1, φ'/φ'' is either positive and nondecreasing in (a, b), or negative and nonincreasing there. Thus, φ''/φ' is either positive and nonnincreasing in (a, b), or negative and nondecreasing there. In the first case we set $C(x) = -\log|\varphi'(x)|$ (in $[a, b]$). Then $C(x)$ is continuous in $[a, b]$ and $C'(x) < 0$ in (a, b). Also $C'(x)$ is nondecreasing in (a, b), and, therefore, $C(x)$ is convex in $[a, b]$. Either for every $x \in [a, b]$, $\varphi(x) = \varphi(a) + \int_a^x \exp\{-C(t)\}dt$, or for every $x \in [a, b]$, $\varphi(x) = \varphi(a) - \int_a^x \exp\{-C(t)\}dt$. In the second case, we set $C(x) = \log|\varphi'(x)|$ (in $[a, b]$). Then $C(x)$ is continuous in $[a, b]$, $C'(x) < 0$ in (a, b), and, again, $C(x)$ is convex in $[a, b]$. Either for every $x \in [a, b]$, $\varphi(x) = \varphi(a) + \int_a^x \exp C(t)dt$, of for every $x \in [a, b]$, $\varphi(x) = \varphi(a) - \int_a^x \exp C(t)dt$.

Sufficiency. (α) and (β) clearly hold. Also, by the convexity of $C(t)$, $C'(t)$ is nondecreasing in (a, b). Now, either throughout (a, b), $\varphi'/\varphi'' = \{C'(t)\}^{-1}$, or throughout (a, b), $\varphi'/\varphi'' = -\{C'(t)\}^{-1}$. In the first case, φ' and φ are monotone in opposite senses, and φ'/φ'' is nonincreasing in (a, b). In the second case, φ' and φ are monotone in the same sense, and φ'/φ'' is nondecreasing in (a, b). In either case, by Theorem 1, $M_{\varphi'} \leqq M_{\varphi}$.

Theorem 4 can be modified by replacing in it "convex" by "strictly convex," and "$M_{\varphi'} \leqq M_{\varphi}$" by "$M_{\varphi'} < M_{\varphi}$."

THEOREM 5. *Let* φ *be strictly monotone in* $[a, b]$ *and three-times differentiable in* (a, b). *Let* φ' *be continuous in* $[a, b]$ *(where* $\varphi'(a)$

and $\varphi'(b)$ are right and left hand derivatives, respectively). Let $\varphi'' \neq 0$ throughout (a, b). A necessary and sufficient condition for $M_{\varphi'} \leqq M_\varphi$ to hold is that $\varphi''^2 \geqq \varphi'\varphi'''$ throughout (a, b) if φ' and φ are monotone in the same sense, and that $\varphi''^2 \leqq \varphi'\varphi'''$ throughout (a, b) if φ' and φ are monotone in opposite senses.

Theorem 5 follows easily from Theorem 1 by considering the derivative of φ'/φ''.

Similarly, under the hypotheses of Theorem 5, $M_{\varphi'} < M_\varphi$ holds, if $\varphi''^2 > \varphi'\varphi'''$ throughout (a, b) and φ and φ' are monotone in the same sense, and also if $\varphi''^2 < \varphi'\varphi'''$ throughout (a, b) and φ and φ' are monotone in opposite senses.

As an example, let $a = 0$, $b = \pi/2$, $\varphi(x) \equiv \cos x$. φ and φ' are monotone in the same sense in $[0, \pi/2]$, and $\varphi''^2 = \cos^2 x > -\sin^2 x = \varphi'\varphi'''$ throughout $(0, \pi/2)$. Therefore, $M_{-\sin x} < M_{\cos x}$, i.e., $M_{\sin x} < M_{\cos x}$.

6. In a previous paper [3] the authors studied, for given positive $q_1, q_2, \cdots, q_n$ (with $\sum_{\nu=1}^n q_\nu = 1$), the ratio

$$(5)\quad \begin{cases} F(x_1, x_2, \cdots, x_n) \\ \quad = M_\chi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n)/M_\psi(x_1, x_2, \cdots, x_n \mid q_1, q_2, \cdots, q_n) \end{cases}$$

where $0 < a$, $\psi(x) \equiv x^r$, $\chi(x) \equiv x^s$ $(r < s,\ rs \neq 0)$.

Their purpose was to find an upper bound for F in

$$I = \{(x_1, x_2, \cdots, x_n):\ a \leqq x_k \leqq b,\ k = 1, 2, \cdots, n\}\ .$$

A crucial step was to show that if X^* is a point of I such that $F(X^*) = \max\{F(X) : X \in I\}$, then X^* is necessarily a vertex of I. In particular, X^* cannot be an interior point of I. This last property holds under quite general conditions:

THEOREM 6. *Let ψ and χ be elements of Φ, differentiable in (a, b), and satisfying $\psi'\chi' \neq 0$ there. Assume $0 \notin [a, b]$, $M_\psi < M_\chi$. Let $q_1, \cdots, q_n$ $(n > 1)$ be given positive numbers with $\sum_{\nu=1}^n q_\nu = 1$, and let I be as in the last paragraph. Let F of* (5) *attain its maximum in I at a point $X^* = (x_1^*, \cdots, x_n^*)$ of I. Then X^* is not an interior point of I.*

Proof. Suppose that some x_j^* satisfies $a < x_j^* < b$. Then $(\partial F/\partial x_j)_{x_\nu = x_\nu^*,\ \nu=1,2,\cdots,n} = 0$, i.e.,

$$\Big[\psi^{-1}\Big(\sum_{\nu=1}^n q_\nu\psi(x_\nu^*)\Big)\Big]^{-2}\Big[q_j\chi'(x_j^*)\psi^{-1}\Big(\sum_{\nu=1}^n q_\nu\psi(x_\nu^*)\Big)\Big/\chi'\Big(\chi^{-1}\Big(\sum_{\nu=1}^n q_\nu\chi(x_\nu^*)\Big)\Big)$$
$$- q_j\psi'(x_j^*)\chi^{-1}\Big(\sum_{\nu=1}^n q_\nu\chi(x_\nu^*)\Big)\Big/\psi'\Big(\psi^{-1}\Big(\sum_{\nu=1}^n q_\nu\psi(x_\nu^*)\Big)\Big)\Big] = 0\ .$$

Thus

$$\chi'(x_j^*)/\psi'(x_j^*) = \left[\chi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \chi(x_\nu^*)\Big)\chi'\Big(\chi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \chi(x_\nu^*)\Big)\Big)\right] \times \Big/ \left[\psi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \psi(x_\nu^*)\Big)\psi'\Big(\psi^{-1}\Big(\sum_{\nu=1}^{n} q_\nu \psi(x_\nu^*)\Big)\Big)\right].$$

Let C denote the right hand side of the last equality. If both x_j^* and x_k^* are interior points of $[a, b]$, then $\chi'(x_j^*)/\psi'(x_j^*) = C = \chi'(x_k^*)/\psi'(x_k^*)$, and hence, by the strict monotonicity of χ'/ψ' [see the end of § 2], $x_j^* = x_k^*$. Thus, if X^* were an interior point of I, we would have $x_1^* = x_2^* = \cdots = x_n^*$, and therefore

$$1 = F(x_1^*, x_2^*, \cdots, x_n^*) = \max\{F(X) : X \in I\} > 1 .$$

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

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Pacific Journal of Mathematics

Vol. 14, No. 3 July, 1964